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of an L^2-Stochastic Process

by

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1. Introduction

By an L^2-stochastic process, we understand simply a collection X_t, $-\infty < t < \infty$, of real valued random variables (i.e. measurable functions) on a measure space $(\Omega, F, P) : P(\Omega) = 1$, with $\int X_t^2 dP (= EX_t^2) < \infty$ for each t. In the present paper we will not discuss any "sample path properties," and it will not matter whether P is complete. In fact, we may and shall consider random variables which are equal except on P-null sets as identical. We assume for convenience throughout that $\int X_t dP (= EX_t) = 0$, that the covariance $\Gamma(s,t) = E(X_s X_t)$ is continuous, and finally that for $\lambda > 0$, $\int_0^\infty e^{-\lambda s}\Gamma(s,s)ds < \infty$. Let $H(t)$ denote the Hilbert space closure of $\{X_s, s < t\}$. We note that $X_t \in H(t)$, and that $H(t)$ is, in an obvious sense, left-continuous in t.

The particular class of processes which is our concern are those which are orthogonalizable, in the sense that there exists an L^2-integral representation

$$1) \qquad X_t = \int_{-\infty}^{t} F(t,u)dY(u) + V_t$$

where Y is an L^2-valued measure $(E(\Delta_1 Y \Delta_2 Y) = 0$ if $\Delta_1 \cap \Delta_2 = \phi)$, and also $V_t \in H(-\infty)$ $(= \bigcap_u H(u))$ and $E(V_t(\Delta Y)) = 0$ for all finite Δ. Here we choose $Y(u) - Y(0)$ to be L^2-left-continuous in u, and the integral 1) does not include any jump in Y at time t. Also, if $d\sigma^2(u) = dEY^2(u)$ $(= E(dY(u))^2)$ then $\int_{-\infty}^{t} F^2(t,u)d\sigma^2(u) < \infty$. If, in addition, the collection $\{V_s, \Delta Y; \Delta, s \le t\}$ has Hilbert space closure

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$H(t)$ for each t, then we call 1) a Lévy canonical representation. Necessary and sufficient conditions on Γ for such a representation were obtained by P. Lévy [5] and T. Hida [2], among others (in Hilbert space language, the requirement is that X_t have multiplicity at most 1). Here it will suffice to observe that, apparently, all L^2-processes of any intrinsic interest do satisfy the conditions. From now on, therefore, we assume the existence of a canonical representation 1).(2)

This canonical representation is of course not unique. For any measurable function $\beta(u) \neq 0$, with β^2 locally $d\sigma^2$-integrable, we may replace $(F(t,u), dY(u))$ by $(\beta^{-1}(u)F(t,u), \beta(u)dY(u))$. On the other hand, this is the full extent of the nonuniqueness in $dY(u)$. To see this, let $\mathbb{P}(Z;H)$ denote the projection of an L^2-random variable Z onto a closed subspace H. Then, in 1), $\{X(t) - \mathbb{P}(X(t); H(t_1)), \ t_1 < t < t_2\}$ generates the same Hilbert space as $\{Y(t) - Y(t_1), \ t_1 < t < t_2\}$, because both are orthogonal to $H(t_1)$ and, together with $H(t_1)$, generate $H(t_2)$. Now if Y_1 and Y_2 denote Y for two distinct representations 1) of the same X, with corresponding $d\sigma_1^2$ and $d\sigma_2^2$, then $Y_1(B_1) (= \int_{B_1} dY_1)$ and $Y_2(B_2)$ are orthogonal whenever B_1 and B_2 are disjoint bounded Borel sets. This follows by the above for disjoint finite unions of intervals, hence for each such B_1 it holds for all bounded Borel sets B_2 disjoint from B_1 by L^2-approximation using $E(Y_2(B_2) - Y_2(B_2'))^2 = d\sigma_2^2(B_2 \Delta B_2')$. Hence, finally, by the monotone class theorem, it is true for all bounded Borel sets B_1 and B_2 disjoint from B_1. Now we can write $Y_2(-n,n) = \int_{-n}^{n} f_n(u)\, dY_1(u)$ for an f_n unique up to $d\sigma_1^2$-null sets. Then for $B \subset (-n,n)$ we have :

$$Y_2(-n,n) = \int_B f_n(u)\, dY_1(u) + \int_{(-n,n)-B} f_n(u)\, dY_1(u) \quad ,$$

where the first term on the right is orthogonal to $Y_2((-n,n)-B)$. It follows

(2) The results below are extended to the general case in [3], with considerable loss of explicitness. The present paper was motivated by a remark of J. L. Doob.

that this term is $Y_2(B)$. Thus, letting $n \to \infty$ we obtain an f, unique up to $d\sigma_1^2$ - null sets, with $Y_2(B) = \int_B f(u)\, dY_1(u)$ for all bounded B. This is a relation of the asserted type (of course, we also have the trivial non-uniqueness that $F(t,u)$ may be changed on a $d\sigma^2$ - null set of u for each t, without changing dY).

We can think of 1) as a linear analysis of $X(t)$ in terms of its past evolution $H(s)$, $s \leq t$. The object here is to relate this to the futures $X(t + s)$, $s \geq 0$. Since these cannot be known at time t, we must be content with their prediction in terms of $H(t)$. It is well known from Hilbert space theory that the best prediction of $X(t + s)$, in the sense of minimizing $E(X(t + s) - Y)^2$ over $Y \in H(t)$, is simply

Notation. $R(t + s,t) = \mathbb{P}(X(t + s); H(t))$.

2. Statement of the Problem

The problem which we propose to solve here is now to obtain $(F(t,u),dY(u))$ from $R(t + s,t)$ when t, u, and s vary appropriately. Let us note first that the converse problem is very simple. To obtain R we note that there must exist some representation

$$2) \qquad R(t + s,t) = \int_{-\infty}^{t} G(t + s,u)dY(u) + V,$$

because every element of $H(t)$ is so represented. But $V = \mathbb{P}(X(t + s)\,;\, H(-\infty))$ implies that $V = V_{s+t}$, and then we need only observe that in the decomposition

$$X(t + s) = \left(\int_{-\infty}^{t} F(t + s,u)dY(u) + V_{s+t}\right) + \int_{t}^{t+s} F(t + s,u)dY(u)$$

the last term is orthogonal to $H(t)$. Hence we have $G(t + s,u) = F(t + s,u)$ in 2). The problem below is, however, not as simple. Even if X_t is "wide-sense stationary" (i.e. $\Gamma(s,t)$ depends only on $s - t$) the known solution (from [1, XII, Theorem 5.3]) depends on the spectral representation of X_t. Thus it expresses ΔY in the "frequency domain". This does not easily give an expression in the "time domain," as required here (for example, the solution may require derivatives of X, hence it cannot be expressed in integral form over X_s, $s \leq t$). In any case, the spectral method does not extend to the general process 1)).

Stated more precisely, our problem is this: given $R(t' + s, t')$ for $s \geq 0$ and $t' < t$, in terms of X_u, $u \leq t'$, to construct $F(t', u)$ and $dY(u)$, $u < t$, $t' < t$, for a canonical representation 1). We observe why t' must be introduced -- if $X_{t+s} = X_t$ for all s, then $R(t + s, t) = X_t$ for all s and there is no hope of obtaining either F or dY from this. Actually, our problem has two distinct parts. Since $F(t', u)$ is nonrandom, we seek to determine it, not from observation of $R(\cdot,\cdot)$, but from the covariance of $R(\cdot,\cdot)$. We are assuming that an expression for R in terms of $X(\cdot)$ is known, and we may assume without loss of generality that it is linear in $X(\cdot)$. Therefore, the covariance of R may be calculated from Γ, and our hypothesis justifies its use. On the other hand, Y is "random", and to calculate it in an interval we must use the "observed values" of R, rather than only its covariance.

The same determination problem has been studied by P. Lévy in several papers, but without using R. It is of course possible in theory to determine F and dY directly from Γ and $X_{t'}$, $t' < t$. The direct attempt leads, however, to a singular Fredholm equation for F, which has no unique solution [Lévy, 4]. On the other hand, the corresponding problem with t replaced by a discrete parameter n is not difficult, and is solved in [4, Section 4.1]. It thus appears that with a discrete parameter the canonical representation naturally precedes solution of the prediction problem, while with a continuous parameter it is the other way around.

3. A Class of Wide-sense Martingales

The solution to be given here hinges on the following quantities, which may appear a little complicated at first sight, but which are probably as simple as the problem admits.

Definition 1.[3] For $\lambda > 0$ and $t \geq 0$, let

$$M_\lambda(t) = P_\lambda(t) - P_\lambda(0) + \lambda \int_0^t (X(u) - P_\lambda(u))du, \tag{3}$$

where $P_\lambda(t) = \lambda \int_0^\infty e^{-\lambda s} R(t+s,t)ds$, and the integrals are in the L^2-sense on (Ω,F,P).

The existence of these integrals follows from our hypotheses on Γ. Indeed, since X_t is L^2-continuous, $R(t+s,t)$ is L^2-continuous in s, and $ER^2(t+s,t) \leq \Gamma(t+s, t+s)$. Then $P_\lambda(t) = \mathbb{P}(\lambda \int_0^\infty e^{-\lambda s}X(t+s)ds; H(t))$, where the integral on the right exists because

$$E^{\frac{1}{2}}\left(\int_0^\infty e^{-\lambda s}X(t+s)ds\right)^2 \leq \int_0^\infty e^{-\lambda s}\Gamma^{\frac{1}{2}}(t+s)ds,$$

which is finite by another application of Schwartz' inequality. It follows readily that $P_\lambda(t)$, and also $M_\lambda(t)$, are L^2-left-continuous in t, and L^2-continuous in λ. It will be shown that, for suitable λ, $M_\lambda(t)$ can serve as $Y(t) - Y(0)$ in 1) for $t \geq 0$. It is clear that $M_\lambda(t) \in H(t)$, and we next show that it has orthogonal increments. This follows immediately from

Theorem 2. For each $\lambda > 0$, $M_\lambda(t)$ is a wide-sense martingale with respect to $H(t)$; i.e. $\mathbb{P}(M_\lambda(t+s); H(t)) = M_\lambda(t)$, $0 \leq t,s$.

Proof. We use the fact that L^2-integration commutes with projection to write

$$\mathbb{P}(M_\lambda(t_2) - M_\lambda(t_1); H(t_1)) =$$

$$\lambda \int_{t_2}^\infty ((e^{-\lambda(v-t_2)} - e^{-\lambda(v-t_1)})\mathbb{P}(X(v); H(t_1))dv$$

$$- \lambda \int_{t_1}^{t_2} (e^{-\lambda(u-t_1)} - 1)\mathbb{P}(X(u); H(t_1))du$$

[3] This notation differs slightly from that of [3], where X_t was Gaussian and $P_\lambda(t)$ was right-continuous. Here we use $P_\lambda(t-)$ instead.

$$- \lambda^2 \int_{t_1}^{t_2} \int_u^{t_2} e^{-\lambda(v-u)} \mathbf{P}\,(X(v);\ H(t_1))dvdu$$

$$- \lambda^2 \int_{t_1}^{t_2} \int_{t_2}^{\infty} e^{-\lambda(v-u)} \mathbf{P}\,(X(v);\ H(t_1))dvdu.$$

Combining the first and last terms of this expression, and interchanging order of integration, it becomes simply

$$\lambda \int_{t_2}^{\infty} (e^{-\lambda(v-t_2)} - e^{-\lambda(v-t_1)} - \lambda \int_{t_1}^{t_2} e^{-\lambda(v-u)} du) \mathbf{P}\,(X(v);\ H(t_1))dv$$

$$+ \lambda \int_{t_1}^{t_2} (1 - e^{-\lambda(v-t_1)} - \lambda \int_{t_1}^{v} e^{-\lambda(v-u)} du) \mathbf{P}\,(X(v);\ H(t_1))dv.$$

Here both integrands are 0, completing the proof.

Returning to 1), it will be convenient to choose $F(t,u)$, for given $dY(u)$, to be continuous in $t \geq u$ for each u. To see that this is always possible, we observe that we have

$$X(t) = \int_{-\infty}^{t} \left(\frac{dE(X(t)Y(u))}{d\sigma^2(u)} \right) dY(u) + V_t$$

for any Radon-Nikodym derivative of $dE(X(t)Y(u))$ with respect to $d\sigma^2(u)$ on $(-\infty,t)$, where the absolute continuity follows by Schwartz' inequality. Here it is not difficult to choose

$$\left| \frac{dE(X(t_2)Y(u))}{d\sigma^2(u)} - \frac{dE(X(t_1)Y(u))}{d\sigma^2(u)} \right| \leq E^{\frac{1}{2}}(X(t_2) - X(t_1))^2$$

for all $u \leq t_1 < t_2$. Thus, in fact, we obtain continuity in t, uniformly in u for bounded t.

From now on, we assume that $F(t,u)$ is continuous in t as above. The connection of $M_\lambda(t)$ with the canonical representation 1) is as follows.

<u>Theorem 3</u>. For $\lambda > 0$ and $t \geq 0$ we have

$$M_\lambda(t) = \int_0^t \left\{ \lambda \int_0^\infty e^{-\lambda s} F(u + s,u)ds \right\} dY(u),$$

where the inner integral exists for $d\sigma^2$-a.e. u, and is in $L^2(d\sigma^2)$.

Proof. Substitution of 2) with $G = F$ into Definition 1 of P_λ gives

$$P_\lambda(t) = \lambda \int_0^\infty e^{-\lambda s} \left(\int_{-\infty}^t F(t + s,u)dY(u)\right)ds + \lambda \int_0^\infty e^{-\lambda s} V_{t+s}\, ds.$$

We need to interchange order of integration on the right. To justify this, note first that

$$\int_{-\infty}^t \left(\int_0^\infty e^{-\lambda s} F^2(t + s,u)ds\right) d\sigma^2(u)$$

$$= \int_0^\infty e^{-\lambda s} E\left(\int_{-\infty}^t F(t + s,u)dY(u)\right)^2 ds$$

$$\leq \int_0^\infty e^{-\lambda s}\, EX^2(t + s)ds,$$

and the last expression is finite by our hypothesis on Γ. Then it follows from Schwartz' Inequality that the left side of

$$\int_{-\infty}^t \left(\int_0^\infty e^{-\lambda s}|F(t + s,u)|ds\right)^2 d\sigma^2(u)$$

$$\leq \lambda^{-1} \int_{-\infty}^t \int_0^\infty e^{-\lambda s} F^2(t + s,u)ds d\sigma^2(u)$$

is also finite, so that the L^2-integral $\int_{-\infty}^t \left(\int_0^\infty e^{-\lambda s} F(t + s,u)ds\right) dY(u)$ exists. Clearly it is in the Hilbert space closure of $\{\Delta Y(u);\ \Delta \subset (-\infty,t]\}$. But for $v_1 < v_2 \leq t$, by Fubini's Theorem we have

$$E[(Y(v_2) - Y(v_1)) \int_0^\infty e^{-\lambda s}\left(\int_{-\infty}^t F(t + s,u)dY(u)\right)ds]$$

$$= \int_{v_1}^{v_2} \left(\int_0^\infty e^{-\lambda s} F(t + s,u)ds\right) d\sigma^2(u)$$

$$= E[(Y(v_2) - Y(v_1)) \int_{-\infty}^t \left(\int_0^\infty e^{-\lambda s} F(t + s,u)ds\right) dY(u)],$$

where the double integral in the middle expression exists by the above inequality.

Thus the integrals may be interchanged, and we have from Definition 1

$$\lambda^{-1}M_\lambda(t) = \int_0^t \left(\int_0^\infty e^{-\lambda s}F(t + s,u)ds\right)dY(u)$$

$$+ \int_0^t \int_0^v (F(v,u) - \lambda \int_0^\infty e^{-\lambda s}F(v + s,u)ds)dY(u)dv.$$

Reasoning similar to the preceding shows that the second term on the right is

$$\int_0^t \int_u^t (F(v,u) - \lambda \int_0^\infty e^{-\lambda s}F(v + s,u)ds)dv\ dY(u),$$

whence the coefficient of $dY(u)$ in $\lambda^{-1}M_\lambda(t)$ is

$$\int_0^t e^{-\lambda s}F(t + s,u)ds + \int_u^t (F(v,u) - \lambda \int_0^\infty e^{-\lambda s}F(v + s,u)ds)dv$$

$$= \int_0^t e^{-\lambda s}F(t + s,u)ds + \int_u^t F(v,u)dv$$

$$- \lambda \int_0^\infty e^{-\lambda s} \left(\int_u^t F(v + s,u)dv\right)ds.$$

Now the last term on the right may be integrated by parts for $d\sigma^2$-a.e. u to become

$$- \int_u^t F(v,u)dv - \int_0^\infty e^{-\lambda s}(\frac{d}{ds} \int_{u+s}^{t+s} F(w,u)dw)ds.$$

Combining the last two expressions yields

$$\lambda^{-1}M_\lambda(t) = \int_0^t \left(\int_0^\infty e^{-\lambda s}F(u + s,u)ds\right)dY(u),$$

as was to be shown.

We next state two Corollaries, of which the first is now trivial, while the second is immediate but rich in content.

Corollary 4. The incremental process

$$M_\lambda(t_2) - M_\lambda(t_1) = P_\lambda(t_2) - P_\lambda(t_1) + \lambda \int_{t_1}^{t_2} (X_u - P_\lambda(u))du,$$

$-\infty < t_1 < t_2 < \infty$, is in $H(t_2)$ and has orthogonal increments. We have

$$M_\lambda(t_2) - M_\lambda(t_1) = \int_{t_1-}^{t_2} \left(\int_0^\infty \lambda e^{-\lambda s} F(u + s,u)ds\right) dY(u)$$

for any Lévy canonical representation 1).

Corollary 5. If X_t is wide-sense stationary, so that we may choose $F(t,u) = F(t - u)$ and $d\sigma^2(u) = \sigma^2 du$ in 1), then

$$M_\lambda(t_2) - M_\lambda(t_1) = (\lambda \int_0^\infty e^{-\lambda s} F(s)ds)(Y(t_2) - Y(t_1)),$$

where dY is a process of wide-sense stationary orthogonal increments. Given a single observation of $M_\lambda(t_2) - M_\lambda(t_1)$ for all $\lambda > 0$ (fixed $t_1 < t_2$ and $w \in \Omega$), if it does not vanish identically then it determines F up to a constant factor. If V_t is known to vanish, then Γ is similarly determined.

Proof. The equivalence of wide-sense stationarity with the assertions on F and $d\sigma^2$ is well-known ([1, ℓoc.sit]). The rest is immediate from Theorem 3 and the uniqueness theorem for Laplace transforms.

4. Solution of the Main Problem, and Example.

We return now to the determination of F and dY in the non-stationary case. In practice, the key to our methods is the calculation of $EM_\lambda^2(t)$ from Γ. This follows by a simple formula when R, and hence P_λ, are known. The proof is in [3, Theorem 3.1], and as it is rather intricate we omit the details.

Lemma 6. For $\lambda > 0$ and $t_1 < t_2$, we have

$$EM_\lambda^2(t_2) - EM_\lambda^2(t_1) = EP_\lambda^2(t_2) - EP_\lambda^2(t_1) + 2\lambda \int_{t_1}^{t_2} E(X_u P_\lambda(u) - P_\lambda^2(u))\, du.$$

This bring us to the main theorem, which in sense is a proof without a theorem (the content depends on what is meant here by "effectively").

<u>Theorem 7</u>. A canonical pair $(F(t,u),\ dY(u))$ is determined effectively in an interval $u_1 < u < u_2$ by $E(M_\lambda^2(u) - M_\lambda^2(u_1))$ and $dM_\lambda(u)$, $\lambda > 0$, $u_1 < u < u_2$.

<u>Proof</u>. For notational convenience we take $u_1 = 0$, $u_2 = t$. By Corollary 4 we have for any canonical (F_o, dY_o),

$$6)\qquad EM_\lambda^2(t) = \int_0^t (\lambda \int_0^\infty e^{-\lambda s} F_o(u+s,u)ds)^2 d\sigma_o^2(u).$$

Hence for every $\lambda > 0$ the measure $dEM_\lambda^2(u)$ is absolutely continuous with respect to $d\sigma_o^2(u)$. Our problem is to obtain a linear combination (possibly infinite) $\sum_i c_i M_{\lambda_i}(u)$ to serve as $Y(u)$. In fact, we will determine a λ_o such that $Y(u) = M_{\lambda_o}(u)$ is possible. However, as a subsequent Example 8 shows, it is sometimes more convenient in practice to use a linear combination rather than fixing λ_o. The only requirement is that the variance should determine a measure equivalent to $d\sigma_o^2$ as u varies, which is a requirement not depending on the (unknown) $d\sigma_o^2$. Then we will have

$$\sum_i c_i M_{\lambda_i}(u) = \int_0^u \left(\sum_i c_i \int_0^\infty \lambda_i e^{-\lambda_i s} F_o(u+s,u)ds \right) dY_o(u)$$

and hence we can use $d(\sum_i c_i M_{\lambda_i}(u))$ as $dY(u)$ in a new canonical representation.

We next show that, in fact, for all λ excepting a certain countable set the measures $dEM_\lambda^2(u)$ and $d\sigma_o^2$ are equivalent in (o,t). By 6) they are equivalent in (o,t) if and only if

$$7)\qquad 0 = d\sigma_o^2\{0 < u < t: \int_0^\infty e^{-\lambda s} F_o(u+s,u)ds = 0\},$$

and we can assume without loss of generality that the Laplace transform exists for all u. Then for each u it can vanish at most on a countable set of λ without making $F_o(u+s,u) = 0$ for all $s \geq 0$, since it is analytic in λ. It follows from this that for any continuous measure $d\nu(\lambda)$ we have

$$8) \qquad 0 = dv\{\lambda : \int_0^\infty e^{-\lambda s} F_o(u+s,u)ds = 0\}$$

except on a $d\sigma_o^2$-null set of u. Indeed, if $d\sigma_o^2\{0 < u < t: F_o(u + s,u) = 0$ for all $s \geq 0\} \neq 0$, then denoting the set in brackets by A we would have for $0 < t' < t$ $X_{t'} = V_{t'} + \int_{A^c \cap (o,t')} F_o(t',u)dY_o(u)$, and it would follow that $\int_{A^c \cap (o,t)} dY_o(u) \notin H(t)$, which contradicts the definition of a canonical representation. From 8) it follows by Fubini's Theorem that 7) holds for all λ except in a dv-null set for every continuous dv. On the other hand, the right side of 7) is upper-semicontinuous in λ because the integral is continuous in λ (so that $d\sigma_o^2$ applied to the complement is lower-semicontinuous). Hence for every $\varepsilon > 0$ the set of λ for which $d\sigma_o^2\{0 < u < t: \int_0^\infty e^{-\lambda s} F_o(u + s,u)ds) = 0\} \geq \varepsilon$ is closed and null for every dv. Since every uncountable closed set supports a nontrivial continuous measure (since it contains a monotone image of the Cantor set) the above set must be countable. Letting $\varepsilon \to 0$, we see that 7) holds outside a countable set of λ.

It remains to "effectively" determine a λ_0 outside this set (which is of course easy in "practical" cases). In any case, we may proceed as follows. We first obtain a measure which is equivalent to the (as yet unknown) $d\sigma_0^2$ in any one of various ways. For example, $d\left(\int_1^2 EM_\lambda^2(t)d\lambda\right)$ is always such a measure since the exceptional set of λ is erased. Denoting this by $d\sigma_1^2$, we next obtain a Radon-Nikodym derivative $\frac{dEM_\lambda^2(t)}{d\sigma_1^2(t)}$ which is continuous in λ for all t. By 6) this will always be continuous along the rationals for $d\sigma_1^2$-almost-all t, whence we can extend it to all λ by continuity, and use 0 on the exceptional set (if any). Now, as before, $d\sigma_1^2\{0 < u < t: \frac{dEM_\lambda^2(u)}{d\sigma_1^2(u)} = 0\}$ exceeds any $\varepsilon > 0$ on an at most countable, closed set. An element not in such a set can be effectively determined by systematically checking all points $0 < \lambda = kN^{-1} \leq N$ until such an element is found. Hence, choosing $\varepsilon_n \to 0$, we can determine by induction on n a nested sequence of closed intervals of the complements. Then any λ_0 in the intersection will satisfy $d\sigma_1^2\{0 < u < t: \frac{dEM^2_{\lambda_o}(u)}{d\sigma_1^2(u)} = 0\} = 0$. Consequently, the same equation holds with σ_0^2 in place of σ_1^2, and $dY(u) = dM_{\lambda_0}(u)$ is a possible choice of dY for a canonical representation 1).

It remains only to determine $F(t,u)$ for known dY, as was done above Theorem 3. Thus we have $F(t,u) = \frac{dE(X(t)Y(u))}{d\sigma^2(u)}$, which is computed from Γ, our assumed expression for R, and Lemma 6 (the covariance $E(M_{\lambda_1}(u)M_{\lambda_2}(u))$ is also in [3] if needed for the case $dY(u) = d(\sum_i c_i M_{\lambda_i}(u))$). We remark that, since $F(t,t)$ is not involved in the representation of X_t, we should define $F(t,t) = \lim_{r \to t+} F(r,t)$ in order to obtain the right-continuity of F in $t \geq u$ for each u.

We conclude with an example which involves a process X_t elsewhere studied by P. Lévy.

Example 8. Suppose that $X_t = \int_0^t (2t - u)dW(u)$, where $W(u)$ is a Wiener process. Lévy ([4],[5]) has checked that this representation is canonical, and he also has obtained infinitely many other noncanonical representations 1) of the same process X_t. However, our approach is from the opposite direction, in the sense that we should begin from the covariance. In this case, we have easily $\Gamma(s,t) = 3s^2t - \frac{2}{3}s^3$ for $0 \leq s \leq t$, and we take $X_s = \Gamma = 0$ for $s \leq 0$. We claim next that the predictors (i.e. projections) are given by 0 for $t \leq 0$ and $R(t+s,t) = (1 + 2st^{-1})X_t - 2st^{-2}\int_0^t X_u du$ for $t > 0$. Since our method takes this as starting point, we will not discuss the derivation, but only remark that such assertions are easily checked. One need only use Γ and a little integral calculus to show that $E(X_v(R(t+s,t) - X_{t+s})) = 0$ for $v \leq t$. It follows without difficulty that for $t > 0$,

$$M_\lambda(t) = (1 + 2(\lambda t)^{-1})X_t - 2\lambda^{-1}t^{-2}\int_0^t X_u du$$

$$- 2\int_0^t (u^{-1}X_u - u^{-2}\int_0^u X_v dv)du$$

$$= (1 + 2(\lambda t)^{-1})X_t - 2(\lambda^{-1}t^{-2} + t^{-1})\int_0^t X_u du.$$

We want to choose a convenient linear combination of $dM_\lambda(u)$ to use as $dY(u)$. We observe that $M_1(t) - M_2(t) = t^{-1}X_t - t^{-2}\int_0^t X_u du$, and it is straightforward to check that $E(M_1(t) - M_2(t))^2 = t$. Thus $Y(t) = M_1(t) - M_2(t)$ is a Wiener process, and we can use it for a canonical representation of X_t if it generates $H(t)$. This must be checked here only if we do not assume a priori that a canonical representation exists. Otherwise it suffices that $d\sigma^2(u)$ be absolutely continuous with respect to du, which is practically obvious. In any case, we can easily solve for X_t in the present case, obtaining $X_t = \int_0^t (M_1(u) - M_2(u))du + t(M_1(t) - M_2(t))$. Now straightforward computation yields $\frac{d}{du} E(X_t(M_1(u) - M_2(u)) = 2t - u$, hence our representation is $X_t = \int_0^t (2t - u)d(M_1(u) - M_2(u))$. Of course, this is entirely equivalent to the original representation of P. Lévy, but here dW is expressed in terms of dX instead of conversely.

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