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Strong existence, uniqueness and non-uniqueness in an equation involving local time

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Strong Existence, Uniqueness and Non-uniqueness in an Equation Involving Local Time

by

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1. Introduction

In [12] Protter and Sznitman proved if $|\alpha|>1$, $\beta\in\mathbb{R}$ and B_t is a Brownian motion, then

$$(1.1) \qquad X_t + \alpha L_t^0(X) = B_t + \beta L_t^0(B)$$

holds if and only if $\alpha=\beta$ and $X=B$. Here $L_t^0(X)$ is the symmetric local time of the semimartingale X . They posed the problem of investigating solutions of (1.1) when $|\alpha|\leq 1$. The case $\beta=0$ had already been studied by Harrison and Shepp [4] who showed that (1.1) has a unique solution, distributed as a skew Brownian motion. In this paper we study existence, uniqueness and the structure of solutions of (1.1) for general $\beta\in\mathbb{R}$ and $|\alpha|\leq 1$.

Note first that by replacing (B,X) with $(-B,-X)$, we may assume, without loss of generality, that $\alpha\in(0,1]$ (recall that we are working with the symmetric local time). Moreover it is easy to see that nothing is lost by assuming $B_0=0$. Solutions to (1.1), which are adapted to the natural filtration of B, F_t^B , are shown to exist for all $\alpha\in(0,1]$, $\beta\in\mathbb{R}$. If $\alpha\in(0,1)$, the solution is unique if and only if $\beta\leq\alpha/(1+\alpha)$ or $\beta\geq\alpha/(1-\alpha)$ (Theorem 3.4). If $\alpha=1$, then uniqueness is established for $\beta\leq\frac{1}{2}$, while non-uniqueness is proved for $\beta>\frac{1}{2}$

(Corollary 4.3 and Theorems 4.7 and 4.9). Moreover whenever non-uniqueness is established, $\mathcal{F}_t^B$ adapted minimal and maximal solutions of (1.1) are constructed.

A technique of [4] is used to transform (1.1) into an equation of the form

$$(1.2) \qquad dY_t = \sigma(Y_t)d(B + \beta L(B))_t$$

where σ is discontinuous, and also degenerate if $\alpha=1$. Due to the particular nature of σ, existence and uniqueness results for (1.2) may be obtained by studying the simpler equation

$$(1.3) \qquad dY_t = \sigma(Y_t)dB_t + \beta dL_t(B) \quad .$$

It is the study of these transformed equations that lead to our interest in (1.1). In Section 2 the weak existence of a solution to (1.3) is established using nonstandard analysis.

For $0<\alpha<1$, a technique of LeGall [10] is used to prove pathwise uniqueness and hence strong existence for (1.3). A different method must be used for $\alpha=1$ but strong existence and pathwise uniqueness still hold in (1.3) even though σ may be degenerate (see Theorem 4.4). The case $\alpha\epsilon(0,1)$ is studied in Section 3, while $\alpha=1$ is treated in Section 4. In Section 5 the corresponding results are stated without proof for the related equation

$$X_t + \alpha L_t^{0+}(X) = B_t + \beta L_t^0(B) \ ,$$

where $L_t^{0+}(X)$ denotes the "right local time" of the semi-martingale X at 0, i.e.

$$L_t^{0+}(X) = \lim_{\varepsilon\to 0+} \varepsilon^{-1} \int_0^t 1_{(X_s\in[0,\varepsilon])} d\langle X,X\rangle_s$$

$$L_t^{0-}(X) = \lim_{\varepsilon\to 0+} \varepsilon^{-1} \int_0^t 1_{(X_s\in[-\varepsilon,0])} d\langle X,X\rangle_s$$

$$L_t^{0}(X) = \tfrac{1}{2}(L_t^{0+}(X) + L_t^{0-}(X)) \quad .$$

If there is no ambiguity (for example if $X=B$) we simply write $L_t^0(X)$ for the local time at zero.

We shall always work on a probability space $(\Omega,\mathcal{F},\mathcal{F}_t,P)$ satisfying the usual conditions. B_t will always be an $\mathcal{F}_t$-Brownian motion with $B_0=0$, and L_t or $L_t(B)$ will be its local time at 0 . C will denote a constant whose exact value may change from line to line.

2. A Weak Existence Theorem

In this section nonstandard analysis is used to prove an existence theorem for one-dimensional stochastic differential equations of the form

$$Y_t = \int_0^t \sigma(Y_s)dB_s + V_t \ ,$$

where V has sample paths of bounded variation and $\sigma:\mathbb{R}\to\mathbb{R}$ may be degenerate and discontinuous. Results of this type have been proved in d-dimensions by Kosciuk [9] (also using nonstandard analysis). We give a separate proof here since the result we need is not quite covered by Kosciuk's theorem and because the existence of local time makes the proofs in one dimension much simpler.

The reader skilled in weak convergence arguments will undoubtedly be able to give a standard proof but perhaps will also appreciate the brevity of the nonstandard approach. A good introduction to nonstandard probability thoery may be found in Loeb [11], while further material is in Keisler [8] and Hoover and Perkins [5]. Although specific references to [5] and [8] are given freely, the proof may well be inaccessible to the reader who is unfamiliar with nonstandard probability theory.

Let W denote Wiener measure on $C[0,\infty)$, the space of continuous functions on $[0,\infty)$ with its Borel sets for the compact-open topology, and let $\mathcal{C}_t$ denote the σ-algebra generated by the coordinate mappings up to t and the W-null sets.

Definition An $\mathcal{F}_t^B$ adapted process of finite variation with continuous paths is a measurable mapping $V:C[0,\infty)\to C[0,\infty)$ such that $V(\cdot)(t)$ is $\mathcal{C}_t$-measurable for all t and $t\to V(t)$ has finite variation on compacts W-a.s.

If V is as above, then a weak solution of

$$(2.1) \qquad Y_t = \int_0^t \sigma(Y_s)dB_s + V_t(B)$$

is a probability space $(\Omega,\mathcal{F},\mathcal{F}_t,P)$, satisfying the usual conditions, that carries an $\mathcal{F}_t$ Brownian motion B and an optional process Y for which (2.1) holds.

Theorem 2.1 Assume $\sigma:\mathbb{R}\to\mathbb{R}$ is bounded, has limits from the left and right, and $\sigma(x)=0$ whenever $|\sigma(x^+)|\wedge|\sigma(x^-)|=0$. Let V be an $\mathcal{F}_t^B$ adapted process of finite variation with continuous paths. Then there is a weak solution of (2.1).

Proof. Let $(\Omega,\mathcal{F},\mathcal{F}_t,P)$ be an adapted Loeb space carrying an $\mathcal{F}_t$-Brownian motion, B (see [5, Def. 3.1]). Let Δt be a positive infinitesimal and define $T = \{k\Delta t | k\in {}^*\mathbb{N}|$. Points in T are denoted by $\underline{s}$, $\underline{t}$, etc. By [5, Th.7.6], there is an internal filtration, $\{\mathcal{B}_{\underline{t}}, \underline{t}\in T\}$, and a $\{\mathcal{B}_{\underline{t}}\}$-semimartingale lifting of $(V_t(B), B_t)$ which we denote by $(\overline{V}_t, \overline{B}_t)$. We define an internal process $\overline{Y}_{\underline{t}}$ inductively by

$$(2.2) \qquad \overline{Y}(\underline{t}+\Delta t) = \sum_{\underline{s}\leq\underline{t}} {}^*\sigma(\overline{Y}_{\underline{s}})(\overline{B}(\underline{s}+\Delta t)-\overline{B}(\underline{s})) + \overline{V}(\underline{t}+\Delta t) ,$$

where ${}^*\sigma: {}^*\mathbb{R} \to {}^*\mathbb{R}$ is the nonstandard extension of σ. If $\overline{M}(\underline{t}) = \overline{Y}(\underline{t}) - \overline{V}(\underline{t})$ then $\overline{M}$ is a $\mathcal{B}_{\underline{t}}$-martingale and has S-continuous paths by the continuity theorem for internal martingales (see [5, Th.8.5]). Therefore we may define a continuous local martingale, M, by $M=st(\overline{M})$ (see [5, Th.5.2]) and a semimartingale, Y, by $Y=st(\overline{Y}) = M+V$. Here st is the standard part map on the space of continuous functions with the compact-open topology (see Keisler [8, Prop.1.17]).

Let

$$[\overline{B},\overline{B}](\underline{t}) = \sum_{\underline{s}<\underline{t}} (\overline{B}(\underline{s}+\Delta t)-\overline{B}(\underline{s}))^2 ,$$

let $L([\overline{B},\overline{B}])$ be the Loeb measure on T induced by this internal increasing process, and define $[\overline{M},\overline{M}]$ and $L([\overline{M},\overline{M}])$ in a similar way. If H is the countable set of discontinuities of σ, then

$$0 = \int_0^\infty I(Y(s)\in H)\,d[M,M](s)$$

$$= \int_{ns(T)} I(Y({}^o\underline{s})\in H)\,dL([\overline{M},\overline{M}]) \qquad \text{(by [5, Lemma 2.7 and Th.6.7])}$$

Therefore

$$(2.3) \qquad 0 = \int_{ns(T)} I(Y({}^o\underline{s})\in H)\,{}^o{}^*\sigma(\overline{Y}_{\underline{s}})^2\,dL(L[\overline{B},\overline{B}]) .$$

Note that since $Y_{{}^o\underline{s}} = {}^o\overline{Y}_{\underline{s}}$ for all $\underline{s}\in s(T)$ a.s. ,

$$\int_{ns(T)} I(\sigma(Y_{^o\underline{s}}) \neq {}^o{*}\sigma(\overline{Y}_{\underline{s}}))dL([\overline{B},\overline{B}])$$

$$(2.4)\quad = \int_{ns(T)} I(Y_{^o\underline{s}} \in H, \sigma({}^o\overline{Y}_{\underline{s}}) \neq {}^o{*}\sigma(\overline{Y}_{\underline{s}}))dL([\overline{B},\overline{B}])\ .$$

The hypotheses on σ imply that $|{}^o{*}\sigma(\overline{Y}_{\underline{s}})|>0$ whenever $\sigma({}^o\overline{Y}_{\underline{s}}) \neq {}^o{*}\sigma(\overline{Y}_{\underline{s}})$ and hence (2.3) shows that (2.4) is zero. This means that ${*}\sigma(\overline{Y}_{\underline{s}})$ is a $\overline{B}$-lifting of $\sigma(Y_s)$ (see [5, Def. 7.4]). Therefore we may take standard parts in (2.2) and use the nonstandard characterization of the stochastic integral ([5, Def. 7.14 and Th. 7.15(b)]) to see that Y is a solution of (2.1). □

<u>Remarks</u> (1) By making only minor changes in the above one can prove weak existence for solutions of

$$(2.5)\qquad Y_t = \int_0^t \sigma(Y_s)dB_s + \int_0^t f(Y_s)dV_s$$

where f is bounded and continuous, σ is as above and V is an F_t^B adapted process of bounded variation with right-continuous paths. Indeed the above method will show the existence of a class of rich probability spaces, (Ω,F,F_t,P) on which there are solutions of (2.5) for every F_t Brownian motion, B. The assumption that $\sigma(x^+)$ and $\sigma(x^-)$ exist may also be weakened (see [9]).

(2) The hypotheses of the above theorem are satisfied by $\sigma_1(x)=I(x>0)$ but not by $\sigma_2(x)=I(x\geq 0)$. Indeed, it is easy to see that no weak solution of (2.1) can exist if $\sigma=\sigma_2$ and $V\equiv 0$.

3. The Case $0<\alpha<1$.

As in [4] we transform (1.1) into a stochastic differential equation. In the applications of the following theorem, the semimartingale Z will be $B + \beta L(B)$, $\beta\in\mathbb{R}$.

Let

$$\operatorname{sign}(x) = \begin{cases} 1 & x>0 \\ 0 & x=0 \\ -1 & x<0 \end{cases}$$

$$s(x) = x + \alpha|x|$$

$$f(x) = 1 + \alpha\operatorname{sign}(x) \qquad \text{(the symmetric derivative of } s).$$

Theorem 3.1 Let $0<\alpha<1$, and let Z be a continuous semimartingale with $Z_0=0$. Then a semimartingale $\tilde{X}_t$ satisfies

$$(3.1) \qquad \tilde{X}_t + \alpha L_t^0(\tilde{X}) = Z_t$$

if and only if $X_t = s(\tilde{X}_t)$ satisfies

$$(3.2) \qquad X_t = \int_0^t f(X_s)dZ_s \ .$$

If $\tilde{X}$ satisfies (3.1) and in addition

$$\int_0^{\cdot} I(\tilde{X}_s=0)dZ_s=0 \ ,$$

then

$$L_t^0(\tilde{X}) = \frac{1}{1-\alpha} L_t^{0+}(\tilde{X}) = \frac{1}{1+\alpha} L_t^{0-}(\tilde{X})$$

Proof. The generalized Itô formula shows that (since $X_t=s(\tilde{X}_t)$) ,

$$X_t = \int_0^t f(\tilde{X}_s)\,dX_s + \alpha L_t^0(\tilde{X})$$

$$(3.3) \qquad = \int_0^t f(\tilde{X}_s)\,d(X_s + \alpha L_s^0(\tilde{X}))$$

Suppose first that $\tilde{X}$ satisfies (3.1). Using (3.3) and the fact that $f(\tilde{X}_s) = f(X_s)$, we have

$$X_t = \int_0^t f(X_s)\,dZ_s \quad .$$

If X_t satisfies (3.2), then

$$\int_0^t f(X_s)\,dZ_s = X_t = \int_0^t f(X_s)\,d(\tilde{X} + \alpha L^0(\tilde{X}))_s \quad ,$$

the last by (3.3), and as $f(X_s)$ does not vanish it follows that

$$Z_t = \tilde{X}_t + \alpha L_t^0(\tilde{X}) \quad .$$

The condition $\int_0^t I(\tilde{X}_s=0)\,dZ_s = 0$ implies that

$$\tfrac{1}{2}(L_t^{0+}(\tilde{X}) - L_t^{0-}(\tilde{X})) = \int_0^t I(\tilde{X}_s=0)\,d\tilde{X}_s \qquad \text{(see [14, p.29])}$$

$$= -\alpha L_t^0(\tilde{X})$$

$$= -\frac{\alpha}{2}(L_t^{0+}(\tilde{X}) + L_t^{0-}(\tilde{X})) \quad .$$

Rearranging, we have that

$$L_t^{0+}(\tilde{X}) = (1-\alpha)L_t^0(\tilde{X}) \quad , \quad L_t^{0-}(\tilde{X}) = (1+\alpha)L_t^0(\tilde{X}) \quad . \qquad \square$$

<u>Theorem 3.2</u> <u>Let</u> $\sigma:\mathbb{R}\to\mathbb{R}$ <u>be measurable, positive, bounded and bounded away from</u> 0 , <u>and have finite quadratic variation on compacts. Let</u> V_t^1 , V_t^2 <u>be</u> F_t^B <u>adapted processes of finite</u>

variation with continuous paths, and $V_0^1=V_0^2=0$.

(a) There exist unique solutions X_t^i to the equations

$$(3.4.i) \qquad X_t^i = \int_0^t \sigma(X_s^i)dB_s + V_t^i(B) \ .$$

Moreover the X_t^i are F_t^B-adapted.

(b) $L^0(X^1-X^2) = 0$.

If, in addition, V^1-V^2 is non-decreasing, then

(c) $X^1 \geq X^2$,

(d) $\int_0^\infty I(X_s^1=X_s^2)d(V^1-V^2) = 0$.

Proof. In [10, Lemma 2.1] Le Gall has shown that if X^1 and X^2 are solutions to (3.4) with $V^1=V^2=0$ then $L^0(X^1-X^2)=0$. (His result is actually more general than this.) Only minor alterations are needed to deal with general V^i , proving (b).

For (a) it is enough to consider the first equation. By Theorem 2.1 there is a Brownian motion B and a solution X^1 of (3.4.1) defined on some (Ω,F,F_t,P) . If Y^1 is another solution of (3.4.1), then $L^0(X^1-Y^1) = 0$ by (b), and so, as X^1-Y^1 is a martingale null at 0 , it follows that $X^1=Y^1$. Therefore there is pathwise uniqueness in (3.4.1) and hence X^1 is F_t^B adapted by an extension of the Yamada-Watanabe theorem [13] - see Jacod and Memin [7, T2.25].

Now let V^1-V^2 be non-decreasing. By Tanaka's formula

$$(X_t^1-X_t^2)^- = -\int_0^t I(X_s^1<X_s^2)d(X^1-X^2)_s \ ,$$

so that

$$E((X_t^1-X_t^2)^-) = -E\int_0^t I(X_s^1<X_s^2)d(V^1-V^2)_s$$

$$\leq 0 \quad .$$

Therefore $X^1 \geq X^2$, and by (b)

$$0 = \tfrac{1}{2}(L_t^{0+}(X^1-X^2) - L_t^{0-}(X^1-X^2))$$

$$= \int_0^t I(X_s^1-X_s^2 =0)d(X^1-X^2)_s$$

$$= \int_0^t I(X_s^1=X_s^2)d(V^1-V^2)_s \quad . \qquad \square$$

Now set $Z_t = B_t + \beta L_t^0(B)$. By Theorem 3.1, to study solutions of (1.1) it suffices to consider

$$(3.5) \qquad Y_t = \int_0^t f(Y_s)d(B + \beta L(B))_s \; .$$

Let $\tilde{B}_t = s(B_t)$. Theorem 3.1 implies that

$$(3.6) \qquad \tilde{B}_t = \int_0^t f(\tilde{B}_s)d(B + \alpha L(B))_s$$

$$= \int_0^t f(\tilde{B})_s dB_s + \alpha L_t(B) \; ,$$

so that $\tilde{B}$ is a solution of (3.5) with $\alpha=\beta$, and is the unique solution of (3.4) with $V_t = \alpha L_t(B)$. More generally, we suppress the dependence on α , and let Y^γ , $\gamma \in \mathbb{R}$, denote the unique $\mathcal{F}_t^B$ adapted solution of (3.4) with $V_t = \gamma L_t(B)$. In particular, $\tilde{B} = Y^\alpha$.

Theorem 3.3 Let $0<\alpha<1$.

(a) If $\gamma_1 \le \gamma_2$ then $Y^{\gamma_1} \le Y^{\gamma_2}$. If $\gamma_1 < \gamma_2$ then $Y^{\gamma_1} \ne Y^{\gamma_2}$.

(b) $\int_0^\infty 1_{(Y_s^\gamma=0)} dL_s(B) = 0$ for all $\gamma \ne \alpha$.

(c) If $\beta \le \frac{\alpha}{1+\alpha}$ then $Y^{\beta(1-\alpha)}$ is the unique solution of (3.5), and satisfies $Y^{\beta(1-\alpha)} \le \tilde{B}$.

(d) If $\beta \ge \frac{\alpha}{1-\alpha}$ then $Y^{\beta(1+\alpha)}$ is the unique solution of (3.5), and satisfies $Y^{\beta(1+\alpha)} \ge \tilde{B}$.

(e) If $\frac{\alpha}{1+\alpha} < \beta < \frac{\alpha}{1-\alpha}$, then

(i) $Y^{\beta(1+\alpha)}$ and $Y^{\beta(1-\alpha)}$ are both solutions of (3.5), and if $\alpha=\beta$ $\tilde{B}$ is also a solution of (3.5).

(ii) $Y^{\beta(1-\alpha)} \le \tilde{B} \le Y^{\beta(1+\alpha)}$, and these three processes are distinct.

(iii) If Y is any solution of (3.5), $Y^{\beta(1-\alpha)} \le Y \le Y^{\beta(1+\alpha)}$.

Proof We write L_t for $L_t(B)$. (a) is immediate from Theorem 3.2(c). By 3.2(d), setting $V^1 = \gamma L_t$, and $V^2 = \alpha L_t$, if $\gamma<\alpha$

$$0 = \int_0^t 1_{(Y_s^\gamma=\tilde{B}_s)} d((\gamma-\alpha)L)_s$$

$$= (\gamma-\alpha) \int_0^t 1_{(Y_s=0)} dL_s ,$$

which proves (b) in the case $\gamma<\alpha$; the case $\gamma>\alpha$ is exactly the same. If $\gamma<\alpha$ then as $Y^\gamma \le Y^\alpha = \tilde{B}$, and as $\{\tilde{B}=0\} = \{B=0\}$, $\int_0^t f(Y_s^\gamma) dL_s = (1-\alpha)L_t + \alpha\int_0^t 1_{(Y_s^\gamma=0)} dL_s = (1-\alpha)L_t$ by (b). Thus

$$Y_t^\gamma = \int_0^t f(Y_s^\gamma) d(B_s+\beta L_s) + (\gamma-\beta(1-\alpha))L_t ,$$

and similarly if $\gamma>\alpha$

$$Y_t^\gamma = \int_0^t f(Y_s^\gamma)d(B_s + \beta L_s) + (\gamma - \beta(1+\alpha))L_t .$$

Hence Y^γ satisfies (3.5) if and only if either

(i) $\gamma<\alpha$ and $\gamma=\beta(1-\alpha)$

or (ii) $\gamma>\alpha$ and $\gamma=\beta(1+\alpha)$

Thus if $\beta\leq\alpha/(1+\alpha)$ the only possible value of γ is $\beta(1-\alpha)$, and if $\beta\geq\alpha/(1-\alpha)$, then $\gamma=\beta(1+\alpha)$, while if $\alpha/(1+\alpha)<\beta<\alpha/(1-\alpha)$, $\gamma=\beta(1\pm\alpha)$. To establish uniqueness in (c), let Y be any solution to (3.5). Then since $\beta f(Y_s)\leq(1+\alpha)\beta\leq\alpha$, by Theorem 3.2(c), $Y\leq\tilde{B}$. Hence by 3.2(d),

$$\begin{aligned} 0 &= \int_0^t 1_{(Y_s=\tilde{B}_s)}(f(Y_s)\beta-\alpha)dL_s \\ &= \int_0^t 1_{(Y_s=0)}(\beta-\alpha)dL_s , \end{aligned}$$

and so

$$\begin{aligned} Y_t &= \int_0^t f(Y_s)dB_s + \int_0^t [(1-\alpha)1_{(Y_s<0)} + 1_{(Y_s=0)}]\beta dL_s \\ &= \int_0^t f(Y_s)dB_s + (1-\alpha)\beta L_t . \end{aligned}$$

Uniqueness in (d) is proved in the same manner. As for (e), (i) and (ii) have already been proved, while if Y is any solution of (3.5) then Y is also a solution of (3.4) with $V_t = \beta\int_0^t f(Y_s)dB_s$. Therefore, as $V_t-\beta(1-\alpha)L_t$ is non-decreasing, by Theorem 3.2(c) $Y\geq Y^{\beta(1-\alpha)}$, and similarly $Y\leq Y^{\beta(1+\alpha)}$.

The following theorem is an immediate consequence of 3.1 and 3.3.

Theorem 3.4 Let $0<\alpha<1$, and $X_t^\gamma = s^{-1}(Y_t^\gamma)$.

(a) If $\gamma_1 \le \gamma_2$ then $X^{\gamma_1} \le X^{\gamma_2}$. If $\gamma_1 < \gamma_2$, $X^{\gamma_1} \ne X^{\gamma_2}$.

(b) $\int_0^\infty 1_{(X_s^\gamma=0)} dL_s(B) = 0$ for $\gamma \ne \alpha$

(c) If $\beta \le \frac{\alpha}{1+\alpha}$ then $X^{\beta(1-\alpha)}$ is the unique solution of (1.1), and satisfies $X^{\beta(1-\alpha)} \le B$.

(d) If $\beta \ge \frac{\alpha}{1-\alpha}$ then $X^{\beta(1+\alpha)}$ is the unique solution of (1.1), and satisfies $X^{\beta(1+\alpha)} \ge B$.

(e) If $\frac{\alpha}{1+\alpha} < \beta < \frac{\alpha}{1-\alpha}$ then

(i) $X^{\beta(1+\alpha)}$ and $X^{\beta(1-\alpha)}$ are solutions of (1.1), and if $\alpha=\beta$, B is also a solution.

(ii) $X^{\beta(1-\alpha)} \le B \le X^{\beta(1+\alpha)}$, and these three processes are distinct.

(iii) If X is any solution of (1.1), $X^{\beta(1-\alpha)} \le X \le X^{\beta(1+\alpha)}$.

4. The case $\alpha=1$.

If $\alpha=1$, Theorem 3.1 breaks down as $s(x)$ is no longer one to one. Nonetheless, it is still useful to consider X_t^+ separately, where X_t is a solution of (1.1).

Proposition 4.1 Let $Z = M+V$ be the canonical decomposition of a continuous semimartingale satisfying $Z_0=0$. If X is a solution of

(4.1) $$X_t + L_t^0(X) = Z_t ,$$

then

(4.2) $$X_t^+ = \int_0^t I(X_s^+>0)\,dZ_s + \tfrac{1}{2}\int_0^t I(X_s=0)\,dV_s$$

<u>and</u>

(4.3) $$X_t = Z_t - \sup_{s\le t}(Z_s - X_s^+) .$$

<u>Proof</u>. Apply Tanaka's formula to (4.1) to obtain

$$X_t^+ = \int_0^t I(X_s>0)dZ_s + \tfrac{1}{2}\int_0^t I(X_s=0)dZ_s - \tfrac{1}{2}L_t^0(X) + \tfrac{1}{2}L_t^0(X)$$

$$= \int_0^t I(X_s^+>0)dZ_s + \tfrac{1}{2}\int_0^t I(X_s=0)dV_s .$$

To prove (4.3) note first that by (4.1),

$$X_t^- = -Z_t + X_t^+ + L_t^0(X) .$$

Since $X_t^- \ge 0$ and $L_t^0(X)$ only increases on the zero set of X^-, it follows that $(X_t^-, L_t^0(X))$ is the unique solution of the reflection problem for $-Z_t + X_t^+$ (see El Karoui and Chaleyat-Maurel [1]). Therefore

$$X_t^- = -Z_t + X_t^+ + \sup_{s\le t}(Z_s - X_s^+)$$

and (4.3) is immediate. □

We construct solutions to (4.1) with $Z = B+\beta L(B)$ by first finding a candidate, X^+, for a solution to (4.2), then defining X by (4.3) and finally checking that X is in fact a solution of (4.1). Our first candidate for X^+ is 0.

<u>Theorem 4.2</u> <u>Let</u> $Z = M+V$ <u>be the canonical decomposition of a continuous semimartingale satisfying</u> $Z_0=0$. <u>Let</u> $S_t = \sup_{s\le t} Z_s$, $X_t^0 = Z_t - S_t$ <u>and assume that</u>

$$(4.4) \qquad \int_0^{\cdot} I(S_t = Z_t)\,dV_t \equiv 0 \ .$$

(a) X^0 <u>is the unique non-positive solution of (4.1)</u>. <u>Moreover,</u> X^0 <u>is the minimal solution of (4.1),</u> <u>i.e., if</u> X <u>is any solution of (4.1) then</u> $X^0 \le X$.

(b) <u>If</u> V <u>is non-increasing, then</u> X^0 <u>is the unique</u> <u>solution of (4.1)</u>.

<u>Proof</u>. (a) As $(-X^0, S)$ is the unique solution of the reflection problem for $-Z$, Prop.I.2.1 of [1] implies that

$$S_t = \tfrac{1}{2}L_t^{0+}(-X^0) + \int_0^t I(S_u = Z_u)\,dV_u$$

$$= L_t^0(-X^0) \qquad \text{(by (4.4))}$$

$$= L_t^0(X^0) \quad .$$

Therefore

$$X^0 + L^0(X^0) = Z - S + S = Z \ .$$

It is clear from (4.3) that X^0 is the minimal, and unique non-positive solution of (4.1).

(b) If V is non-increasing, it follows easily from (4.2) that $E(X_t^+) \le 0$, for any solution of (4.1). Therefore $X = X^0$ by (a). □

<u>Corollary 4.3</u> <u>Let</u>

$$X_t^0 = B_t + \beta L_t(B) - \sup_{s \le t}(B_s + \beta L_s(B)) \ .$$

<u>Then</u> $X = X^0$ <u>is the minimal, and unique non-positive solution of</u>

(4.5) $X_t + L_t^0(X) = B_t + \beta L_t(B)$.

<u>Moreover if</u> $\beta \le 0$, X^0 <u>is the unique solution of (4.5)</u>.

<u>Proof</u>. We must prove that (4.4) holds with $Z = B + \beta L(B)$. Let τ_t be the right-continuous inverse of $L_t(B)$, and set $Z_u=0$ for $u<0$. Then, by an argument in Emery and Perkins [3, Prop.1], if t is fixed, $(\tau_t, Z(\tau_t - \cdot) - \beta t)$ is equal in law to $(\tau_t, B(\cdot \wedge \tau_t) - \beta L_{\cdot \wedge \tau_t}(B))$. Therefore

$$(4.6) \qquad P(S_{\tau_t} = Z_{\tau_t}) = P(\sup_{u \le \tau_t} Z(\tau_t - u) = \beta t)$$

$$= P(\sup_{u \le \tau_t} B(u) - \beta L_u(B) = 0) \ .$$

A simple scaling argument shows that

$$P(\sup_{u \le \varepsilon} B_u - \beta L_u(B) = 0)$$

is independent of ε and hence must be zero by the 0-1 law. It follows from (4.6) that $P(S_{\tau_t} = Z_{\tau_t}) = 0$ for each t and therefore

$$\int_0^\infty I(S_u = Z_u) dL_u(B) = \int_0^\infty I(S_{\tau_t} = Z_{\tau_t}) dt = 0 \quad \text{a.s.}$$

Hence (4.4) holds with $Z = B + \beta L(B)$ and Theorem 4.2 implies the required result. □

In order to obtain a maximal solution of (4.5) for $\beta > \frac{1}{2}$, we construct another candidate for X^+ .

<u>Theorem 4.4</u> <u>Let</u> $\beta > \frac{1}{2}$. <u>There is a unique solution</u> Y <u>of</u>

$$(4.7) \qquad Y_t = \int_0^t I(Y_s>0)dB_s + \beta L_t(B) .$$

Y is F_t^B adapted,

$$L_t^{0+}(Y) = 2\beta\int_0^t I(Y_s=0)dL_s(B) = 0 ,$$

and if $\hat{Y}$ is a solution of

$$(4.8) \qquad \hat{Y}_t = \int_0^t I(Y_s>0)dB_s + V_t ,$$

where $\beta L_t(B)-V_t$ is adapted, continuous and non-decreasing, then $\hat{Y}\leq Y$. If in addition $\frac{1}{2}L_t(B)-V_t$ is non-decreasing, then $\hat{Y}\leq B^+$.

Note that (4.7) is not covered by the classical existence and uniqueness results, as the diffusion coefficient is discontinuous and degenerate. Before proving the theorem, we establish two lemmas, the first of which is interesting in its own right.

Lemma 4.5. For $\beta>0$, let

$$T_\beta = \inf\{t: B_t + \beta L_t = -1\} .$$

(a) $P(L_{T_\beta}(B)>t) = (1 + \beta t)^{-\frac{1}{2}\beta}$ for $t\geq 0$.

(b) If $\gamma>(4\beta)^{-1}$, there is a $C_\gamma>0$ such that $P(T_\beta>t)\geq C_\gamma t^{-\gamma}$ for $t\geq 1$.

Proof. Let τ_t be the right-continuous inverse of L_t , and define a Poisson point process with state space $(0,\infty)$ by

$$E(t) = \begin{cases} \sup\limits_{u\in[\tau_{t-},\tau_t]} B_u^- & \text{if } \tau_{t-}<\tau_t \text{ and this sup is } >0 \\ \delta & \text{otherwise} \end{cases}$$

(see Itô [6]). Using the fact that $L_{T(-x)}$ has an exponential law with mean $2x$ (here $T(-x) = \inf\{t: B_t=-x\}$), it is easily shown that the characteristic measure of E, μ, satisfies $\mu([x,\infty)) = (2x)^{-1}$ for $x>0$. Therefore if $A\subset[0,\infty)\times(0,\infty)$ and $N(A)$ denotes the cardinality of $\{t: (t,E(t))\in A\}$, then $N(A)$ has a Poisson distribution with parameter $m\times\mu(A)$, where m denotes Lebesgue measure. In particular, since

$$L_{T_\beta} = \inf\{u: -E(u)+\beta u\le -1\} = \inf\{u: E(u)\ge 1+\beta u\}$$

one has

$$\begin{aligned} P(L_{T_\beta}>t) &= P(N(\{(u,x):u\le t,\ x\ge 1+\beta u\})=0) \\ &= \exp\{-\int_0^t M([1+\beta u,\infty))du\} \\ &= \exp\{-\int_0^t \frac{1}{2(1+\beta u)}\,du\} \\ &= \exp\{-\frac{1}{2\beta}\log(1+\beta t)\} \\ &= (1+\beta t)^{-\frac{1}{2}\beta}. \end{aligned}$$

If $\gamma>(4\beta)^{-1}$ and $t\ge 1$, then

$$\begin{aligned} P(T_\beta>t) &\ge P(T_\beta>\tau(t^{2\beta\gamma})) - P(\tau(t^{2\beta\gamma})\le t) \\ &\ge P(L_{T_\beta}>t^{2\beta\gamma}) - P(L_1\ge t^{2\beta\gamma-\frac{1}{2}}) \\ &\ge (1+\beta t^{2\beta\gamma})^{-\frac{1}{2}\beta} - \exp\{-t^{4\beta\gamma-1}/2\} \end{aligned}$$

Therefore $P(T_\beta > t) \geq c_\gamma t^{-\gamma}$. □

<u>Lemma 4.6</u> <u>Let</u> H <u>be a non-negative previsible process, and</u> Y <u>be a solution of</u>

$$Y_t = \int_0^t 1_{(Y_s>0)} dB_s + \int_0^t H_s dL_s \ .$$

Then

(a) $Y \geq 0$

(b) <u>If</u> $D_t = \inf\{s \geqslant t : B_s = 0,\ H_s \neq 0\}$ <u>then on</u> $\{Y_t = 0\}$, $Y_s = 0$ <u>for</u> $t \leq s \leq D_t$.

(c) <u>Let</u> $\tau_t = \sup\{s<t : Y_s = 0\}$. <u>Then on</u> $\{Y_t > 0\}$

$$Y_t = B_t + \int_{\tau_t}^t H_s dL_s \ .$$

(d) <u>If</u> $H_s \equiv \beta > \frac{1}{2}$, <u>then</u> $L^{0+}(Y-B^+) = 0$, $Y \geq B^+$ <u>and</u> $Y \neq B^+$.

<u>Proof</u>. (a) By Tanaka's formula,

$$Y_t^- = -\int_0^t 1_{(Y_s<0)} H_s dL_s + \tfrac{1}{2} L_t^{0-}(Y) \ .$$

Thus Y^- is of integrable variation, and therefore $L^0(Y^-) = L^{0-}(Y) = 0$. Hence, as H is non-negative, $EY_t^- \leq 0$, so that $Y^- = 0$.

(b) As $Y \geq 0$ it is enough to show that for each $s>t$, $E\, 1_{(Y_t=0)} Y_{s\wedge D_t} = 0$. However, $\int_t^{D_t} H_s dL_s(B) = 0$ on $\{Y_t = 0\}$, and so

$$E\, 1_{(Y_t=0)} Y_{s\wedge D_t} = E\, 1_{(Y_t=0)} \int_t^{s\wedge D_t} 1_{(Y_s>0)} dB_s$$

$$= 0 \ .$$

(c) Fix $t>0$, and enlarge $(\mathcal{F}_t)$ to make the random time τ_t a stopping time. As τ_t is honest (the end of an optional set) stochastic integrals take the same value in both filtrations. Hence, as $B_{\tau_t} = 0$ by (b),

$$Y_t = Y_{\tau_t} + \int_{\tau_t}^{t} 1_{(Y_s>0)} dB_s + \int_{\tau_t}^{t} H_s\, dL_s$$

$$= B_t + \int_{\tau_t}^{t} H_s\, dL_s\ .$$

(d) Let $\varepsilon>0$, $T_0^\varepsilon=0$, and

$$S_n^\varepsilon = \inf\{t>T_{n-1}^\varepsilon : Y_t - B_t^+>\varepsilon\}$$

$$T_n^\varepsilon = \inf\{t>S_n^\varepsilon : Y_t - B_t^+=0\}\ .$$

Writing $\tau(S_n^\varepsilon)$ for $\tau_{S_n^\varepsilon} = \sup\{s<S_n^\varepsilon: Y_s=0\}$, by (c), since $Y_{S_n^\varepsilon} > 0$

(4.9) $$Y_{S_n^\varepsilon} - B^+_{S_n^\varepsilon} = -B^-_{S_n^\varepsilon} + \beta(L_{S_n^\varepsilon} - L_{\tau(S_n^\varepsilon)})$$

It follows that $B_{S_n^\varepsilon}= 0$, for if $R = (\sup\{s\le S_n^\varepsilon: B_s=0\})\vee T_{n-1}^\varepsilon$, by (c); $Y_R - B_R^+ = Y_{S_n^\varepsilon} - B_{S_n^\varepsilon} \ge Y_{S_n^\varepsilon} - B^+_{S_n^\varepsilon}$, so that $R = S_n^\varepsilon$.

Let $B_t = B(S_n^\varepsilon+t)$ be a Brownian motion, and

$$V_n(\varepsilon) = \inf\{t: B_t^n + \beta L_t(B^n) = -\varepsilon\}.$$

We have, therefore, for $S_n^\varepsilon \le t \le T_\varepsilon^n$

$$Y_t = Y_{S_n^\varepsilon} + B_t - B_{S_n^\varepsilon} + \beta(L_t - L_{S_n^\varepsilon})$$

$$= \varepsilon + B^n_{t-S_n^\varepsilon} + \beta L_{t-S_n^\varepsilon}(B^n)\ ,$$

so that $T_n^\varepsilon = S_n^\varepsilon + V_n(\varepsilon)$. Let

$$N(\varepsilon,t) = \sum_{i=1}^{\infty} I(S_i^\varepsilon \le t) ,$$

then for $\varepsilon > \frac{1}{2}$,

$$t+1 \ge E(\sum_i ((T_i^\varepsilon - S_i^\varepsilon)\wedge 1) I(S_i^\varepsilon \le t))$$

$$= E(\sum_i I(S_i^\varepsilon \le t) E(V_i(\varepsilon)\wedge 1 \mid F_{S_i^\varepsilon}))$$

$$= E(N(\varepsilon,t)) E(V_1(\varepsilon)\wedge 1)$$

$$= \varepsilon^2 E(N(\varepsilon,t)) E(V_1(1)\wedge \varepsilon^{-2}) \quad \text{(scaling)}$$

$$\ge C\varepsilon^2 E(N(\varepsilon,t)) \int_1^{\varepsilon^{-2}} t^{-\gamma} dt , \text{ where } \gamma \in (\tfrac{1}{4\beta}, \tfrac{1}{2}) \text{ by Lemma 4.5}$$

$$\ge C\varepsilon^{2\gamma} E(N(\varepsilon,t)) .$$

In particular it follows that $\lim_{\varepsilon \to 0+} \varepsilon E(N(\varepsilon,t)) = 0$ for all $t \ge 0$. The downcrossing characterization of local time (see El Karoui[2]) implies that

$$\tfrac{1}{2} L_t^{0+}(Y-B^+) = \lim_{\varepsilon \to 0+} \varepsilon N(\varepsilon,t) \quad (\text{in } L^1)$$

$$= 0 .$$

It remains only to show that $Y \ge B^+$. Apply Tanaka's formula to (4.8) to obtain

$$(Y_t - B_t^+)^- = \int_0^t I(B_s > 0 = Y_s) dB_s - (\beta - \tfrac{1}{2}) \int_0^t I(Y_s \le B_s^+) dL_s^0(B) .$$

As $\beta>\frac{1}{2}$, we see that $E((Y_t-B_t^+)^-)\le 0$. Finally $Y\ne B^+$ is obvious. □

Proof of Theorem 4.4. Let $\beta>\frac{1}{2}$ and assume Y is a solution of (4.7). Let $\beta L_t(B) - V_t$ be an adapted non-decreasing process and suppose that $\hat{Y}$ is a solution of (4.8) (with respect to the same Brownian motion). Then

$$\begin{aligned} L_t^{0+}(Y-\hat{Y}) &= \lim_{\varepsilon\to 0+} \varepsilon^{-1} \int_0^t I(Y_s-\hat{Y}_s\in(0,\varepsilon))(I(Y_s>0)-I(\hat{Y}_s>0))^2 ds \\ &\le \lim_{\varepsilon\to 0+} \varepsilon^{-1} \int_0^t I(Y_s\in(0,\varepsilon))I(\hat{Y}_s\le 0)ds \\ &\le \lim_{\varepsilon\to 0+} \varepsilon^{-1} \int_0^t I(Y_s\in(0,\varepsilon))ds \\ &= L_t^{0+}(Y) \\ &= L_t^{0+}(Y) - L_t^{0-}(Y) \qquad \text{(as } Y\ge 0) \end{aligned}$$

Therefore

$$L_t^{0+}(Y-\hat{Y}) \le 2\beta\int_0^t I(Y_s=0)dL_s(B) \qquad \text{(by Yor [14])}. \tag{4.9}$$

As $Y\ge B^+$ by Lemma 4.6, we also have

$$\begin{aligned} 0 = L_t^{0+}(Y-B^+) &= L_t^{0+}(Y-B^+) - L_t^{0-}(Y-B^+) \\ &= (\beta-\tfrac{1}{2})\int_0^t I(Y_s=0)dL_s(B) \\ &\ge (\beta-\tfrac{1}{2})(2\beta)^{-1}L_t^{0+}(Y-\hat{Y}) \qquad \text{(by (4.9))}. \end{aligned}$$

So $L_t^{0+}(Y-\hat{Y})=0$ and by using Tanaka's formula one can easily see that $Y\ge\hat{Y}$, just as in the proof of $Y\ge B^+$ (see Lemma 4.6(c)).

In particular, if Y and $\hat{Y}$ are both solutions of (4.7) then $Y \geq \hat{Y} \geq Y$ and so pathwise uniqueness holds in (4.7). By Theorem 2.1 there is a weak solution, and so by Theorem 2.25 of [7], Y is F_t^B adapted.

Finally, let $\frac{1}{2}L_t(B) - V_t$ be non-decreasing. Then if, for $\beta > \frac{1}{2}$, Y^β is the unique solution to (4.7), $Y^{\beta_1} \leq Y^{\beta_2}$ when $\beta_1 < \beta_2$, since $(\beta_2 - \beta_1)L_t(B)$ is non-decreasing. Similarly, $\hat{Y} \leq Y^\beta$, so that $\hat{Y} \leq \lim_{\beta \downarrow \frac{1}{2}} Y^\beta$. Also, $B^+ \leq Y^\beta$ for each $\beta > \frac{1}{2}$, as $(\beta - \frac{1}{2})L_t(B)$ is non-decreasing, and $E(\lim_{\beta \downarrow \frac{1}{2}} Y_t^\beta - B_t^+) = \lim_{\beta \downarrow \frac{1}{2}} E(Y_t^\beta - B_t^+) = \lim_{\beta \downarrow \frac{1}{2}} (\beta - \frac{1}{2}) E\, L_t(B) = 0$. So $B^+ = \lim_{\beta \downarrow \frac{1}{2}} Y^\beta \geq \hat{Y}$.

<u>Theorem 4.7</u> <u>Let</u> $\beta > \frac{1}{2}$, <u>and</u> Y^β <u>be the unique solution of (4.7). Let</u>

$$X_t^1 = B_t + \beta L_t - \sup_{s \leq t}(B_s + \beta L_s - Y_s^\beta) .$$

<u>Then</u>

(a) X^1 <u>is the maximal solution of (4.5), and is distinct from the minimal solution,</u> X^0, <u>constructed in Corollary 4.3, and from</u> B, <u>which if</u> $\beta=1$, <u>is also a solution.</u>

(b) X^1 <u>is</u> F_t^B <u>adapted.</u>

(c) $(X^1)^+ = Y^\beta$, $(X^1)^+ \geq B^+ \geq (X^0)^+ = 0$, <u>and these three processes are distinct.</u>

<u>Proof</u> We fix $\beta > \frac{1}{2}$, and write Y for Y^β, X for X^1. We show first that $X^+ = Y$. By Lemma 4.6(b), if $Y_t > 0$, then

$Y_t=B_t + \beta(L_t-L_{\tau_t})$, and so $X_t-Y_t = \beta L_{\tau_t} - \sup_{s\le t}(B_s+\beta L_s-Y_s)$ $= \beta L_{\tau_t} - \sup_{s\le\tau_t}(B_s+\beta L_s-Y_s)$. By Theorem 4.4, $Y\ge B^+$, and therefore, if $s\le\tau_t$, $B_s+\beta L_s-Y_s \le B_s^+-Y_s+\beta L_s \le \beta L_s \le \beta L_{\tau_t}$. Therefore $X_t-Y_t=0$ if $Y_t>0$, and as it is clear from the definition of X that $X-Y\le 0$, it follows that $Y=X^+$. The remainder of (c) follows immediately.

(b) is an immediate consequence of Theorem 4.4.

For (a), let $M_s=B_s+\beta L_s-Y_s$, and note that, by (4.7), M is a martingale. We have

$$X_t^- = \sup_{s\le t} M_s - M_t \ ,$$

and therefore, by Prop.I.2.1 of [1], $\frac{1}{2}L_t^{0+}(X^-) = \sup_{s\le t} M_s$. Now $L^{0+}(X) = L^{0+}(X^+) = 0$, by Theorem 4.4, and $L^{0-}(X) = L^{0-}(-X^-)$ $= L^{0+}(X^-)$; therefore $L^0(X) = \frac{1}{2}L^{0+}(X) + \frac{1}{2}L^{0-}(X) = \sup_{s\le t} M_s$,and

$$X_t + L_t^0(X) = Y_t + M_t - \sup_{s\le t} M_s + \sup_{s\le t} M_s$$

$$= B_t + \beta L_t \ .$$

If Z is another solution of (4.5), by (4.2) and Theorem 4.4 $X^+\ge Z^+$, and so $X\ge Z$ by (4.3).

We now turn to the case $0<\beta\le\frac{1}{2}$. The following result is in part a refinement of the estimate used to prove (4.4).

Lemma 4.8. Let

$$\Lambda(\beta) = \{t>0:\ B_t=0, B_t+\beta L_t(B)\ge B_s+\beta L_s(B) \text{ for } 0\le s\le t\} \ .$$

Then for $0<\beta\le\frac{1}{2}$, $\Lambda(\beta)=\emptyset$ a.s.

Proof. As $\Lambda(\beta)\subseteq\Lambda(\frac{1}{2})$ for $\beta<\frac{1}{2}$, it is enough to prove the result for $\Lambda(\frac{1}{2})$. If $t\in\Lambda(\frac{1}{2})(\omega)$, then $B_t^+(\omega)+\frac{1}{2}L_t(B,\omega) \le B_s^+(\omega)+\frac{1}{2}L_s(B,\omega)$ for $s\le t$, and $B_t^+(\omega)+\frac{1}{2}L_t(B,\omega)=\frac{1}{2}L_t(B,\omega) \le B_u^+(\omega)+\frac{1}{2}L_u(B,\omega)$ for $u\ge t$, so that t is a point of increase of $B_\cdot^+(\omega)+\frac{1}{2}L_\cdot(B,\omega)$.

We shall now show that $B^++\frac{1}{2}L(B)$ has no (non-zero) points of increase. Let W denote a Brownian motion, with $W_0=0$. Points of increase are not removed by time-change, so, time-changing $B^++\frac{1}{2}L(B)$ by the inverse of $\int_0^\cdot I_{(B_s>0)}ds$, it follows that $B^++\frac{1}{2}L(B)$ has points of increase if and only if $|W|+\frac{1}{2}L^{0+}(|W|)$ does. Now $\frac{1}{2}L^{0+}(|W|)=L^0(W)$, and, if $S_t=\sup_{s\le t} W_s$, $(|W|,L^0(W))$ is equal in law to $(S-W,S)$, and so $|W|+\frac{1}{2}L^{0+}(|W|)$ is equal in law to $2S-W$, which, by Pitman's result (see [16]) is a 3-dimensional Bessel process. Thus for $t>0$, the law of $|W_t|+\frac{1}{2}L_t^{0+}(|W|)$ is absolutely continuous with that of Brownian motion, and hence, by the result of Dvoretsky, Erdos and Kakutani [15], $|W_t|+\frac{1}{2}L_t^{0+}(|W|)$ has no non-zero points of increas

Theorem 4.9. Let $0<\beta\le\frac{1}{2}$. Then (4.5) has a unique solution.

Proof. Let X be a solution of (4.5). By Corollary 4.3 it is sufficient to prove that X is non-positive. Let $Y_t=X_t^+$; by Proposition 4.1

$$Y_t = \int_0^t 1_{(Y_s>0)}dB_s + \int_0^t (1_{(Y_s>0)} + \tfrac{1}{2}1_{(X_s=0)})\beta dL_s .$$

Thus, as $\frac{1}{2}L_t(B) - \int_0^t (1_{(Y_s>0)} + \frac{1}{2}1_{(X_s=0)})\beta dL_s(B)$ is non-decreasing, by the final part of Theorem 4.4 $Y_t\le B_t^+$. Y therefore satisfies

$$Y_t = \int_0^t 1_{(Y_s>0)}dB_s + \tfrac{1}{2}\int_0^t 1_{(X_s=0)}\beta dL_s(B) .$$

Let $t>0$, and $\tau_t = \sup\{s<t: Y_s=0\}$. By Lemma 4.6(b), on $\{\tau_t<t\}$ $B_{\tau_t}=0$, and by (c), using the fact that $B_s>0$ for $\tau_t<s<t$, we have

$$Y_t = B_t^+ \quad \text{on} \quad \{Y_t>0\} .$$

However, if $S = \inf\{t: Y_t<B_t^+\}$, then $B_S^+-Y_S \geq \tfrac{1}{2}L_S(B)$, so that $S=0$ a.s. Therefore, by the section theorem, for any $\varepsilon>0$ there exists a stopping time T such that $P(T=\infty)<\varepsilon$, and $0<T<\varepsilon$, $Y_T=0$, $B_T^+>0$ on $\{T<\infty\}$. Let $R=\inf\{s>T: Y_s>0\}$. By Lemma 4.6(b), on $\{R<\infty\}$ $X_R=B_R=0$, so that, by (4.3), on $\{R<\infty\}$

$$0 = B_R + \beta L_R(B) - \sup_{s\leq R}(B_s + \beta L_s(B) - Y_s)$$

$$\leq \beta L_R(B) - \sup_{T\leq s\leq R}(B_s + \beta L_s(B)) .$$

Therefore on $\{R<\infty\}$

$$\beta(L_R(B) - L_T(B)) \geq \sup_{T\leq s\leq R}(B_s + \beta L_s(B) - \beta L_T(B)) .$$

Now let $U = \inf\{s>T: B_s=0\}$; then $X_U \leq - B_T^+$, so that $R>U$. Hence $R(\omega)$ is in the set $\Lambda(\beta)$ for the Brownian motion $B_{U+\cdot}$, and so by Lemma 4.8, $R=\infty$. Thus $P(Y1_{(\varepsilon,\infty)}\neq 0)<\varepsilon$, and as ε is arbitrary, $Y=0$, completing the proof of the theorem.

5. $X + \alpha L^{0+}(X) = B + \beta L(B)$

If instead of (1.1) we consider the equation

$$(5.1) \qquad X_t + \alpha L_t^{0+}(X) = B_t + \beta L_t(B)$$

the results are slightly different. In fact, the same proofs go through with some minor changes and we only state the theorem.

Suppressing dependence on α , we define

$$r(x) = \begin{cases} (2\alpha+1)x , & \text{if } x>0 \\ x , & \text{if } x\le 0 \end{cases}$$

$$g(x) = \begin{cases} 2\alpha+1 , & \text{if } x>0 \\ 1 , & \text{if } x\le 0 \end{cases}$$

If $\alpha>-\frac{1}{2}$, the proofs of Theorems 3.2 and 3.3 go through without change to show the existence of a unique, and F_t^B adapted solution, $\hat{Y}^\gamma$ of

$$\hat{Y}_t^\gamma = \int_0^t g(\hat{Y}_s^\gamma)dB_s + \gamma L_t(B) \qquad (\gamma\in\mathbb{R}).$$

In particular, $\hat{Y}^\alpha = r(B_t)$ by Tanaka's formula. For $\beta<-\frac{1}{2}$, let Z^β denote the unique, and F_t^B adapted solution of

$$Z_t^\beta = -\int_0^t I(Z_s>0)dB_s - \beta L_t(B)$$

(apply Theorem 4.4 to $-B$).

<u>Theorem 5.1</u>. (a) <u>If</u> $\alpha<-\frac{1}{2}$, <u>then (5.1) holds if and only if</u> $\beta=\alpha$ <u>and</u> $X=B$.

(b) <u>Let</u> $\alpha>-\frac{1}{2}$.

(i) If $\beta\in(\frac{\alpha}{2\alpha+1},\alpha]$, <u>then</u> $r^{-1}(\hat{Y}^{\beta(2\alpha+1)})$ <u>and</u> $r^{-1}(\hat{Y}^\beta)$ <u>are the (distinct) maximal and minimal solutions of (5.1), respectively</u>.

(ii) <u>If</u> $\beta>\alpha$, $X = r^{-1}(\hat{Y}^{\beta(2\alpha+1)})$ <u>is the unique solution of (5.1)</u>.

(iii) <u>If</u> $\beta \le \frac{\alpha}{2\alpha+1}$, $X = r^{-1}(\hat{Y}^{\beta})$ <u>is the unique solution of (5.1).</u>

(c) <u>Let</u> $\alpha = -\frac{1}{2}$.

(i) $X_t^0 = B_t + \beta L_t(B) - \inf_{s \le t}(B_s + \beta L_s(B))$ <u>is the maximal and unique non-negative solution of (5.1).</u>

(ii) <u>If</u> $\beta > -\frac{1}{2}$, X^0 <u>is the unique solution of (5.1).</u>

(iii) <u>If</u> $\beta \le -\frac{1}{2}$ <u>then</u>

$$X_t^1 = B_t + \beta L_t(B) - \inf_{s \le t}(B_s + \beta L_s(B) + Z_s^{\beta})$$

<u>is the minimal solution of (5.1), and satisfies</u> $(X^1)^- = Z^{\beta}$. <u>In particular</u> X^1 <u>is distinct from</u> X^0 .

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