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A SIMPLE REMARK ON THE CONDITIONED SQUARE FUNCTIONS FOR MARTINGALE TRANSFORMS

by N.Kazamaki

1. Let $X=(X_n,F_n)$ be a fixed uniformly integrable martingale defined on a probability space (Ω,F,P), and denote its difference sequence by $x=(x_n)$, $x_n=X_n-X_{n-1}, n\geq 1, X_o=0$. If $f=(f_n), f_n=\sum_{k=1}^{n} v_k x_k$, is a martingale transform of x, then the conditioned square function of f is $s(f)=\left\{\sum_{k=1}^{\infty} v_k^2\, E[x_k^2|F_{k-1}]\right\}^{1/2}$. Denote by $\underline{\underline{M}}$ the collection of all martingale transforms f of x. Let now $T_o=0\leq T_1\leq T_2\leq \dots$ be F_n-stopping times. Then (X_{T_n}) is a martingale over (F_{T_n}).because X is uniformly integrable. For simplicity we put $G_n=F_{T_n}$. Since for each $k\geq 1$ $v_{T_{k-1}+1}$ is G_{k-1}-mesurable, $\hat{f}_n=\sum_{k=1}^{n} v_{T_{k-1}+1}(X_{T_k}-X_{T_{k-1}})$ defines a new martingale transform. It should be noted that $\hat{f}=f$ if $T_k=k$, and that $\hat{f}=f^S$ if $T_k=k\wedge S$ for some stopping time S. Here f^S is the martingale transform stopped at S. Now we let

$$U(f)=\left\{\sum_{k=1}^{\infty} v_{T_{k-1}+1}^2\, E[(X_{T_k}-X_{T_{k-1}})^2|G_{k-1}]\right\}^{1/2}$$

for $f_n=\sum_{k=1}^{n} v_k x_k$ in $\underline{\underline{M}}$. This is none other than the conditioned square function $s(\hat{f})$. It follows at once that U is a symmetric and quasi-linear operator on $\underline{\underline{M}}$. It seems to be interesting to investigate this operator, but to the best of our knowledge no papers on the subject have been published. In this paper we shall give an L^p-estimate between $U(f)$ and $s(f)$.

2. We start with these remarks: <u>the operator U is not local</u>, and <u>$\|U(f)\|_p$ can not always be compared with $\|s(f)\|_p$</u>. For example, let $w=(w_n)$ be an

independent sequence satisfying $P(w_k=-1)=P(w_k=1)=1/2, k\geq 1$, and W the martingale with difference sequence w. Define now $X_{2n+1}=X_{2n}=\sum_{k=1}^{n}\frac{1}{k}w_k, X_1=X_0=0$. Then $x_{2n+1}=0$ and $x_{2n}=\frac{1}{n}w_n$ so that $s(X)=(\sum_{n=1}^{\infty}\frac{1}{n^2})^{1/2}$. Thus X is an L^2-bounded martingale. If $T_k=2k, v_{2k+1}=1$ and $v_{2k}=0$ for each $k\geq 1$, then $s(f)=0$ but $U(f)=(\sum_{n=1}^{\infty}\frac{1}{n^2})^{1/2}$. This implies that U is not local and that $\|U(f)\|_p\leq C_p\|s(f)\|_p$ does not hold in general. On the other hand, if $T_k=2k, v_{2k+1}=0$ and $v_{2k}=1$ for each $k\geq 1$, then $U(f)=0$ and $s(f)=(\sum_{k=1}^{\infty}\frac{1}{k^2})^{1/2}$.

Moreover, in what follows we assume that the martingale X is locally square integrable.

PROPOSITION 1. Let $f_n=\sum_{k=1}^{n}v_kx_k, n\geq 1$, be a martingale transform in $\underline{\underline{M}}$.

(1) If $|v_{T_{k-1}+1}|\leq |v_j|$ on $\{T_{k-1}<j\leq T_k\}$ for every j and k, then

$$\|U(f)\|_p\leq\sqrt{\frac{p}{2}}\,\|s(f)\|_p\ ,\ 2\leq p<\infty$$

(2) If $|v_{T_{k-1}+1}|\geq |v_j|$ on $\{T_{k-1}<j\leq T_k\}$ for every j and k, then

$$\|U(f)\|_p\geq\sqrt{\frac{p}{2}}\,\|s(f)\|_p\ ,\ 0<p\leq 2.$$

PROOF. We show only the part (1), the second part being proved similarly. Let now $2\leq p<\infty$, and suppose that for every k, $|v_{T_{k-1}+1}|\leq|v_j|$ for $T_{k-1}<j\leq T_k$. An easy computation shows that $E[x_j^2|F_{(j-1)\vee T_{k-1}}]=E[x_j^2|F_{j-1}]$ on $\{T_{k-1}<j\leq T_k\}$ and $E[(X_{T_k}-X_{T_{k-1}})^2|G_{k-1}]=\sum_{j=1}^{\infty}E[x_j^2\, I_{\{T_{k-1}<j\leq T_k\}}|G_{k-1}]$. Therefore we have

$$\|U(f)\|_p=E[\Big\{\sum_{k=1}^{\infty}v_{T_{k-1}+1}^2\ E[\sum_{j=1}^{\infty}E[x_j^2|F_{j-1}]I_{\{T_{k-1}<j\leq T_k\}}|G_{k-1}]\Big\}^{p/2}]^{1/p}$$

$$\leqslant E[\{\sum_{k=1}^{\infty} E[\sum_{j=1}^{\infty} v_j^2 E[x_j^2 | F_{j-1}]\ I_{\{T_{k-1} < j \leqslant T_k\}} | G_{k-1}]\}^{p/2}]^{1/p}$$

$$= E[\{\sum_{k=1}^{\infty} E[s_{T_k}(f)^2 - s_{T_{k-1}}(f)^2 | G_{k-1}]\}^{p/2}]^{1/p}$$

$$\leqslant \sqrt{\frac{p}{2}}\ E[(\sum_{k=1}^{\infty} \{s_{T_k}(f)^2 - s_{T_{k-1}}(f)^2\})^{p/2}]^{1/p}$$

$$\leqslant \sqrt{\frac{p}{2}}\ \| s(f) \|_p .$$

We considered in [1] the special case v=1.

REMARK. Let f be any martingale transform in $\underline{\underline{M}}$ as above. Define the following stopping times: $k \geqslant 1$

$$T_o=0,\ T_k = \mathrm{Min}\{ j > T_{k-1} ;\ |v_{j+1}| < |v_{T_{k-1}+1}| \}$$

$$S_o=0,\ S_k = \mathrm{Min}\{ j > S_{k-1} ;\ |v_{j+1}| > |v_{S_{k-1}+1}| \}$$

Then we get $|v_j| \geqslant |v_{T_{k-1}+1}|$ on $\{T_{k-1} < j \leqslant T_k\}$ and $|v_j| \leqslant |v_{S_{k-1}+1}|$ on $\{S_{k-1} < j \leqslant S_k\}$.

PROPOSITION 2. For any f in $\underline{\underline{M}}$ there exist martingale transforms $f^{(1)}$ and $f^{(2)}$ in M such that

1^o. $f=f^{(1)}+f^{(2)}$

2^o. for each i=1,2 $\| U(f^{(i)}) \|_p \leqslant \sqrt{\frac{p}{2}}\ \| s(f^{(i)}) \|_p$, $2 \leqslant p < \infty$.

PROOF. Let $f_n = \sum_{k=1}^{n} v_k x_k, v_o=0$ and define

$$v_n^{(1)} = \sum_{k=1}^{n} (v_k - v_{k-1})^+,\ v_n^{(2)} = -\sum_{k=1}^{n} (v_k - v_{k-1})^- .$$

Then each $v^{(i)}$ is a previsible process so that the martingale transform $f^{(i)}$ defined by $f_n^{(i)} = \sum_{k=1}^{n} v_k^{(i)} x_k$ belongs to $\underline{\underline{M}}$. As $v_n = v_n^{(1)} + v_n^{(2)}$ for each n, we get $f = f^{(1)} + f^{(2)}$. It is clear that $|v_n^{(i)}| \leq |v_{n+1}^{(i)}|$ for each i=1,2. This completes the proof.

REFERENCE

[1]. N.Kazamaki, An inequality for the conditioned square functions on martingales. Tohoku Math. Journ., (to appear).