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§1. Introduction

1. Notations and assumptions.

Let E be a locally compact, separable metric space, and use $\mathfrak{B}$ to denote the Borel field over E. Suppose that $P_t(x,\cdot)$ is a sub-Markovian transition function on $(E,\mathfrak{B})$ satisfying

i) $(t,x) \to P_t\phi(x)$ is continuous on $[0,\infty) \times E$ for each $\phi \in C_0(E) \equiv \{\phi : \phi \in C(E)$ has compact support$\}$,

ii) there exists a $\psi_0 \in C_0^+(E)$ such that for each $K \subset\subset E$ (i.e. K is a compact subset of E) there is a $t > 0$ and an $\varepsilon > 0$ for which $\varepsilon I_K \leq P_t\psi_0$.

Let

$$X^+ = \{\mu : \mu \text{ and } \mu P_t \text{ are non-negative Radon measures on } (E,\mathfrak{B}) \text{ for each } t > 0\}.$$

We endow X^+ with the vague topology.

For $\lambda \in R$, we put

$$\mathfrak{S}_\lambda = \{\mu \in X^+ : e^{-\lambda t}\mu P_t \uparrow \mu \text{ as } t \downarrow 0\}$$

$$\tilde{\mathfrak{S}}_\lambda = \{\mu \in \mathfrak{S}_\lambda : \mu \neq 0\}, \quad \mathfrak{S}_\lambda(g) = \{\mu \in \mathfrak{S}_\lambda : \mu(g) = 1\}$$

$$\lambda_\pi = \inf\{\lambda \in R : \tilde{\mathfrak{S}}_\lambda \neq \phi\}$$

One basic theorem of [9] is the following:

2. Theorem. $\lambda_\pi \in (-\infty,0]$,

$$\mathfrak{S}'_{\lambda_\pi} \equiv \{\mu \in X^+ : e^{\lambda_\pi t}\mu \geq \mu P_t \text{ for each } t > 0\} \neq \{0\}.$$

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We note that if $\mu \in \mathfrak{C}'_{\lambda_\pi}$ and $\mu \neq 0$, then $\nu \equiv \lim_{t \downarrow 0} e^{\lambda_\pi t} \mu P_t$ satisfies

$$e^{-\lambda_\pi t} \nu P_t \leq e^{\lambda_\pi t} \mu P_t \leq \nu .$$

From this fact and (1.ii) it is easy to see that $\nu \in \mathfrak{C}_{\lambda_\pi}$ and $\nu \neq 0$, hence $\tilde{\mathfrak{C}}_{\lambda_\pi} \neq \phi$.

In order to describe the number λ_π, the second author showed in [9] that λ_π is closely related to the rate at which the process exists from open sets and also to the spectrum of the operators $\{P_t\}$. His conclusions confirm that the number λ_π is a critical point. In some sense, it is a border between recurrence and transience. These considerations led him to rephrase a conjecture of D. Sullivan as follows.

3. Conjecture. [9, (3.1)]. Under reasonably general hypotheses about $\{P_t : t > 0\}$, there exists a positive Radon measure μ satisfying

$$e^{\lambda_\pi t} \mu = \mu P_t \quad \text{for each} \quad t > 0 .$$

The second author already proved this conjecture in some cases. One of his general results is:

4. Theorem. If $\{e^{-\lambda_\pi t} P_t : t > 0\}$ is recurrent in the sense that there is no positive Radon measure ν for which $\int_0^\infty e^{-\lambda_\pi t} \nu P_t \, dt$ is a Radon measure, then each $\mu \in \mathfrak{C}_{\lambda_\pi}$ satisfies $e^{\lambda_\pi t} \mu = \mu P_t$, $t \geq 0$. In particular, there is a positive Radon measure which is $\{e^{-\lambda_\pi t} P_t : t > 0\}$ invariant.

Unfortunately, this conjecture is not true in general. The original counterexample was found in the context of Markov chains. This example, along with other related examples, is given in the next section. Later, S.R.S. Varadhan suggested a method of producing a counterexample with a diffusion process. We now present an example based on Varadhan's idea.

Let Δ the ordinary Laplacian on R^3. Choose a smooth $\rho : R^3 \to (0,\infty)$ so that the diffusion generated by $L = \frac{1}{2} \rho\Delta$ explodes with positive probability (cf. Exercise 10.3.3 on p. 260 of [10]). Denote by

$\{P_t : t > 0\}$ the minimal Markov semigroup generated by L (i.e. $\{P_t : t > 0\}$ is the semigroup associated with the process which is "killed" when it explodes). Set $m(dy) = 1/\rho(y)dy$. Then $P_t f(x) = \int_{R^3} p(t,x,y)f(y)m(dy)$, $f \in C_b(R^3)$ where $p: (0,\infty) \times R^3 \times R^3 \to (0,\infty)$ is smooth and is symmetric with respect to x and y . Moreover, if $\tilde{L}$ denotes the Friedrich's extension in $L^2(m)$ of $L\big|_{C_0^\infty(R^3)}$, then $P_t = e^{t\tilde{L}}\big|_{C_b(R^3)}$. (These facts can be checked directly or as a consequence of the results in [6]). Combining these observations with (2.6) in [9] , one concludes that $\lambda_\pi = \sup\{(\phi,L\phi)_{L^2(m)} : \phi \in C_0^\infty(R^3)$ and $\|\phi\|_{L^2(m)} = 1\}$. But $(\phi,L\phi)_{L^2(m)} = (\phi,1/2\Delta\phi)_{L^2(\text{Lebesgue})}$, and so $\lambda_\pi = 0$. Thus, we will have a counterexample once we show that there is no non-zero Radon measure μ satisfying $\mu P_t = \mu$, $t > 0$. But if $\mu P_t = \mu$, $t > 0$, then $\mu(dy) = f(y)dy$ where $f \in C^\infty(R^3)^+$ and $\Delta(\rho\cdot f) = 0$. Hence, $\rho\cdot f$ would be constant, and so we would conclude that $mP_t = m$, $t > 0$. In particular, we would have

$$\int_{R^3} g(x)(1-P_t 1(x))m(dx) = \int_{R^3} g(x)m(dx) - \int_{R^3} P_t g(x)m(dx) = 0$$

for all $t > 0$ and $g \in C_b(R^3)$. Since this would mean that $P_t 1 = 1$ for all $t > 0$, we see that no such μ exists.

5. Definition. Let $\lambda \in R$ be given. Each μ in $\mathfrak{E}_\lambda$ is called a λ-excessive measure. $\mu \in \mathfrak{E}_\lambda$ is said to be a λ-invariant measure if

$$e^{\lambda t}\mu = \mu P_t \quad \text{for each } t > 0 \quad .$$

Denote the set of all λ-invariant measures by $\mathfrak{I}_\lambda$. We also write $\tilde{\mathfrak{I}}_\lambda = \mathfrak{I}_\lambda \cap \tilde{\mathfrak{E}}_\lambda$, $\mathfrak{I}_\lambda(g) = \mathfrak{E}_\lambda(g) \cap \mathfrak{I}_\lambda$ and put

$$P_t^\lambda = e^{-\lambda t}P_t \quad , \quad t > 0 \quad .$$

Finally, a non-negative $\mathfrak{B}$-measurable function h is called a λ-excessive function, if

$$P_t^\lambda h \uparrow h \quad \text{as} \quad t \downarrow 0$$

and $h < \infty$ a.e. with respect to $P_t(x,\cdot)$ for each $t > 0$ and $x \in E$.

In §3 we will give a limit procedure for computing the elements of $\mathfrak{I}_0$ by the Dynkin's machine [5] . As for $\mathfrak{I}_{\lambda_\pi}$ when $\lambda_\pi < 0$, we already know from (4) that it is enough to study non-recurrent $\{P_t^{\lambda_\pi} : t > 0\}$, and in that case we are able to reduce the study of $\mathfrak{I}_{\lambda_\pi}$ to the study of $\mathfrak{I}_0$ for a new transition function.

In §4 , we use the following lemma from [8] :

<u>6. Lemma</u>. Replace (1.i) with the assumption that for each $t > 0$, $P_t\phi_0$ is positive everywhere. Let $\lambda \in R$ and $\mu \in \mathfrak{S}_\lambda$. Then $\mu \in \mathfrak{I}_\lambda$ if and only only if there is a $T > 0$ for which $\mu\phi_0 = \mu P_T^\lambda \phi_0$.

This lemma allows us to focus on the discrete case which is discussed in §4 . In particular, we will show how to extend a result due to Harris [7] and Veech [11] .

§2. Counterexamples From Markov Chains

Let $E = \{0,1,2,\cdots\}$. We call $P(t) = (P_{ij}(t) : i,j \in E)$ $(t > 0)$ a sub-Markov transition function, if

$$P_{ij}(t) \geq 0 , \quad \sum_{j\in E} P_{ij}(t) \leq 1 , \quad P_{ij}(t+s) = \sum_{k\in E} P_{ik}(t)P_{kj}(s)$$

and $\lim_{t\downarrow 0} P_{ij}(t) = \delta_{ij}$ $(\delta_{ii} = 1 , \delta_{ij} = 0 \ (i \neq j))$ for any i and j in E
It is well known that the following limits:

$$\lim_{t\downarrow 0} \frac{P_{ij}(t)}{t} = q_{ij} \quad (i \neq j) \quad \text{and} \quad \lim_{t\downarrow 0} \frac{1-P_{ii}(t)}{t} = q_i$$

all exist. We set $q_{ii} = -q_i$. Then

$$0 \leq q_i \leq \infty , \quad 0 \leq q_{ij} < \infty \quad (i \neq j) , \quad \sum_{j\neq i} q_{ij} \leq q_i \tag{7}$$

The matrix $Q = (q_{ij})$ is called a Q-matrix. A sub-Markov transition function with this matrix (q_{ij}) is called a Q-process.

In this context, (1.i) and (1.ii) become the assumption that for all $i,j \in E$, $P_{ij}(t) > 0$. Thus we make this assumption about $P(t)$. Also, it is natural to fix

$$g(x) = I_{\{0\}}(x) \ .$$

8. Theorem. If $\mu \in \mathfrak{C}_\lambda(g)$, then

$$\lambda > -q_i \quad \text{and} \quad (\lambda+q_i)\mu_i \geq \sum_{j\neq i} \mu_j q_{ji}$$

for each $i \in E$. In particular, we have

$$\lambda_\pi \geq -\inf_{i\in E} q_i \quad .$$

Proof. Let

(9) $$\tilde{P}_{ij}(t) = \mu_j P^\lambda_{ji}(t)/\mu_i \quad , \quad i,j \in E$$

It is easy to check that $\tilde{P}(t)$ is a Q-process with Q-matix $\tilde{Q} = (\tilde{q}_{ij})$:

$$\tilde{q}_i = -\tilde{q}_{ii} = \lambda_\pi + q_i \quad , \quad \tilde{q}_{ij} = \mu_j q_{ji}/\mu_i \quad (i \neq j)$$

hence the assertions follow from (7) and the fact that $\tilde{P}(t) > 0$.

In order to remove time from our consideration, we will need the next lemma.

10. Lemma. Assume that $P(t)$ is a Q-process, totally stable (i.e. $q_i < \infty$ for each $i \in E$), and satisfies the forward Kolmogorov equations:

(11) $$P'_{ij}(t) = -P_{ij}(t)q_j + \sum_{k\neq j} P_{ik}(t)q_{kj}$$

Also assume that $\mu = \mu P(t)$. Then

(12) $$\mu_j q_j = \sum_{i\neq j} \mu_i q_{ij} \ , \quad \forall j \in E \quad .$$

Proof. [8] We have

(13) $$\sum_{i=0}^{N} \mu_i P'_{ij}(t) = -q_j \sum_{i=1}^{N} \mu_i P_{ij}(t) + \sum_{i=1}^{N} \sum_{k\neq j} \mu_i P_{ik}(t) q_{kj}$$

The sum $\sum_{i=0}^{N} \mu_i P_{ij}(t)$ is non-negative, continuous in t and it monotonically increases to μ_j as $N \to \infty$. Similarly, the second sum on the right side in (13) is non-negative, continuous in t and it monotonically increases to $\sum_{i\neq j} \mu_i q_{ij}$, which is finite by (8) (cf. [1, II§3, Theorem 1]). Hence, by Dini's theorem, these sums converge uniformly for t in a finite interval. Consequently, differentiation and summation can be interchanged in (13) when

$N = \infty$ and so (12) follows.

14. Lemma. Equations $\mu_j q_{ij} = \sum_{i\neq j} \mu_i q_{ij}$ $(j \in E)$ have a positive solution (μ_i) if and only if the equation $\nu = \nu\overline{P}$ has a positive solution (ν_i), where $\overline{P}_{ii} = 0$, $\overline{P}_{ij} = q_{ij}/q_i$ $(i \neq j)$. Moreover, we can pass from one to the other by taking $\nu_i = \mu_i q_i$ $(i \in E)$.

Proof. Obvious.

15. Theorem*). Let $Q = (q_{ij})$ be a totally stable, irreducible and conservative (i.e. $q_i = \sum_{j\neq i} q_{ij}$ for each $i \in E$) Q-matrix. Suppose that there is precisely one Q-process and that it is transient. Then, in order that $\tilde{\mathfrak{J}}_0 \neq 0$, the following condition is necessary: there exists an infinte subset $\{i_1, i_2, \cdots\}$ of distinct integers such that

$$(16) \qquad \overline{P}_{i_2 i_1} > 0 \;,\; \overline{P}_{i_3 i_2} > 0 \;,\; \cdots \;,\; \overline{P}_{i_{n+1} i_n} > 0$$

where $\overline{P} = (\overline{P}_{ij})$ is defined in (14). In particular, if $\lambda_\pi = 0$, then this gives a necessary condition for $\mathfrak{J}_{\lambda_\pi} \neq \emptyset$.

Proof. Because of (10) and (14), we need only consider the solutions to $v = v\overline{P}$. But now our condition comes from Harris' observation [7, Theorem 1].

17. Example. Take $\overline{P}_{00} = 0$; $\overline{P}_{0i} = P_i > 0$ $(i \geq 1)$, $\sum_{i=1}^{\infty} P_i < 1$; $P_{i0} = 1$ $(i \geq 1)$. It is clear that this $\overline{P}$ does not satisfy the condition (16). So the equation $v = v\overline{P}$ has no positive solution. This fact is also very easy to check directly.

We now take $0 < q_i \downarrow 0$ as $i \uparrow \infty$, $q_{ij} = \overline{P}_{ij} q_i$ $(i \neq j)$. With this Q-matrix, the Q-process is unique (since $Q = (q_{ij})$ is bounded). Hence the unique Q-process $P(t)$ satisfies the Kolmogorov forward equations. Moreover, $P(t)$ is transient since $\overline{P}$ is. Finally, (8) implies that $\lambda_\pi = 0$ since $q_i \downarrow 0$. We therefore see that $\tilde{\mathfrak{J}}_{\lambda_\pi} \neq \tilde{\mathfrak{J}}_0 = \emptyset$.

*) We will give a more general result later (see (32)).

Notice that $P(t)$ is symmetric, because $\overline{P}$ is symmetric with respect to $\{\mu_0 = 1 , \mu_i = P_i , i \geq 1\}$ and therefore $Q = (q_{ij})$ is symmetric with respect to $(\mu_i/q_i = i \in E)$ which, by uniqueness, means that $P(t)$ is. On the other hand, $\sum_{j\in E} P_{ij}(t) < 1$ $(\forall i \in E)$, hence, we now have a counterexample which is symmetric but also a stopped Q-process.

To get an example of a non-stopped (conservative) Q-process for which the conjecture fails, we proceed as follows.

<u>18. Example</u> [2]. Take $\overline{P}_{i,i+1} = P_i > 0$, $\overline{P}_{i0} = 1 - P_i$ $(i \in E)$. It is easy to see that there is a (unique) positive solution to $v = v\overline{P}$ $(v_0 = 1)$ if and only if $\lim_{n\to\infty} \prod_{k=0}^{n} P_k = 0$. We now take (P_i) satisfying $\lim_{n\to\infty} \prod_{k=0}^{n} P_k \neq 0$ and take $0 < q_i \downarrow 0$ as $i \uparrow \infty$. By (10) , (14) and (8) we see that $\lambda_\pi = 0$ and $\tilde{\mathfrak{J}}_{\lambda_\pi} = \tilde{\mathfrak{J}}_0 = \emptyset$.

We note that if we take $p_0 = 1$ then $\sum_{j\in E} P_{ij}(t) = 1$ $(\forall i \in E)$, i.e. $P(t)$ is non-stopped, since $Q = (q_{ij})$ is conservative and bounded.

Before moving on from Markov chains, we note that in the chain setting Theorem (4) can be improved. Namely,

<u>19. Theorem</u>. If $\{e^{-\lambda_\pi t} P_{ij}(t)\}$ is recurrent in the sense that $\int_0^\infty e^{-\lambda_\pi t} P_{ii}(t)dt = \infty$ for each $i \in E$, then there is precisely one $\mu \in \mathfrak{J}_{\lambda_\pi}(g)$ and μ satisfies:

$$\mu_i = \lim_{n\to\infty} \Big[\sum_{r=1}^{n} P_{0i}^{(r)} e^{-\lambda_\pi r} \Big] / \Big[\sum_{r=0}^{n} P_{00}^{(r)} e^{-\lambda_\pi r} \Big] \tag{20}$$

for each $i \in E$,

where $(P_{ij}^{(r)}) = (P_{ij}(1))^r$.

<u>Proof</u>. The existence comes from (4) . To prove the uniqueness and (20) notice that if $\mu \in \mathfrak{J}_{\lambda_\pi}(g)$ then the corresponding $\tilde{P}(t)$ defined in (9) is a recurrent process and

$$\frac{\sum_{r=1}^{n} \tilde{P}_{i0}(rt)}{\sum_{r=0}^{n} \tilde{P}_{00}(rt)} = \frac{1}{\mu_i} \cdot \frac{\sum_{r=1}^{n} P_{0i}^{(r)}(t) e^{-\lambda_\pi tr}}{\sum_{r=0}^{n} P_{00}^{(r)}(t) e^{\lambda_\pi tr}} \qquad (\forall i \ , \ \forall t)$$

Hence

$$\mu_i = \frac{\sum_{r=1}^{n} P_{0i}^{(r)} e^{-\lambda_\pi r}}{\sum_{r=0}^{n} P_{00}^{(r)} e^{-\lambda_\pi r}} \left(\frac{\sum_{r=1}^{n} \tilde{P}_{i0}^{(r)}}{\sum_{r=0}^{n} \tilde{P}_{00}^{(r)}} \right)^{-1} , \quad \forall i$$

But $\lim_{n\to\infty} \left(\sum_{r=1}^{n} \tilde{P}_{i0}^{(r)} \right)^{-1} \left(\sum_{r=0}^{n} \tilde{P}_{00}^{(r)} \right) = 1$ by [1; I.§9. Theorem 5).

§3. Minimal λ_π-invariant Measures

We begin this section with a description of the minimal elements of $\mathfrak{I}_0$.

Fix a strictly positive function $g \in \mathfrak{B}$. Denote the set of all extreme points of $\mathfrak{C}_0(g)$ by $\mathfrak{C}_0^e(g)$. As in [4] or [5] , we can endow a convex measurable structure to $\mathfrak{C}_0(g)$: the σ-algebra in $\mathfrak{C}_0(g)$ is generated by the sets $\{\mu \in \mathfrak{C}_0(g) : \mu(B) < u\}$, $B \in \mathfrak{B}$, $u \in R$. A measurable subset $\mathfrak{D} \subset \mathfrak{C}_0(g)$ is called a face if for every probability measure on $\mathfrak{C}_0(g)$ the measure μ_η given by

$$\text{(21)} \qquad \mu_\eta(B) = \int_{\mathfrak{C}_0^e(g)} \tilde{\mu}(B)\eta(d\tilde{\mu}) \quad , \quad B \in \mathfrak{B}$$

is in $\mathfrak{D}$ when and only when η is concentrated on $\mathfrak{D}$.

<u>22. Lemma</u>. $\mathfrak{I}_0(g)$ is a face of $\mathfrak{C}_0(g)$.

<u>Proof</u>. By (6) , we have for any $t > 0$:

$$\mathfrak{I}_0(g) = \bigcap_{\phi \in C_0(E)} \{\mu \in \mathfrak{C}_0(g) : \mu(\phi) = \mu P_T(\phi)\}$$

Hence $\mathfrak{I}_0(g)$ is measurable in $\mathfrak{C}_0(g)$. It is clear that μ_η defined in (21) belongs to $\mathfrak{I}_0(g)$ if η is concentrated on $\mathfrak{I}_0(g)$. We now assume that μ_η defined in (21) belongs to $\mathfrak{I}_0(g)$. Then

$$\mu(\phi) = \int_{\mathfrak{C}_0^e(g)} \tilde{\mu}(\phi)\eta(d\tilde{\mu}) = \int_{\mathfrak{C}_0^e(g)} \tilde{\mu}P_T(\phi)\eta(d\tilde{\mu})$$

hence

$$\text{(23)} \qquad \int_{\mathfrak{C}_0^e(g)\setminus\mathfrak{I}_0(g)} (\tilde{\mu}(\phi)-\tilde{\mu}P_T(\phi))\eta(d\tilde{\mu}) = 0 \ , \quad \forall \phi \in C_0(E) \quad .$$

Put

$$\mathfrak{D}' = \mathfrak{C}_0^e(g) \setminus \mathfrak{J}_0(g)$$

$$\mathfrak{D}'_\phi = \{\tilde{\mu} \in \mathfrak{D}' : \tilde{\mu}(\phi) - \tilde{\mu}P_T(\phi) > 0\}$$

then, by (23), we have

$$\eta(\mathfrak{D}) = \eta(\bigcup_{\phi \in C_0(E)} \mathfrak{D}'_\phi)$$

$$\leq \sum_{\phi \in C_0(E)} \eta(\mathfrak{D}'_\phi) = 0$$

Therefore η is concentrated on $\mathfrak{J}_{0,g}$.

It was shown in [4; 6.1] that the set of all extreme points of a face is just $\mathfrak{D} \cap \mathfrak{C}_0^e(g)$. Hence the set of all extreme points of $\mathfrak{J}_0(g)$, denoted by $\mathfrak{J}_0^e(g)$, is the subset $\mathfrak{J}_0^e(g) \cap \mathfrak{C}_0^e(g)$ of $\mathfrak{C}_0^e(g)$.

Let M be the class of non-negative measures. We say that $m \in M$ is minimal, if the relation $m = m_1 + m_2$, $m_1, m_2 \in M$ implies that m_1 and m_2 are proportional to m. It is now easy to see that μ is a minimal elements of $\tilde{\mathfrak{C}}_0$ if and only if μ is an extreme point of $\mathfrak{C}_0(g)$ for some g Thus, we may use [5; Lemma 2.2, Theorem 2.1 and Theorem 2.2] to give some limit procedures for computing the minimal elements of $\tilde{\mathfrak{J}}_0$.

Set

$$E_c = \{x \in E : \int_0^\infty P_t g(x)dt = \infty\}$$

$$E_d = \{x \in E : \int_0^\infty P_t g(x)dt < \infty\}$$

$$\mu_c = \mu\big|_{E_c} , \quad \mu_d = \mu\big|_{E_d} , \quad \mu \in \tilde{\mathfrak{C}}_0$$

$$\mathfrak{J}_{0,c} = \{\mu \in \tilde{\mathfrak{J}}_0 : \mu = \mu_c\} ,$$

$$\mathfrak{J}_{0,d} = \{\mu \in \tilde{\mathfrak{J}}_0 : \mu = \mu_d\} .$$

__24. Theorem.__ Let μ be a minimal element of $\tilde{\mathfrak{J}}_0$.

i) If $\mu(g) < \infty$, then either μ belongs to $\mathfrak{J}_{0,c}$ or μ belongs to $\mathfrak{J}_{0,d}$;

ii) If $\mu \in \mathfrak{J}_{0,c}$, $\phi, \psi \in L^1(\mu)$ and $\mu(\phi) \neq 0$, then

$$\frac{\mu(\phi)}{\mu(\psi)} = \lim_{u\to\infty} \frac{\int_0^u P_t(\phi(x)dt}{\int_0^u P_t\psi(x)dt}$$

for μ-almost all x ,

iii) If $\mu \in \mathfrak{I}_{0,d}$, then there exists a probability measure P on the space E^∞ of all sequences $x_1,x_2,\cdots,x_k,\cdots$ in E such that if

$\phi,\psi \in L^1(\mu)$ and $\mu(\psi) \neq 0$, then

$$\frac{\mu(\phi)}{\mu(\psi)} = \lim_{k\to\infty} \frac{\int_0^\infty P_t(x_k)dt}{\int_0^\infty P_t\psi(x_k)dt}$$

and

$$\lim_{k\to\infty} \frac{\int_0^\infty P_t\phi(x_k)dt}{\int_0^\infty P_t\psi(x_k)dt} = 0$$

for P-almost all sequences $\{x_k\}$ and $s \in (0,\infty)$.

In order to use these results to study $\tilde{\mathfrak{I}}_{\lambda_\pi}$, we now reduce the general case to the case where $\lambda_\pi = 0$.

Let f be a λ_π-excessive function which is finite and trictly positive everywhere. Then we may define

(25) $$\hat{P}_t(x,dy) = f(x)^{-1}P_t(x,dy)f(y)$$

It is easy to check that $\hat{P}_t(x,\cdot)$ is a sub-Markovian transition function.

Denote the set of all non-trivial invariant measures for $\{\hat{P}_t : t > 0\}$ by $\hat{\mathfrak{I}}_0$.

<u>26. Theorem</u>. $\tilde{\mathfrak{I}}_{\lambda_\pi} \neq \emptyset$ is equivalent to $\hat{\mathfrak{I}}_0 \neq \emptyset$. In detail, the corresponding between $\mu \in \mathfrak{I}_{\lambda_\pi}$ and $\nu \in \hat{\mathfrak{I}}_0$ is the following:

$$\nu(dx) = f(x)\mu(dx)$$

<u>Proof</u>. If $\nu \in \hat{\mathfrak{I}}_0$ and $\nu(dx) = f(x)\mu(dx)$, then

$$\int_B f(x)\mu(dx) = \nu(B) = \int \nu(dx)\hat{P}_t(x,B)$$

$$= \int \mu(dx)\int_B P_t^{\lambda_\pi}(x,dy)f(y)$$

$$= \int_B (\mu P_t^{\lambda_\pi})(dx)f(x) \quad \text{for each} \quad B \in \mathfrak{B} \ .$$

Hence $\mu = \mu P_t^{\lambda_\pi}$, that is $\mu \in \tilde{\mathfrak{I}}_{\lambda_\pi}$. The converse can be proved similarly.

Since we have a complete answer to the problem "$\tilde{\mathfrak{I}}_{\lambda_\pi} \neq \emptyset$?" in case that $\{P_t^{\lambda_\pi} : t > 0\}$ is recurrent, it is important to construct a λ_π-excessive function only when $\{P_t^{\lambda_\pi} : t > 0\}$ is non-recurrent. We will assume slightly more than non-recurrence. Namely, we assume that

$$\int_0^\infty P_t^{\lambda_\pi}\psi_0(x)dt < \infty \quad \text{for each} \quad x \in E \ . \tag{27}$$

In many cases (cf. [8]) , (27) is equivalent to non-recurrence.

<u>28. Lemma</u>. Under the condition (27) , the function f defined by

$$f(x) = \int_0^\infty P_t^{\lambda_\pi}\psi_0(x)dt$$

is a λ_π-excessive, finite and positive everywhere function.

<u>Proof</u>. The positive property comes from (1.i) and (1.ii) . The λ_π-excessive property follows from

$$P_t^{\lambda_\pi}f(x) = \int_t^\infty P_s^{\lambda_\pi}\psi_0(x)ds \uparrow f(x) \quad \text{as} \quad t \downarrow 0 \ .$$

Sometimes it is convenient to use the following decomposition:

For a strictly positive function $g \in \mathfrak{B}$, put

$$\hat{E}_c = \{x \in E : \int_0^\infty P_t^{\lambda_\pi}g(x)dt = \infty\}$$

$$\hat{E}_d = \{x \in E : \int_0^\infty P_t^{\lambda_\pi}g(x)dt < \infty\}$$

$$\hat{\mu}_c = \mu\big|_{\hat{E}_c} \ , \quad \hat{\mu}_d = \mu\big|_{\hat{E}_d} \ , \quad \mu \in \tilde{\mathfrak{E}}_{\lambda_\pi}$$

By [5; Theorem 3.2] , we have the following:

<u>29. Theorem</u>. If $\mu \in \tilde{\mathfrak{C}}_{\lambda_\pi}$, $\mu(g) < \infty$ and $\hat{\mu}_d = 0$, then $\mu \in \tilde{\mathfrak{J}}_{\lambda_\pi}$.

This is an improvement of (4) . Indeed, for each $\mu \in \tilde{\mathfrak{C}}_{\lambda_\pi}$, we may choose a strictly positive function $g \in C(E)$ (for example, $g(x) \equiv \int_0^\infty P_t^\lambda \psi_0(x)dt$ $(\lambda > 0)$), such that $\mu(g) < \infty$. Suppose that $\{P_t^{\lambda_\pi} : t > 0\}$ is recurrent and $\mu(\hat{E}_d) \neq 0$, then there exists $0 < c_1 < c_2 < \infty$ and a compact subset K such that if

$$G \equiv \{x : c_1 \leq \int_0^\infty P_t^{\lambda_\pi} g(x)dt \leq c_2\} \cap K$$

then $0 < \mu(G) < \infty$. Put $\nu = \mu|_G$, then ν is a Radon measure and

$$\int_0^\infty \nu P_t^{\lambda_\pi} g dt \begin{cases} \geq c_1\mu(G) > 0 \\ \leq c_2\mu(K) < \infty \end{cases} .$$

For each $\phi \in C_0^+(E)$, we have

$$\begin{aligned} \int_0^\infty \nu P_t^{\lambda_\pi} \phi dt &\leq \frac{\|\phi\|}{a} \int_0^\infty \nu P_t^{\lambda_\pi} g dt \\ &\leq \frac{\|\phi\| c_2}{a} \mu(K) < \infty \end{aligned}$$

where $a = \inf_{x \in \mathrm{supp}(\phi)} g(x) > 0$. Hence $\int_0^\infty \nu P_t^{\lambda_\pi} dt$ is a Radon measure. This is a contradiction.

In particular, we have

<u>30. Corollary</u>. If $P(t)$ is recurrent, then $\lambda_\pi = 0$ and $\tilde{\mathfrak{J}}_0 \neq \emptyset$. In fact, for each $\mu \in \tilde{\mathfrak{C}}_0$, $\hat{\mu}_d = 0$.

§4. Markov Chains

We first discuss the discrete time case.

<u>31. Theorem</u>. Suppose that $P = (P_{ij})$ is an irreducible matrix on E and satisfies

$$\sum_{n=0}^\infty P^n < \infty$$

Define

$${}_{H}P_{ij}^{(0)} = \delta_{ij}$$

$${}_{H}P_{ij}^{(n)} = \sum_{k_1,\cdots,k_{n-1} \in H} P_{ik_1} P_{k_1k_2} \cdots P_{k_{n-1},j} \qquad (n \geq 1)$$

$$L_{ki}(j) = \sum_{r=j}^{\infty} \sum_{n=1}^{\infty} {}_{i}P_{kr}^{(n)} P_{ri} + P_{ki}$$

where $H \subset E$. Then the equation $\mu = \mu P$ has a positive solution if and only if there exists an infinite subset K of E such that

$$\lim_{j\to\infty} \lim_{K \ni k\to\infty} L_{ki}(j)/L_{ki}(0) = 0$$

<u>Proof</u>. This theorem was proved by Harris [7] and Veech [11] in the case that P is a strictly stochastic matrix. Their arguments are also available for us. We have need only to point out some changes.

Define

$$Q = \sum_{n=1}^{\infty} P^n$$

$$\phi_{ij} = (\sum_{n=0}^{\infty} {}_{i}P_{ii}^{(n)})^{-1}$$

$$\theta_{ij} = \sum_{n=1}^{\infty} {}_{\{i,j\}}P_{ij}^{(n)} \qquad (i \neq j)$$

Then, it is easy to see that $0 < \theta_{ij}, \phi_{ij} < \infty$ and that

$${}_{H}P_{ij}^{(n)} = \sum_{\ell \notin H} {}_{H}P_{i\ell}^{(m)} {}_{H}P_{\ell j}^{(n-m)} \qquad (i,j \in E \ , \ 0 \leq m \leq n)$$

$${}_{i}P_{ij}^{(n)} = \sum_{m=1}^{n} {}_{\{i,j\}}P_{ij}^{(m)} {}_{i}P_{jj}^{(n-m)}$$

$$\begin{aligned}\theta_{ij} &= \theta_{ij}(\sum_{n=0}^{\infty} {}_{i}P_{jj}^{(n)})\theta \\ &= \sum_{m=1}^{\infty} {}_{\{i,j\}}P_{ij}^{(m)} \sum_{n=0}^{\infty} {}_{i}P_{jj}^{(n)} \phi_{ji} \\ &= \sum_{n=1}^{\infty} \sum_{m=1}^{n} {}_{\{i,j\}}P_{ij}^{(m)} {}_{i}P_{jj}^{(n-m)} \phi_{ji}\end{aligned}$$

$$= \Big(\sum_{n=1}^{\infty} {}_iP_{ij}^{(n)}\Big)\phi_{ji}\ , \quad i \neq j$$

$$\sum_{n=2}^{\infty} \sum_{i \in \{k_1,\cdots,k_{n-1}\}} P_{kk_1}P_{k_1k_2} \cdots P_{k_{n-1}j}$$

$$= \sum_{n=2}^{\infty} \sum_{m=1}^{n-1} \sum_{k_1\cdots k_{m-1}} P_{kk_1}P_{k_1k_2} \cdots P_{k_{m-1}i} \ \cdot$$

$$\cdot \sum_{k_{m+1}\neq i,\cdots,k_{n-1}\neq i} P_{ik_{m+1}} \cdots P_{k_{n-1}j}$$

$$= \sum_{n=2}^{\infty} \sum_{m=1}^{n-1} P_{ki}^{(m)}\ {}_iP_{ij}^{(n-m)}$$

$$= \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} P_{ki}^{(m)}\ {}_iP_{ij}^{(n)}$$

$$= Q_{ki} \sum_{n=1}^{\infty} {}_iP_{ij}^{(n)}$$

and

$$Q_{kj} = \sum_{n=2}^{\infty} \sum_{i\in\{k_1,\ldots,k_{n-1}\}} P_{kk_1}P_{k_1k_2} \cdots P_{k_{n-1}j} + \sum_{n=1}^{\infty} {}_iP_{kj}^{(n)}$$

$$= Q_{ki} \sum_{n=1}^{\infty} {}_iP_{ij}^{(n)} + \sum_{n=1}^{\infty} {}_iP_{kj}^{(n)}$$

$$= Q_{ki}\ \theta_{ij}/\phi_{ji} + \sum_{n=1}^{\infty} {}_iP_{kj}^{(n)}\ , \quad i \neq j$$

We now arrive at the same decomposition as in [7] :

$$\frac{Q_{ki}}{Q_{k0}} = \sum_{r=0}^{j-1} \frac{Q_{kr}}{Q_{k0}} P_{ri} + \frac{Q_{ki}}{Q_{k0}} \sum_{r=j}^{\infty} \Big(\frac{\theta_{ir}}{\phi_{ri}}\Big)P_{ri} + \frac{L_{ki}(j)}{Q_{k0}}\ .$$

We can now state the last result.

<u>32. Theorem</u>. Let $Q = (q_{ij})$ be a totally stable and irreducible Q-matrix, $P(t)$ a Q-process such that $\overset{\lambda}{P}{}^{\pi}(t)$ is non-recurrent. Define

$$f_i = \int_0^{\infty} \overset{\lambda}{P}{}^{\pi}_{ij}(t)dt < \infty$$

$$\hat{P}_{ij}(t) = f_i^{-1} \overset{\lambda}{P}{}^{\pi}_{ij}(t) f_j$$

$$\hat{P}_{ii} = 0 \quad , \quad \hat{P}_{ij} = f_i^{-1} q_{ij} f_j (\lambda_\pi + q_i)^{-1} \qquad (i \neq j)$$

then $\hat{P}_t$ is a Q-process with Q-matrix $\hat{Q} = (q_{ij})$:

$$\hat{q}_i = \lambda_\pi + q_i \quad , \hat{q}_{ij} = f_i^{-1} q_{ij} f_j \qquad (i \neq j)$$

$\hat{P}(t)$ satisfies the forward Kolmogorov equations with $\hat{Q}$ if and only if $P(t)$ satisfies the forward Kolmogorov equations with Q . Finally, if $P(t)$ satisfies the equations, then, in order that $\tilde{\mathfrak{J}}_{\lambda_\pi} \neq \emptyset$ the following condition is necessary: there exists an infinite subset K of E such that

$$\lim_{j\to\infty} \lim_{K \ni k \to \infty} \hat{L}_{ki}(j)/\hat{L}_{ki}(0) = 0$$

where for fixed i and j , $\hat{L}_{ki}(j)$ is the minimal non-negative solution to

$$x_k = \sum_{\ell \neq i} \hat{P}_{k\ell} x_\ell + \sum_{r=j}^{\infty} \delta_{kr} \hat{P}_{ri} \quad , \quad k \in E$$

This can be obtained by the formula

$$\hat{L}_{ki}(j) = \sum_{n=1}^{\infty} x_k^{(n)}$$

where

$$x_k^{(1)} = \sum_{r=j}^{\infty} \delta_{kr} \hat{P}_{ri} \qquad , \quad k \in E$$

$$x_k^{(n+1)} = \sum_{\ell \neq i} \hat{P}_{k\ell} x_\ell^{(n)} \qquad , \quad n \geq 1 \ , \ k \in E$$

<u>Proof</u>. 1^0 . As mentioned in (8) , it is easy to check the first assertion.

2^0 . We now prove the second assertion.

$$\sum_k \hat{P}_{ik}(t) \hat{q}_{kj} = f_i^{-1} \sum_{k \neq j} P_{ik}^{\lambda_\pi}(t) q_{kj} f_j - f_i^{-1} e^{-\lambda_\pi t} P_{ij}(t) f_j \ .$$
$$\cdot \ (\lambda_\pi + q_j)$$

$$= f_i^{-1} e^{\lambda_\pi t} f_j \Big(\sum_{k \neq j} P_{ik}(t) q_{kj} - P_{ij}(t)(\lambda_\pi + q_j) \Big)$$

$$= f_i^{-1} e^{-\lambda_\pi t} f_j (P'_{ij}(t) - \lambda_\pi P_{ij}(t))$$

$= \hat{P}'_{ij}(t)$.

3^0 . By (26) , $\tilde{\mathfrak{J}}_{\lambda_\pi} \neq \emptyset \Leftrightarrow \hat{\mathfrak{J}}_0 \neq \emptyset$. Thus, if $\tilde{\mathfrak{J}}_{\lambda_\pi} \neq \emptyset$, then by (10) and (14) , there is a positive solution to $\nu = \nu\hat{P}$. Notice that $\hat{P}$ is transient, it is not hard to prove that the condition given here is equivalent to the one given in (31) .

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