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GIRSANOV TYPE FORMULA FOR A LIE GROUP VALUED BROWNIAN MOTION

by

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Let G be a Lie group of $d \times d$ matrices and X^1 be a G-valued continuous semimartingale, P^1 be its distribution on $\Omega = \mathcal{C}([0,1],G)$. Let $X^2 = AX^1$ be the left translate of X^1 by a G-valued adapted continuous process with finite variation paths, and P^2 be the distribution of X^2 on Ω. The question analogous to the classical Girsanov theorem is : Under what conditions on A, X is $P^2 \ll P^1$, and what is the density $\frac{dP^2}{dP^1}$?

We denote by $\mathcal{G}$ the Lie algebra of G, and by W the sample space $\mathcal{C}([0,1],\mathcal{G})$. Using the pathwise integration formula (see Karandikar [3]) for multiplicative stochastic integration, we may define an " exponential" mapping $\mathcal{E} : W \to \Omega$ and a "logarithm" $\mathcal{L} : \Omega \to W$, which are independent of the choice of the laws and, in a reasonable sense, inverse to each other. Then we may denote by Y^1, Y^2 the processes $\mathcal{L}(X^1)$, $\mathcal{L}(X^2)$ and by Q^1, Q^2 the corresponding laws on W. Next, using the " integration by parts formula " for multiplicative stochastic integration (Karandikar [4]), we show that $Y^2 = Y^1 + B$, where the process B is expressible in terms of A. Therefore the ordinary Girsanov theorem in the additive set-up will give conditions for the absolute continuity $Q^2 \ll Q^1$, and explicit expressions for the density. Returning to Ω by the exponential mapping $\mathcal{E}$, we can in this way solve a multiplicative Girsanov problem.

We study in particular the case where X^1 is a G-valued (multiplicative) brownian motion.

I. GENERALITIES

We first introduce some notation. Let U,V be continuous semimartingales (on some fixed probability space Ω with a filtration $(\mathcal{F}_t)$, not necessarily the same Ω as above), taking values in the space $L(d)$ of all $d \times d$ matrices. We denote by $\langle U,V \rangle$ the $L(d)$ valued process defined by

$$\langle U,V\rangle^i_j = \Sigma_k \langle U^i_k, V^k_j \rangle$$

The paths of $\langle U,V\rangle$ are continuous with finite variation, equal to 0 for t=0 . We denote by $V\cdot U$ and $V\circ U$ the Ito stochastic integral and the Stratonovich stochastic integral of V with respect to U :

$$(V\cdot U)_t = \int_0^t V_s dU_s \quad (\text{ matrix product }) , \quad (V\circ U)_t = \int_0^t V_s \circ dU_s$$

and we denote by $U:V$, $U\overset{\circ}{:}V$ the similar integrals, with matrix products on the right side $((U:V)_t = \int_0^t (dU_s)V_s \ldots)$. As usual, we may express Stratonovich integrals in terms of Ito integrals

$$(1) \qquad V\circ U = V\cdot U + \tfrac{1}{2}\langle V,U\rangle \quad , \quad U\overset{\circ}{:}V = U:V + \tfrac{1}{2}\langle U,V\rangle .$$

These formulas can be verified by looking at the entries and using the 1-dimensional relation (see Ito-Watanabe [2]). In the Lie group - Lie algebra setting, the Stratonovich integrals arise naturally.

We now assume that $U_0=0$. The Ito exponential of U , denoted by $\varepsilon(U)$, is the only solution to the stochastic differential equation

$$(2) \qquad V = I + V\cdot U \qquad .$$

More precisely, this is the <u>left</u> exponential (see Karandikar [3]. The right exponential will not be used here). It can be shown that $\varepsilon(U)=V$ is invertible (see Karandikar [3]) and we can recover U from V by the formula

$$(3) \qquad U = \ell(V) = V^{-1}\cdot V \quad (\text{ hence } \varepsilon(U)=\varepsilon(U') \Rightarrow U=U') .$$

Similarly, we define the Stratonovich (left) exponential $\varepsilon^*(U)$ as the solution to

$$(4) \qquad V = I + V\circ U$$

It can be easily seen that if V is a solution to (4), then $\langle V,U\rangle = V\cdot\langle U,U\rangle$, and therefore $V = I+V\cdot(U + \frac{1}{2}\langle U,U\rangle)$, hence

$$(5) \qquad \varepsilon^*(U) = \varepsilon(U + \tfrac{1}{2}\langle U,U\rangle)$$

and $\varepsilon^*(U)$ is invertible. Just as above, one can recover U from $V=\varepsilon^*(U)$ by the formula

$$(6) \qquad U = \ell^*(V) = V^{-1}\circ V \quad (\text{ hence } \varepsilon^*(U)=\varepsilon^*(U') \Rightarrow U=U') .$$

Let U and U' denote two continuous semimartingales, such that $U_0=U'_0=0$, and let W denote $\varepsilon(U')$. Then we have the <u>integration by parts</u> formula for multiplicative stochastic integrals

$$(7) \qquad \varepsilon(U + U' + \langle U,U'\rangle) = \varepsilon(W\cdot U:W^{-1})\varepsilon(U') \quad .$$

This is a direct consequence of Ito's formula (Karandikar [4]). The same arguments with Stratonovich integrals in place of Ito's integrals will give

(8) $$\varepsilon^*(U + U') = \varepsilon^*(W \circ U \circ W^{-1})\varepsilon^*(U') \text{ with } W=\varepsilon^*(U').$$

Also, (8) can be deduced from (7) and (5).

We are going to apply this formula in the situation described in the introduction. Let X be a continuous semimartingale such that $X_0=I$, and let A be a continuous semimartingale with finite variation paths, such that $A_0=I$. We assume that these two processes take their values in the set of <u>invertible</u> matrices, and define

(9) $$Y_t = \int_0^t X_s^{-1} \circ dX_s$$

(10) $$B_t = \int_0^t (A_s X_s)^{-1} dA_s X_s \quad \text{(Stieltjes integral)}$$

Then the paths of B have finite variation, and we have :

THEOREM 1. $AX = \varepsilon^*(Y+B)$.

<u>Proof</u>. We have $X=\varepsilon^*(Y)$ according to (9) and (6). Similarly, we set $\alpha = A^{-1} \circ A$, so that $A=\varepsilon^*(\alpha)$. Then $AX = \varepsilon^*(\alpha)\varepsilon^*(Y)$, which we try to identify with the right side of (8). We must have $\varepsilon^*(U')=\varepsilon^*(Y)$, hence $U'=Y$, $W=X$. Then we must have $W \circ U \circ W^{-1} = \alpha$, and therefore since $W=X$, $U=X^{-1} \circ \alpha \circ X = X^{-1} \circ (A^{-1} \circ A) \circ X = (AX)^{-1} \circ A \circ X = B$. Note that we didn't really use in this proof the fact that A has finite variation.

II. CONSTRUCTION OF THE MAPS $\mathcal{E}$ AND $\mathcal{L}$

Let $\mathbb{W}$ be the set of all continuous mappings $w : [0,1] \longmapsto L(d)$ such that $w(0)=0$. We denote by Y_t the coordinate mapping $w \mapsto w(t)$ on W , and by $\mathcal{G}_t$ the σ-field $\sigma(Y_s, s \leq t)$.

Let Ω be the set of all continuous mappings $\omega : [0,1] \longmapsto L(d)$ such that $\omega(0)=I$, and $\omega(t)$ is invertible for every t . The coordinate mappings and fields are denoted here by X_t and $\mathcal{F}_t$.

If one is interested in a particular pair $(\mathcal{G},G)$, the mappings in W will be restricted to be $\mathcal{G}$-valued, and those in Ω to be G-valued. This makes no essential difference, as we shall see.

We say that a probability law on W (Ω) is a <u>semimartingale measure</u> if the corresponding coordinate process is a semimartingale (w.r. to the corresponding filtration, made right-continuous and complete).

Our aim in this section consists in constructing Borel mappings $\mathcal{E} : W \to \Omega$, $\mathcal{L} : \Omega \to W$ such that, for any semimartingale measure on W, $X \circ \mathcal{E}$ is a version of the Stratonovich exponential $\varepsilon^*(Y)$, and for any semimartingale measure on Ω , $Y \circ \mathcal{L}$ is a version of the Stratonovich " logarithm" $\ell^*(X)$. These mappings, however, do not depend on the choice of a measure on W or Ω .

For $n \geq 1$, $w \in W$, $\omega \in \Omega$, define $s_i^n(w)$ and $t_i^n(\omega)$ for $i \geq 0$ inductively by

$$s_0^n(w) = t_0^n(w) = 0 \quad \text{and for} \quad i \geq 0$$

$$s_{i+1}^n(w) = \inf\{ s \geq s_i^n(w) : |Y(s,w)-Y(s_i^n(w),w)| \geq 2^{-n} \text{ or } s \geq 1\}$$

$$\begin{aligned} t_{i+1}^n(\omega) = \inf\{ t \geq t_i^n(\omega) : & |X(t,\omega)-X(t_i^n(\omega),\omega)| \geq 2^{-n} \text{ or} \\ & |X^{-1}(t,\omega)-X^{-1}(t_i^n(\omega),\omega)| \geq 2^{-n} \text{ or} \\ & |X^{-1}(t_i^n,\omega)X(t(\omega),\omega) - I| \geq 2^{-n} \text{ or } t \geq 1 \}. \end{aligned}$$

Here the norm $|\ |$ is chosen so that the logarithm of a matrix (i.e. the inverse mapping of the usual matrix exponential exp) is defined on the neighbourhood $|x-I|<1$ of the identity. We now set for $s,t \in [0,1]$

$$\mathcal{L}_n(t,\omega) = \sum_{i \geq 0} \log(X^{-1}(t \wedge t_i^n(\omega),\omega)\ X(t \wedge t_{i+1}^n(\omega),\omega))$$

$$\mathcal{E}_n(s,w) = \prod_{i \geq 0} \exp(Y(s \wedge s_{i+1}^n(w),w) - Y(s \wedge s_i^n(w),w))$$

It is easy to see that if ω is G-valued , then $\mathcal{L}_n(\cdot,\omega)$ is $\mathcal{G}$-valued, and similarly in the other direction. Let

$$W_0 = \{ w \in W : \mathcal{E}_n(\cdot,w) \text{ converges uniformly } \}$$

$$\Omega_0 = \{ \omega \in \Omega : \mathcal{L}_n(\cdot,\omega) \text{ converges uniformly } \}$$

We denote the corresponding limits by $\mathcal{E}(w)=\mathcal{E}(\cdot,w)$ and $\mathcal{L}(\omega)=\mathcal{L}(\cdot,\omega)$; outside W_0 or Ω_0 we set $\mathcal{E}(t,w)=I$, $\mathcal{L}(t,\omega)=0$ for all t . Of course, since the coordinate mappings are denoted by Y_t on W , X_t on Ω , we may write $Y_t(\mathcal{L}(\omega))$ instead of $\mathcal{L}(t,\omega)$, $X_t(\mathcal{E}(w))$ instead of $\mathcal{E}(t,w)$.

THEOREM 2. Let Z be a continuous $L(d)$-valued semimartingale defined on some filtered probability space $(\Theta, \mathcal{H}, (\mathcal{H}_t)_{t \leq 1}, \mu)$.

1) Assume that $Z_0=0$. Then for μ-a.e. $\theta \in \Theta$ the path $Z_\cdot(\theta)$ belongs to W_0 and the path $\varepsilon^*(Z)(\cdot,\theta)$ of the Stratonovich exponential $\varepsilon^*(Z)$ is equal to $\mathcal{E}(Z_\cdot(\theta))$.

2) Assume that $Z_0=I$ and Z takes its values in the set of invertible matrices. Then for μ-a.e. $\theta \in \Theta$ the path $Z_\cdot(\theta)$ belongs to Ω_0 , and the path $\ell^*(Z)(\cdot,\theta)$ of the Stratonovich logarithm $\ell^*(Z)$ is equal to $\mathcal{L}(Z_\cdot(\theta))$.

COROLLARY . Let P be a semimartingale measure on Ω (Q be a semimartingale measure on W). Then the image measure $\mathcal{L}(P)$ is a semimartingale measure on W ($\mathcal{E}(Q)$ a semimartingale measure on Ω) and we have

(11) $$\mathcal{E}(\mathcal{L}(\omega)) = \omega \quad \text{a.s. } P \qquad (\mathcal{L}(\mathcal{E}(w)) = w \quad \text{a.s. } Q).$$

Proof. The proof relative to the exponential is outlined in Karandikar [3] (Sém. Prob. XVI), and fully given in Karandikar [5]. Keeping the notation $t_i^n(\theta)$ for $t_i^n(Z_{\bullet}(\theta))$, the statement amounts to the fact that

$$J_n(t,\theta) = \sum_{i\geq 0} \log(Z^{-1}(t\wedge t_i^n(\theta),\theta)Z(t\wedge t_{i+1}^n(\theta),\theta))$$

converges μ-a.s. to $\ell^*(Z)(t,\theta)$, uniformly in $t\in[0,1]$. We rewrite $J_n(t,\theta)$ as

$$\sum_{i=0}^{\infty} \log(I + Z^{-1}(t\wedge t_i^n)[Z(t\wedge t_{i+1}^n)-Z(t\wedge t_i^n)])$$

$$= \sum_{i=0}^{\infty} Z^{-1}(t\wedge t_i^n)[Z(t\wedge t_{i+1}^n)-Z(t\wedge t_i^n)]$$

$$- \frac{1}{2}\sum_{i=0}^{\infty} (Z^{-1}(t\wedge t_i^n)[Z(t\wedge t_{i+1}^n)-Z(t\wedge t_i^n)])^2$$

+ higher order terms.

By the methods used in Karandikar [3], [5] it can be shown that a.s. J_n converges uniformly to $Z^{-1}\cdot Z - \frac{1}{2}< Z^{-1}\cdot Z, Z^{-1}\cdot Z >$. Denoting this process by U , we have $< U,U > = < Z^{-1}\cdot Z,Z^{-1}\cdot Z >$ and

$$U + \tfrac{1}{2}<U,U> = Z^{-1}\cdot Z .$$

Therefore, applying (5) and (3)

$$\varepsilon^*(U) = \varepsilon(U + \tfrac{1}{2}<U,U>) = \varepsilon(Z^{-1}\cdot Z) = Z$$

Which implies that $U=\ell^*(Z)$ according to (6). The corollary is almost obvious, and left to the reader.

III. THE MAIN RESULT

We may now translate theorem 1 in the situation of path spaces, to get our main result. Let $P=P^1$ be a semimartingale measure on Ω , and let $A(t,\omega)$ be a G-valued, $\mathcal{F}_t$-adapted continuous process with finite variation paths, such that $A(0,\omega)=I$. Let φ be the mapping from Ω to Ω defined by

(12) $$X(t,\varphi(\omega)) = A(t,\omega)X(t,\omega)$$

We denote by P^2 the image law of P^1 under φ .

We now denote by α the process on W ($\mathcal{G}$-valued)

(13) $$\alpha_t(w) = \int_0^t A_s^{-1}(\varepsilon(w))dA_s(\varepsilon(w))$$

and by B_t the process on W ($\mathcal{G}$-valued) .

(14) $$B_t(w) = \int_0^t X_s^{-1}(\varepsilon(w))d\alpha_s(w)X_s(\varepsilon(w)) .$$

Finally, let $Q=Q^1$ be the image of P^1 under $\mathcal{L}$, and Q^2 be the image of Q^1 under the mapping ψ from W to W defined by

(15) $$Y(t,\psi(w)) = B(t,w) + Y(t,w) .$$

Then theorem 1 gives at once the following result :

THEOREM 3 . 1) $\psi(\mathcal{L}(\omega)) = \mathcal{L}(\varphi(\omega))$ <u>a.s.</u> P^1. <u>Hence</u> Q^2 <u>is the image of</u> P^2 <u>under</u> $\mathcal{L}$, <u>and</u> P^2 <u>the image of</u> Q^2 <u>under</u> $\mathcal{E}$.

2) $Q^2 \ll Q^1$ <u>if and only if</u> $P^2 \ll P^1$. <u>Further, if</u> $Q^2 \ll Q^1$, then

$$(16) \qquad \frac{dP^2}{dP^1}(\omega) = \frac{dQ^2}{dQ^1}(\mathcal{L}\omega) \quad \text{a.e. } P^1 .$$

<u>Proof</u>. The first statement follows from theorem 1 and theorem 2, 2), both processes being versions of the Stratonovich logarithm of AX under the law P^1. The other statements follow from the fact that P^1, P^2, Q^1, Q^2 are semimartingale measures, and therefore $\mathcal{E}$ and $\mathcal{L}$ are almost inverse to each other under any of them (theorem 2, (11)).

Let us apply this to the case of brownian motions on G : we choose some euclidean norm on the Lie algebra $\mathcal{G}$, and an orthonormal basis $(D_1,\dots,D_m)$ relative to it (we denote by $\tilde{D}_1,\dots\tilde{D}_m$ the corresponding left invariant vector fields on G). Let Q^1 be the probability law on W for which (Y_t) is a m-dimensional motion in the euclidean space $\mathcal{G}$ (that is, the components Y^i_t of Y_t in the basis D_i are independent real-valued standard brownian motions). Then, according to Ibéro [1], the Stratonovich exponential $\varepsilon^*(Y)$ is a G-valued brownian motion corresponding to the left invariant " laplacian" $L = \Sigma_1^m \tilde{D}_i^2$. Otherwise stated, the law P^1 of this G-valued brownian motion on Ω is the image of Q^1 under $\mathcal{E}$, and Q^1 is the image of P^1 under $\mathcal{L}$.

The classical Girsanov theorem asserts that the law Q^2 of $Y+B$ under Q^1 will be absolutely continuous with respect to Q^1 , if the $\mathcal{G}$-valued process B on W is continuous, of finite variation, with a density b (progressively measurable) such that

$$(17) \qquad E_{Q^1}\left[\int_0^1 b_s \cdot dY_s - \int_0^1 |b_s|^2\, ds \right] = 1$$

Here $|\ |$ is the euclidean norm in $\mathcal{G}$ and $\cdot$ denotes the corresponding scalar product. Besides that, the function under the sign E is equal to the density dQ^2/dQ^1 .

In the multiplicative set up, this corresponds to the absolute continuity with respect to P^1 of the law P^2 of the process AX , where A is a G-valued process given as a (deterministic) multiplicative integral :

$$(18) \quad A(t,\omega) = I + \int_0^t A(u,\omega)d\alpha(u,\omega) \quad , \quad \alpha(t,\omega) = \int_0^t A^{-1}(u,\omega)dA(u,\omega)$$

where the $\mathcal{G}$-valued process $\alpha(t,\omega)$ on Ω is absolutely continuous with a progressively measurable density $a(t,\omega)$, and a,b are related by

(19) $$b(t, \mathcal{L}(\omega)) = X_t^{-1}(\omega)a(t,\omega)X_t(\omega)$$

Therefore, the condition for absolute continuity is the uniform integrability of the local martingale $\varepsilon(M)$, where

(20) $$M_t(\omega) = \int_0^t (X_s^{-1}a_sX_s)\cdot(X_s^{-1}dX_s)$$

and in that case, the density is equal to $\varepsilon(M)_1$. There is no obvious sufficient condition for absolute continuity, except the case where G is compact and the density a is bounded.

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