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MARTINGALES IN MANIFOLDS - DEFINITION, EXAMPLES, AND BEHAVIOUR UNDER MAPS.

by

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1. Introduction

We consider stochastic processes X with continous trajectories taking values in a differential manifold with a connection Γ. We introduce an intrinsic definition of Γ-martingale, using the idea of Γ-convex function. We give a variety of examples, showing how the definition is used in practice. In § 5 we construct a class of martingales on the Lie group Gl(n). Sections § 7 - § 9 study the behaviour of Brownian motion and Γ-martingales under affine maps, harmonic maps, and harmonic morphisms. The affine maps are the ones which preserve the martingale property. For maximum simplicity, we avoid the notions of horizontal lifting and stochastic integration of differential forms.

2. Differential geometry notations

Let M be an n-dimensional manifold of class C^k, $k \geq 2$. Whenever $\pi : E \longrightarrow M$ is a vector bundle of class C^r, $1 \leq r \leq k$, the vector space of C^r sections of E will be written $C^r(E)$; thus $C^r(TM)$ and $C^r(T^*M)$ denote the C^r vector fields and the C^r 1-forms over M respectively. The C^r functions $M \longrightarrow R$ will be written $C^r(M)$ instead of $C^r(M \times R)$.

If (U,φ) is a chart at x, giving a local co-ordinate system $(x^1,\ldots,x^n)$, and if $f \in C^1(M)$, we write $D_i f(x)$ or $D_i|_x f$ for $\frac{\partial f}{\partial x^i}(x)$. Thus $D_i|_x$ is a tangent vector at x. On U, the total differential of f is written :- $df = D_i f \, dx^i$, with the summation convention for repeated indices.

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Suppose $o \leq r \leq k-2$, and consider a C^r linear connection Γ on the cotangent bundle of M ; we regard Γ as an $\mathbb{R}$-linear operator $\nabla_X : C^{r+1}(T^*M) \longrightarrow C^r(T^*M)$ for each $X \in C^{r+1}(TM)$, such that

(a) $\nabla_{fX}\theta = f\nabla_X\theta$

(b) $\nabla_X f\theta = f\nabla_X\theta + (Xf)\theta$

for $f \in C^{r+1}(M)$, $\theta \in C^{r+1}(T^*M)$

In the chart (U,φ), the Christoffel symbols $\Gamma^i_{jk}(\cdot)$ are C^r functions from U to $\mathbb{R}$, defined by :-

$$\Gamma_{jk}^{\,i}\, dx^k = - \nabla_{D_j} dx^i$$

For $f \in C^{r+2}(M)$, the second covariant derivative of f, written ∇df, can be expressed in local co-ordinates as an nxn matrix with $(j,k)^{th}$ entry

(1) $(\nabla df)_{jk} = \langle\nabla_{D_j} df, D_k\rangle = D_{jk}f - \Gamma_{jk}^{i}D_i f$

where $D_{jk} = D_jD_k$ and $\langle\ ,\ \rangle$ represents the duality between TM and T^*M.

When M is Riemannian with metric g, the Laplacian Δf of $f \in C^{r+2}(M)$ is defined to be the trace of ∇df; thus if (g^{ij}) is the inverse of the metric tensor in local co-ordinates, then

$$\Delta f(x) = g^{ij}(x)(\nabla df(x))_{ij}, \quad x \in U$$

If $V \subset M$, we say that f is harmonic (resp. subharmonic) on V if $\Delta f(x) = o$ (resp. $\geq o$) for $x \in V$

The connection Γ on the cotangent bundle induces a connection on the tangent bundle; thus for each $X \in C^{r+1}(TM)$, an $\mathbb{R}$-linear operator

$\nabla_X : C^{r+1}(TM) \longrightarrow C^r(TM)$ is defined by :-

$\langle\theta,\nabla_X Y\rangle + \langle Y,\nabla_X\theta\rangle = X\langle\theta,Y\rangle$ for $\theta \in C^{r+1}(T^*M)$,

and $X,Y \in C^{r+1}(TM)$

Suppose $\gamma : (-\alpha,\beta) \longrightarrow M$ is a C^{r+2} curve in M, for some positive numbers α and β. We say that γ is a geodesic (with respect to Γ) if $\nabla_{\dot\gamma(t)}\dot\gamma(t) = o$, $-\alpha < t < \beta$. Suppose V is open in M and $f \in C^{r+2}(V)$. Such an f is called Γ-convex on V if for all geodesics $\gamma : (-\alpha,\beta) \longrightarrow V$, the composite map

$f \circ \gamma : (-\alpha,\beta) \longrightarrow \mathbb{R}$ has non-negative second derivative (well-defined since $r \geq o$). This definition is consistent with the definition of C^2 convex function when $M = \mathbb{R}^n$ with the Euclidean connection $\Gamma^i_{jk} \equiv 0$, in which case geodesics are just line segments Using the geodesic equation, as found in Kobayashi and Nomizu [10, p. 140], it is easy to check that f is Γ-convex on V if and only if the matrix $(\nabla df)_{jk}$ defined above is positive semi-definite on V, with respect to any local co-ordinate system. From this one easily deduces that the C^{r+2} Γ-convex functions on V are a subset of the C^{r+2} subharmonic functions on V, in the case where M is Riemannian and Γ is the Riemannian connection.

Let M and N be C^k manifolds with connections ${}^M\Gamma$ and ${}^N\Gamma$ respectively, and let $\varphi: M \longrightarrow N$ be a C^k map. To say that "φ pulls back local ${}^N\Gamma$-convex functions to local ${}^M\Gamma$-convex functions" means that for all open sets $V \subset N$ and all ${}^N\Gamma$-convex functions $f : V \longrightarrow \mathbb{R}$ (always of class C^{r+2}), the map $f \circ \varphi$ is ${}^M\Gamma$-convex from $\varphi^{-1}(V)$ to $\mathbb{R}$. The following table gives functorial characterizations of three classes of maps φ, following Ishihara [8]. The abbreviation "Riem" means that M is a Riemannian manifold. The "stochastic characterizations" will be explained later.

In the definition which follows, the restriction that M be finite-dimensional is lifted. The definitions of connection and Γ-convex function carry over to the setting of Banach manifolds. For a full account of connections on vector bundles over Banach manifolds, the reader is referred to Eliasson [5].

3. Definition of a Γ-martingale on M

Let M be a Banach manifold of class C^k, $k \geq 2$, and $(\Omega,F,(F_t)_{t\geq 0},P)$ a filtered probability space satisfying the usual conditions. Let $X = (X_t,F_t)$ be a process taking values in M up to its "lifetime" $\zeta = \zeta(\omega), \omega \in \Omega$.

Henceforward *all processes on* M *will have a.s. continuous trajectories up to their lifetime*: that is, the map $[0,\zeta(\omega)) \longrightarrow M$, $t \mapsto X(t,\omega)$ is continuous for almost all $\omega \in \Omega$. For simplicity, we shall avoid mention of the lifetime in the sequel.

Type of map φ	We require: on M	 on N	Functional characterization: φ pulls back local ... functions	 to local ... functions	Stochastic characterization The image under φ of ...	 is a ...
affine map	${}^{M}\Gamma$	${}^{N}\Gamma$	${}^{N}\Gamma$-convex	${}^{M}\Gamma$-convex	every ${}^{M}\Gamma$-martingale	${}^{N}\Gamma$-martingale
harmonic map	Riem.	${}^{N}\Gamma$	${}^{N}\Gamma$-convex	subharmonic	Brownian motion started anywhere	${}^{N}\Gamma$-martingale
harmonic morphism	Riem.	Riem.	harmonic	harmonic	Brownian motion started anywhere	Time-changed Brownian motion

Note that the affine maps and the harmonic morphisms are both subsets of the harmonic maps.

Following L. Schwartz [13, p. 6] we say that X is a *semimartingale on* M if for all $f \in C^2(M)$, $f \circ X$ is a real-valued semimartingale.

Suppose now that M has a connection Γ of class C^0. A Γ-*martingale tester*, or simply tester, (U,V,W,f) will consist of:

- open sets U,V,W in M with $\overline{U} \subset V \subset \overline{V} \subset W$,
- a C^2 Γ-convex function $f: W \to \mathbb{R}$.

Associated to each tester and each process X is the collection of stopping-times σ_i, τ_i, $i \geq 0$, defined by:

$$\sigma_0 = 0, \; \tau_0 = 0$$

$$\sigma_i = \inf \{t > \tau_{i-1} : X_t \in U\}, \; i \geq 1$$

$$\tau_i = \inf \{t > \sigma_i : X_t \notin \overline{V}\}, \; i \geq 1.$$

We define the previsible set F to be

$$F = \bigcup_{i=1}^{\infty} (\sigma_i, \tau_i]$$

DEFINITION

Let $X = (X_t, F_t)$ be a continuous semimartingale on a manifold M with connection Γ. X *is said to be a* Γ-*martingale if for all testers* (U,V,W,f), *the process* $Y = (Y_t, F_t)$

$$Y_t = \int_0^t 1_F(s) d(f \circ X_s)$$

is a local submartingale.

Remarks

(i) A natural question is: are there enough Γ-convex functions to make the definition meaningful? Given any $x \in M$ we can take a normal co-ordinate system in a neighbourhood of x; then within a smaller neighbourhood it is possible to take a local co-ordinate system consisting of Γ-convex functions which are quadratic functions of the normal co-ordinates - see Ishihara [8, p. 219].

(ii) Strictly speaking, what we have defined is a continuous *local* Γ-martingale.

Footnote: This definition of Γ-martingale arose from a suggestion of J.Eells and K.D.Elworthy in 1980, and is equivalent to Bismut's definition given in [11,p.54].

(iii) The random variable Y_t is always well-defined; for given a C^2 convex function $f:W \to \mathbb{R}$, there exists a C^2 function $f_1:M \to \mathbb{R}$ such that $f_1|_{\overline{V}} = f|_{\overline{V}}$. Hence $f_1 \circ X$ is a real-valued semimartingale, and

$$Y_t = \int_o^t 1_F(s)d(f_1 \circ X_s)$$

which is well-defined because 1_F is a bounded previsible process.

(iv) An equivalent definition would result if one took F to be $\{(s,\omega):X_s(\omega) \in V\}$, which is also previsible, since X is continuous. The present definition, however, is more explicit.

4. Examples

Since the Ito formula for continuous semimartingales in $\mathbb{R}^n$ will be used repeatedly, we state it here for reference.

Ito formula.

Let $X_t = (X_t^1,\ldots,X_t^n)$ be a continuous semimartingale in $\mathbb{R}^n$, and $f:\mathbb{R}^n \to R$ a C^2 function.

Then

$$f(X_t) = f(X_o) + \int_o^t D_i f(X_s)dX_s^i + \frac{1}{2}\int_o^t D_{ij}f(X_s)d < X^i, X^j >_s$$

(I) Euclidean local martingales

Let $M = \mathbb{R}^n$ with the Euclidean connection, Γ. Let $X_t = (X_t^1,\ldots,X_t^n)$ be a continuous semimartingale. If X is a local martingale in the usual sense and (U,V,W,f) is a tester, then

$$\int_o^t 1_F(s)d(f \circ X_s) = \int_o^t 1_F(s)D_i f(X_s)dX_s^i$$

$$+ \frac{1}{2}\int_o^t 1_F(s)D_{ij}f(X_s)d <X^i, X^j>_s.$$

The first integral on the right is a local martingale, and the second is an increasing process since $D_{ij}f(\cdot)$ is positive semidefinite on W. So the left side is a local submartingale, verifying that X is a Γ-martingale.

Conversely by taking $U = V = W = \mathbb{R}^n$, $f(x) = x$ and $f(x) = -x$, we can verify that a Γ-martingale is a local martingale in the usual sense.

(II) Brownian motion

Suppose M is Riemannian with metric g and Riemannian connection Γ. Brownian motion is characterized as the diffusion $B = (B_t, F_t)$ with generator $\frac{1}{2}\Delta$; in other words, for all $f \in C^2(M)$, the process C^f , where

$$C_t^f = f(B_t) - f(B_o) - \frac{1}{2}\int_o^t 1_F(s)\ \Delta(f\cdot\varphi)(B_s)dS$$

is a local martingale. If (U,V,W,f) is a tester, we integrate 1_F against C^f and rewrite the equation as

$$(1)\quad \int_o^t 1_F(s)d(foB_s) = \int_o^t 1_F(s)dC_s^f + \frac{1}{2}\int_o^t \Delta f(B_s)ds$$

We saw in § 2 that f Γ-convex implies f subharmonic, so $\Delta f(B_s) > o$. Thus the right side is a local martingale plus an increasing process. This verifies that $\int 1_F d(foB)$ is a local submartingale. Hence Brownian motion is a Γ-martingale with respect to the Riemannian connection

(III) Processes on geodesics

Let $I = (-\infty,\infty)$ or $(-\alpha,\beta)$, some $\alpha,\beta > o$.

Let $\gamma: I \to M$ be a geodesic with respect to a connection Γ on M. Let $L = (L_t, F_t)$ be a continuous local martingale with values in I. Then the M-valued process $X_t = \gamma \circ L_t$ is a Γ-martingale. This follows because γ is an affine map - see section § 9.

The result also holds when γ is a closed geodesic, for example a great circle on a sphere.

(IV) Image of Brownian motion under a harmonic map

Let M be Riemannian with Riemannian connection ${}^M\Gamma$, and let N be a manifold with a connection ${}^N\Gamma$. The term 'harmonic map' is defined in the table in section § 2; for more information and examples, see the survey by Eells and Lemaire [4]. The following result is due to Meyer [12 , p. 265].

PROPOSITION A

A C^2 *map* $\phi: M \to N$ *is harmonic if and only if for all* $a \in M$, *the process* $\phi \circ B$ *is a* ${}^N\Gamma$*-martingale, where* B *is the Brownian motion on* M *started at* a.

Many proofs are possible, according to the various characterizations of ${}^N\Gamma$-martingale and harmonic map; a new proof is given in section § 7 .

(V) A class of diffusions on M

Suppose L is an elliptic operator on M of the form

$$L = b^i\frac{\partial}{\partial x^i} + \frac{1}{2}\, a^{jk}\frac{\partial^2}{\partial x^j \partial x^k},\ b^i = -\frac{1}{2}\, a^{jk}\Gamma^{\ i}_{jk},\ (a^{jk}) \text{ symmetric, } > 0$$

This expression is intrinsic, and under regularity assumptions the diffusion associated to L is a Γ-martingale. This subject is treated fully in the author's forthcoming paper [3], where a Girsanov theorem is also given.

(VI) An example on the Lie Group Gl(n)

Let M be $Gl(n)$, the Lie group of non-singular nxn matrices, with the canonical left-invariant connection. It will follow from Theorem B of § 5 that if $w = (w_t, F_t)$ is one-dimensional Brownian motion, then

$$X_t = Be^{Aw_t}\ ,\ B \in Gl(n),\ A \in gl(n)$$

defines a Γ-martingale in $Gl(n)$.

(VII) A diffusion on the torus T^2

Let T^2 be the torus imbedded in $\mathbb{R}^3$, with major and minor radii r and a whose ratio $\mu = r/a$ satisfies $1 < \mu \leq \sqrt{2}$. We give T^2 the imbedded Riemannian metric and the Riemannian connection Γ. Parametrize T^2 by angles (θ,φ), where θ moves around the big circle and φ around the little circle. Let E denote the half of the torus that can be seen by a person at the centre, namely $\{(\theta,\varphi): -\pi/2 < \varphi < \pi/2\}$.

Define $\ell: S^1 \longrightarrow [0,1]$ by :-

$$\ell(\varphi) = \begin{cases} 2\cos\varphi\,(\mu - \cos\varphi) & \text{if } -\pi/2 < \varphi < \pi/2 \\ 0 & \text{otherwise} \end{cases}$$

Let $((W_t,W'_t),F_t)$ be two-dimensional Brownian motion. For any $(\theta_o,\varphi_o) \in E$, the following stochastic differential equation on T^2 has a unique solution $H_t = (\theta_t,\varphi_t)$, since the coefficients are bounded and Lipschitz :-

$$H_o = (\theta_o,\varphi_o)$$

$$d\theta = \ell(\varphi)^{1/2}dW + (1-\ell(\varphi))^{1/2}dW' - (\mu-\cos\varphi)^{-1}\ell(\varphi)\sin\varphi\, dt$$

$$d\varphi = \ell(\varphi)^{1/2}dW + (\mu-\cos\varphi)\sin\varphi\, dt$$

Let τ be the first exit time of H from the region E . Then the process H stopped at τ has the following remarkable property :- it is both a Γ-martingale on T^2 and is mapped into an $\mathbb{R}^3$ local martingale by the inclusion map $i : T^2 \hookrightarrow \mathbb{R}^3$. This example is studied in detail in a forthcoming paper.

(VIII) A diffusion on a non-compact surface of revolution

This example will be constructed in § 6 .

5. A class of Γ-martingales on the Lie group $G\ell(n)$

A connected Lie group G admits a canonical invariant connection, which is described in Kobayashi and Nomizu [10, Vol. II, p. 192]. When G is the group $G\ell(n)$ of all non-singular $n \times n$ matrices over $\mathbb{R}$, the connection Γ may be characterized thus:
the geodesics through an element B of $G\ell(n)$ are all maps $\gamma : (-\alpha,\beta) \longrightarrow M$, some $\alpha,\beta > o$, of the form $t \longrightarrow B\exp(At)$ for some A in the Lie algebra $g\ell(n)$ (= all $n \times n$ matrices over R); exp is here the usual exponential function defined by a power series. When W is open in $G\ell(n)$, this gives a convenient expression for the C^2 Γ-convex functions on W:
f is Γ-convex on W if and only if :-

$$\frac{d^2}{dt^2} f\,(B\exp(At))\Big|_{t=o} \geq o, \text{ for all } B \in W,\ A \in g\ell(n).$$

It is convenient in calculations to regard W as a subset of the vector space of <u>all</u> $n \times n$ matrices over R. Thus Df and D^2f are well-defined, and the Γ-convexity condition may be differentiated to give :-

$$(1)\quad Df(B)(BA^2) + D^2f(B)(BA,BA) \geq o \quad \text{for all } B \in W,\ A \in g\ell(n)$$

We now present a method of constructing Γ-martingales from certain martingales in the Lie algebra. Recall first that two local square integrable martingales on the same filtration are said to be <u>orthogonal</u> if their product is a local martingale.

THEOREM B

Let $A_1,\ldots,A_p$ be commuting elements of $g\ell(n)$, i.e. $A_iA_j = A_jA_i$ for all i,j. Suppose $\beta^i = (\beta^i(t),F_t)$ is a real-valued continuous local square-integrable martingale for $i \in \{1,2,\ldots,p\}$, and the family $\{\beta^i : 1 \leq i \leq p\}$ is mutually orthogonal. For any $B \in G\ell(n)$, the process

$$X_t = B\exp(\Sigma_{i=1}^{p} A_i\,\beta^i(t))$$

is a Γ-martingale on $G\ell(n)$.

First we recall a well-known fact about the exponential function :-

LEMMA

If $A_1,\ldots,A_p$ are commuting nxn matrices, then for all $(x_1,\ldots,x_p)$ in $\mathbb{R}^p$, $\exp(\Sigma_{i=1}^p A_i x_i) = \Pi_{i=1}^p \exp(A_i x_i)$

Proof of Theorem B

To simplify notations, take $p = 2$; the method is quite general. Thus let A and G be commuting nxn matrices and let α and γ be orthogonal real-valued continuous local square-integrable martingales. Define a C^∞ function $h : \mathbb{R}^2 \longrightarrow G\ell(n)$ by :-

$h(x,y) = \exp(Ax+Gy)$. Using the Lemma, and regarding h as a map into $g\ell(n)$

$Dh(x,y)(u,v) = h(x,y)(Au+Gv)$,

$$D^2h(x,y)((u,v),(w,z)) = h(x,y)\left((u,v)\begin{pmatrix}A^2 & AG\\ AG & G^2\end{pmatrix}\begin{pmatrix}w\\ z\end{pmatrix}\right)$$

where all the expressions on the right commute. By Ito's formula,

$$(2)\quad h(\alpha_t,\gamma_t) - h(\alpha_o,\gamma_o) = \int_o^t h(\alpha_s,\gamma_s)(Ad\alpha_s+Gd\gamma_s) + \frac{1}{2}\int_o^t h(\alpha_s,\gamma_s)[A^2d\langle\alpha\rangle_s + G^2d\langle\gamma\rangle_s]$$

where $\langle\alpha\rangle_s = \langle\alpha,\alpha\rangle_s$; the AG coefficient does not appear because $\langle\alpha,\gamma\rangle_s = o$ by assumption.

Now suppose $B \in G\ell(n)$ and X is the process $X_t = B\exp(A\alpha_t+G\gamma_t)$. in $G\ell(n)$. Suppose (U,V,W,f) is a tester. Regarding X_t as $g\ell(n)$-valued, we may write formally

$$d(f\circ X_t) = Df(X_t)dX_t + \frac{1}{2}D^2f(X_t)(dX \otimes dX)_t$$

Since $dX_t = Bd(h(\alpha_t,\gamma_t))$, (2) gives:-

$$= Df(X_t)B\,h(\alpha_t,\gamma_t)\{(Ad\alpha_t + Gd\gamma_t) + \frac{1}{2}[A^2d\langle\alpha\rangle_t + G^2d\langle\gamma\rangle_t]\} + \frac{1}{2}D^2f(X_t)(Bdh(\alpha,\gamma)\otimes Bdh(\alpha,\gamma))_t$$

Since α and γ are local martingales, the $d\alpha$ and $d\gamma$ differentials are both local martingale differentials, and the bounded variation part of the last expression is :-

$\frac{1}{2}\{Df(X_t)(X_tA^2) + D^2f(X_t)(X_tA,X_tA)\}d\langle\alpha\rangle_t$

$\frac{1}{2}\{Df(X_t)(X_tG^2) + D^2F(X_t)(X_t)(X_tG,X_tG)\}d\langle\gamma\rangle_t$

By formula (1) for Γ-convexity this can be written as

$$K_t\, d\langle\alpha\rangle_t + L_t\, d\langle\gamma\rangle_t, \text{ where } K_t \geq o,\ L_t \geq o.$$

Since $\langle\alpha\rangle$ and $\langle\gamma\rangle$ are increasing processes, this proves that $d(f\cdot X_t)$ is a local submartingale differential. Hence X is a Γ-martingale.

□

Example If $w = (w_t, F_t)$ is one-dimensional Brownian motion and $A \in g\ell(n)$, then $X_t = \exp(Aw_t)$ is a Γ-martingale on $Gl(n)$.

Acknowledgment The author thanks Professor K. R. Parthasarathy for help in finding this example.

6. An example of a Γ-martingale on a surface of revolution whose local co-ordinate processes are martingales

The definition of Γ-martingale given by Meyer in [12 , p. 54] is that for each local co-ordinate system (x^i) with domain W, the local co-ordinate processes X^i satisfy :-

(1) $dA_t^i + \frac{1}{2}\Gamma_{jk}^i(X_t)d\langle X^j,X^k\rangle_t = o$ on $\{X \in W\}$

where $X_t^i = M_t^i + A_t^i$ is the decomposition of X^i into local martingale and finite variation parts. To see that this condition is sufficient, write

$$d(f(X_t)) = D_i f(X_t)dX_t^i + {}^1/2\, D_{ij}\, f(X_t)d\langle X^i,X^j\rangle_t$$
$$= D_if(X_t)dM_t^i + D_if(X_t)(dA_t^i + {}^1/2\, \Gamma_{jh}^i(X_t)d\langle X^j,X^h\rangle_t)$$
$$+ {}^1/2(\nabla df(X_t))_{ij}\, d\langle X^i,X^j\rangle_t$$

by formula (1) of § 2. When (1) holds and (U,V,W,f) is a tester, then this reduces to $d(f(x_t) = d$ (local martingale + increasing process) which verifies that $\int 1_F d(f\cdot X)$ is a local submartingale.

To prove that (1) is necessary is harder; see Darling [2].

To demonstrate the utility of (1), we give a computational example.

Let $r : \mathbb{R} \longrightarrow (0,\infty)$ be a C^2 convex function, i.e. with non-negative second derivative. Let M be the surface of revolution in $\mathbb{R}^3$ defined by

$$M = \{(x,y,z) : x^2 + y^2 = r(z)^2\}$$

and let g be the embedded metric on M and Γ the Riemannian connection. Take co-ordinates (z,θ) on M, so that $x = (z)\cos\theta$, $y = r(z)\sin\theta$. If we denote $\frac{dr}{dz}$ by $\dot{r}$ and $\frac{d^2r}{dz^2}$ by $\ddot{r}$, g can be expressed as the matrix

$$g = \begin{pmatrix} 1+\dot{r}^2 & 0 \\ 0 & r^2 \end{pmatrix}$$

The Christoffel symbols can be computed from the usual formula in Kobayashi and Nomizu [10, p. 160], and one finds that

$$\Gamma_{11}^{1} = \dot{r}\ddot{r}/(1+\dot{r}^2),\ \Gamma_{22}^{1} = -r\dot{r}/(1+\dot{r}^2),\ \Gamma_{12}^{2} = \dot{r}/r = \Gamma_{21}^{2}$$

and all other Γ_{jk}^{i} are zero.

Let $(X_t) = (z_t,\theta_t)$ be a stochastic process on M. Decompose the semimartingales (z_t) and (θ_t) into their local martingale and finite variation parts: $z_t = M_t + A_t$, $\theta_t = N_t + C_t$. The condition for X to be a Γ-martingale can be written :-

$$dA + \tfrac{1}{2}(\Gamma_{11}^{1}\, d\langle M\rangle + \Gamma_{22}^{1}\, d\langle N\rangle) = 0$$

$$dC + \tfrac{1}{2}(\Gamma_{12}^{2}\, d\langle M,N\rangle + \Gamma_{21}^{2}\, d\langle N,M\rangle) = 0$$

in other words,

$$(1)\quad \begin{cases} dA + \frac{1}{2}(1+\dot{r}^2)^{-1}(\dot{r}\ddot{r}d\langle M\rangle - r\dot{r}d\langle N\rangle) = 0 \\ dC + (\dot{r}/r)\, d\langle M,N\rangle = 0 \end{cases}$$

An interesting example may be constructed as follows. Let (W_t,W_t') be two-dimensional Brownian motion and suppose

$$z_t = M_t = \int_0^t r(z_s)^{1/2}dW_s,\quad \theta_t = N_t = \int_0^t \ddot{r}(z_s)^{1/2}dW_s'$$

$$A_t = C_t = 0$$

Notice that the second definition is valid because $\ddot{r} \geq 0$ by our initial assumption. It is easy to see from (1) that $X_t = (z_t,\theta_t)$ <u>is a Γ-martingale for which both the local co-ordinate processes are real-valued martingales</u>. (This property does <u>not</u> hold in general).

7. Proof of Proposition A: - harmonic maps send Brownian motion to Γ-martingales

The proof gives an example of how the definition of Γ-martingale is used.

Proof. '$\Longrightarrow$'

Say ϕ is harmonic, and (U,V,W,f) is a ${}^N\Gamma$-martingale tester. Then $f \circ \phi : \phi^{-1}(W) \to \mathbb{R}$ is subharmonic, so by formula (1) of § 4,

$$\int_0^t 1_F d(f \circ \phi \circ B_s) = \int_0^t 1_F(s) dC_s^{f\circ\varphi} + \frac{1}{2} \int_0^t 1_F(s)\ \Delta f\ (B_s) ds$$

which is a local martingale plus an increasing process.

'$\Longleftarrow$' Let W be open in N and $f:W \to \mathbb{R}$ be ${}^N\Gamma$-convex, and let V be open in N with $\bar{V} \subset W$. Let U be a relatively compact connected subset of $\phi^{-1}(V)$ in M. It suffices to prove that $f \circ \phi : U \to \mathbb{R}$ is subharmonic (note that it is bounded, by compactness of $\bar{U}$).

Fix $a \in U$, and let $B = (B_t)$ be Brownian motion on M with $B_0 = a$. Let τ be the first exit time of (B_t) from U. Fix $\varepsilon > 0$. Define

$$E = \{(s,\omega) : s < \tau(\omega),\ \Delta^M(f \circ \phi)(B_s) < -\varepsilon\}.$$

We know that $\phi \circ B$ is a ${}^N\Gamma$-martingale on N, so

$$Z_t = f \circ \phi \circ B_{t\wedge\tau}$$

is a local submartingale, and indeed a submartingale since $f \circ \varphi$ is bounded on U.
However by the definition of Brownian motion on M,

$$H_t^{f \circ \phi} := f \circ \phi(B_{t\wedge\tau}) - f \circ \phi(a) - \frac{1}{2}\int_0^{t\wedge\tau} \Delta^M(f \circ \phi)(B_s) ds$$

is a local martingale, and indeed a martingale by boundedness.
Hence

$$\int_0^t 1_E\ dH_s^{f \circ \phi} = \int_0^t 1_E\ dZ_s - \frac{1}{2}\int_0^t 1_E\ \Delta^M(f \circ \phi)(B_s) ds$$

$$\geq \int_0^t 1_E\ dZ_s + \frac{\varepsilon}{2}\int_0^t 1_E\ ds.$$

If m_t is Lebesgue measure on the time interval $[o,t]$, then

$$m_t \otimes P(E) = \int_\Omega \left(\int_o^t 1_E \, ds\right) dP \le \frac{2}{\varepsilon} \mathbb{E} \int_o^t 1_E (dH_s^{f \circ \phi} - dZ_s) \le 0$$

since the stochastic integral on the right side is a supermartingale with starting value zero. Hence $m_t \otimes P(E) = 0$, and this holds for every $\varepsilon > 0$; so the $m_t \otimes P$ -measure of the set

$$\{(s,\omega) : s < \tau(\omega), \Delta^M (f \circ \phi)(B_s) < 0\}$$

is zero, for every t. Hence by continuity of $s \mapsto \Delta^M (f\circ\varphi)(B_s)$,

$$P(\{\Delta^M (f\circ\varphi)(B_s) < o, \text{ some } o \le s < \tau\}) = o$$

Since U is connected, the Brownian motion hits every non-empty open subset of U with positive probability. Moreover $\Delta^M(f \circ \varphi)$ is continuous on U. Hence $\Delta^M(f\circ\varphi)(x) \ge o$ for all $x \in U$.

□

The Proposition has a wide range of uses of which Theorem B (harmonic morphisms) and the following surprising result are examples.

COROLLARY

Let M *be any connected, non-compact Riemannian manifold of dimension* n, *with Riemannian connection* Γ. *There is a proper embedding* $i : M \hookrightarrow \mathbb{R}^{2n+1}$ *such that* $i \circ B$ *is a local martingale in* $\mathbb{R}^{2n+1}$, *where* B *is Brownian motion on* M.

Proof

The result follows from the last proposition, and a theorem of Greene and Wu, [7 , p. 231], which says that such an M has a proper embedding by harmonic functions in $\mathbb{R}^{2n+1}$; (we use here the result obtained in example (1), namely that Γ-martingales in a Euclidean space are the same as local martingales).

□

For other uses of stochastic methods in harmonic maps see Kendall [9].

8. Harmonic morphisms and Brownian motion

Let M and N be Riemannian manifolds of dimensions m and n respectively. Recall from § 2 that a C^2 map $\phi: M \to N$ is said to be a *harmonic morphism* if for each V open in N and each harmonic function $f: V \to \mathbb{R}$, the function $f \circ \phi : \phi^{-1}(V) \to R$ is harmonic.

The following characterization of harmonic morphisms is due to Fuglede, [6 , p. 116].

LEMMA

A C^2 mapping $\phi: M \to N$ is a harmonic morphism if and only if there exists a function $\lambda \geq 0$ on M (necessarily unique and such that λ^2 is C^0) with the property that

$$\Delta^M (f \circ \phi) = \lambda^2 \, [(\Delta^N f) \circ \phi]$$

for all C^2 functions $f: N \to R$.

This enables us to characterize harmonic morphisms as those C^2 maps which preserve the paths of Brownian motion, as proved below. This result is related to those of Bernard, Campbell and Davie [1].

THEOREM C.

A C^2 mapping $\varphi: M \to N$ of Riemannian manifolds is a harmonic morphism if and only if (*) holds for all $a \in M$:

(*) *Let $B = (B_t)$ be Brownian motion on M with $B_o = a$. There exists a continuous increasing process $A = (A_t)$ and a Brownian motion $\tilde{B}$ on N such that*

$$\tilde{B} \circ A = \phi \circ B \, .$$

Remark

If $m \leq n$, then a harmonic morphism ϕ is necessarily constant, and so $A_t = 0$ a.s.

Proof. '$\Longrightarrow$'

Fix $a \in M$. Let $B = (B_t)$ be Brownian motion on M with $B_o = a$. Let λ be as in the lemma. Define a continuous increasing process $A = (A_t)$ by:

$$A_t = \int_o^t \lambda^2 (B_s) ds$$

and define its inverse by

$$C_t = \inf \{u : A_u > u\}.$$

Denote $\phi \circ B$ by $Y = (Y_t)$ on N. We shall prove that $Y \circ C$ is a Brownian motion $\tilde{B}$ on N. Then $\tilde{B} \circ A = Y$ as desired.

It suffices to show that for all C^2 functions $f : N \to R$,

$$L_t^f := f(Y \circ C_t) - f(Y_o) - \frac{1}{2} \int^t \Delta^N f(Y \circ C_s) ds$$

is a local martingale. However under the change of variables $s \mapsto A_u$.

$$\int_o^t \Delta^N f(Y \circ C_s) ds = \int_o^{C_t} \Delta^N f(Y_u) dA_u = \int_o^{C_t} \lambda^2(B_u)\, \Delta^N f(Y_u) du$$

$$= \int_o^{C_t} \Delta^M (f \circ \phi)(B_u) du$$

by the Lemma. By definition of B,

$$H_t^f := f \circ \phi(B_t) - f \circ \phi(a) - \frac{1}{2} \int_o^t \Delta^M (f \circ \phi)(B_s) ds$$

is a (continuous) local martingale; hence so is $(H^f \circ C_t)$; but

$$L_t^f = H^f \circ C_t$$

so the result follows.

'<=='

Let V be open in N and let $f : V \to \mathbb{R}$ be harmonic. Fix $a \in M$ with $\phi(a) \in V$, and assume (*) holds. Let τ be the first exit time of B from $\phi^{-1}(V)$; then $(f \circ \tilde{B}_{t \wedge \tau})$ is a (continuous) local martingale, and so is $(f \circ \tilde{B} \circ A_{t \wedge \tau}) = (f \circ \phi \circ B_{t \wedge \tau})$. Consider $f \circ \phi$ as a C^2 map : $f^{-1}(V) \to R$. It sends Brownian motion stopped at τ to a local martingale, and hence is a harmonic map by Proposition A of §4 . This proves that $f \circ \phi$ is a local harmonic function on M whenever f is a local harmonic function on N. So ϕ is a harmonic morphism.

□

9. Maps preserving the martingale property

If $\phi : V \to W$ is a map between vector spaces, local martingales on V are sent to local martingales on W if and only if ϕ is linear.

If (M,Γ) and (N,Γ') are manifolds with connections, a C^2 map $\phi : M \to N$ is said to be *affine* (or *totally geodesic*) if its derivative $T\phi$ sends parallel vector fields to parallel vector fields. Affine maps are discussed in Kobayashi and Nomizu [10 Ch. VI]. Affine maps are harmonic, and when M has dimension 1, the two are identical; see Eells and Lemaire [4 , p. 9]. Ishihara points out [8 , p. 220] that affine maps are those which pull back local Γ-convex functions to local Γ-convex functions, in the sense of § 2 .

THEOREM D.

A C^2 *map* $\phi : M \to N$ *is affine if and only if* ϕ *sends* Γ*-martingales to* Γ'*-martingales.*

Proof.

'$\Longrightarrow$' Let (U,V,W,f) be a Γ'-martingale tester on N . Since ϕ sends geodesics to geodesics, the definition of Γ-convex in section § 2 shows that $(\phi^{-1}U, \phi^{-1}V, \phi^{-1}W, f \circ \phi)$ is a Γ-martingale tester on M . The result is immediate from the definition of Γ-martingale.

'$\Longleftarrow$'. It suffices to show that if $\alpha, \beta > 0$ and $\gamma : (-\alpha,\beta) \to M$ is a Γ-geodesic, then $\phi \circ \gamma : (-\alpha,\beta) \hookrightarrow N$ is a Γ'-geodesic. Let $W = (W_t, F_t)$ be one-dimensional Brownian motion with $W_0 = 0$, stopped at the first t at which $|W_t| = \min(\alpha,\beta)$. By § 4, (III) Proposition A , $\gamma \circ W$ is a Γ-martingale, so $\phi \circ \gamma \circ W$ is a Γ'-martingale. Applying the Proposition again, $\phi \circ \gamma$ must be harmonic, hence affine since $(-\alpha,\beta)$ has dimension 1. So $\phi \circ \gamma$ is a Γ'-geodesic.

□

Immersed submanifolds

Suppose $i : M \hookrightarrow (N,g)$ is a Riemannian immersion, and Γ and Γ' are the induced Riemannian connections on M and N . The condition for all M-valued Γ-martingales to be Γ'-martingales is that M be a totally geodesic submanifold of N , by the preceding

Theorem. What are the possibilities when i is <u>not</u> totally geodesic? We take as examples the sphere and the torus embedded in Euclidean space.

Examples

(a) Let M be the sphere S^{n-1} embedded in R^n, with the induced metric and Riemannian connection Γ. Let $a \in S^{n-1}$ have co-ordinates $(a_1,\ldots,a_n)$. Suppose X is a Γ-martingale on S^{n-1} with $X_0 = a$. Then X cannot be a local martingale in $\mathbb{R}^n$, unless it is constant a.s. For if so, then $Y_t = a_1X^1(t),+\ldots+a_nX^n(t)$ would be a local martingale, and in fact a martingale (since X is bounded), with $Y_0 = 1$. Hence $E\,Y_t = 1$ for all t. But for any $x = (x^1,\ldots,x^n) \in S^{n-1}$ with $x \neq a$, $a_1x^1+\ldots+a_nx^n < 1$. So $X_t = a$ a.s.

(b) Consider the torus example of § 4 (VII); the embedding $i: T^2 \hookrightarrow R^3$ is <u>not</u> totally geodesic, and yet we constructed a Γ-martingale on T^2 which was also an R^3-martingale.

It seems likely that the existence of such examples for more general embeddings $i: M \hookrightarrow \mathbb{R}^n$ will depend upon the sectional curvatures of M.

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