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1. NOTATIONS AND THEOREMS.

By a system $(\Omega, \underline{\underline{F}}, \underline{\underline{F}}_t, P)$ is meant a complete probability spce $(\Omega, \underline{\underline{F}}, P)$ together with an increasing right continuous family $(\underline{\underline{F}}_t)_{0\leq t\leq\infty}$ of sub-σ-fields of $\underline{\underline{F}}$ with $\underline{\underline{F}}_\infty = \bigvee_{0\leq t<\infty} \underline{\underline{F}}_t$ such that $\underline{\underline{F}}_0$ contains all P-null sets. The reader is assumed to familiar with the basic notations of the general theory of processes as expounded in [3] and [5]. We define $\underline{\underline{M}}^p$, $\underline{\underline{M}}^p_c$, $\underline{\underline{G}}_t$ and $\underline{\underline{H}}$ as the following.

$\underline{\underline{M}}^p$ = the family of all L^p-bounded $\underline{\underline{F}}_t$-martingales.

$\underline{\underline{M}}^p_c = \{ X \in \underline{\underline{M}}^p ;\ X \text{ is continuous} \}$.

$\underline{\underline{G}}_t = \bigcap_{\varepsilon>0} \sigma(X_s ;\ s<t+\varepsilon,\ X \in \underline{\underline{M}}^2_c)$.

$\underline{\underline{H}} = \{ X_\infty ;\ X \in \underline{\underline{M}}^2_c \}$.

It is immediate from the definition of $\underline{\underline{G}}_t$ that all continuous $\underline{\underline{F}}_t$-martingales are $\underline{\underline{G}}_t$-martingales.

Let $X = X^{\oplus} - X^{\ominus}$ denote the Krickeberg decomposition for $X \in \underline{\underline{M}}^1$. That is, $X^{\oplus}$ and $X^{\ominus}$ are positive martingales such that

$$\sup_t E[|X_t|] = E[X^{\oplus}_0] + E[X^{\ominus}_0].$$

In this note, we will investigate the condition for the Krickeberg decomposition preserving the continuity of the paths of martingales, and then the property of paths of continuous martingales. Namely, we will prove the following theorems.

THEOREM 1. The following statements (1) – (5) are equivalent.

(1) $X^{\oplus}$ is continuous for every $X \in \underline{\underline{M}}^\infty_c$.

(2) $X^{\oplus}$ is continuous for every $X \in \underline{\underline{M}}^1_c$.

(3) $\underline{\underline{H}} = L^2(\underline{\underline{G}}_\infty)$.

(4) All bounded $\underline{\underline{G}}_t$-martingales are continuous $\underline{\underline{F}}_t$-martingales.

(5) All $\underline{\underline{G}}_t$-martingales are continuous $\underline{\underline{F}}_t$-martingales.

THEOREM 2. Suppose the statements in THEOREM 1 hold. Let T be a totally inaccessible $\underline{\underline{F}}_t$-stopping time with $P(T<\infty) > 0$, and let X be a continuous $\underline{\underline{F}}_t$-martingale. Then

(6) $t \longrightarrow X_t$ are almost surely constant on some right neighbourhood at $T < \infty$, and

(7) $t \longrightarrow X_t$ are constant on some left neighbourhood at T with positive probability.

In THEOREM 2, we can replace $t \longrightarrow X_t$ by $t \longrightarrow \langle X, X \rangle_t$, because of [2] p.248, LEMMA (4.1). That implies the following corollary.

COROLLARY. If there exist a totally inaccessible $\underline{\underline{F}}_t$-stopping time T with $P(T<\infty) > 0$ and a continuous $\underline{\underline{F}}_t$-martingale M such that $t \longrightarrow \langle M, M \rangle_t$ is strictly increasing, then there exists $X \in \underline{\underline{M}}_c^\infty$ such that $X^{\oplus}$ is not continuous

REMARK. We can construct a system $(\Omega, \underline{\underline{F}}, \underline{\underline{F}}_t, P)$ that satisfies the assumption in COROLLARY as the following. Let M and N be a Brownian motion with $M_0 = 0$ and a Poisson process with $N_0 = 0$ on some probability space $(\Omega, \underline{\underline{F}}, P)$ respectively, and let both processes are independent. We define $\underline{\underline{F}}_t = \sigma(M_s, N_s;\ s \leqq t)$. Then $T = \inf \{ t;\ N_t = 1 \}$ and M have the required properties. That gives the another proof of the result in [6].

2. PROOF OF THEOREM 1.

First of all we are going to give the following lemmas.

LEMMA 1. Let X be a uniformely integrable martingale. If there exists a sequence $\{X^n\}$ of uniformely integrable continuous martingales such that

$$X^n_\infty \longrightarrow X_\infty \text{ in } L^1 \text{ as } n \to \infty,$$

then X is continuous.

PROOF. See [4] p.115 – 116.

LEMMA 2. Let $\underline{\underline{K}}$ be a closed subspace of $L^2(\underline{\underline{G}}_\infty)$ containing all constant functions. If $\sigma(\underline{\underline{K}}) = \underline{\underline{G}}_\infty$ and $f \vee 0 \in \underline{\underline{K}}$ for each $f \in \underline{\underline{K}}$, then $\underline{\underline{K}} = L^2(\underline{\underline{G}}_\infty)$.

PROOF. See [6] LEMMA 1.

LEMMA 3. For each $X \in \underline{\underline{M}}^1$, $X^{\oplus} - X^+$ and $X^{\ominus} - X^-$ are the same potential of the class (D). Here $X^+ = (X_t \vee 0)_t$ and $X^- = ((-X_t) \vee 0)_t$.

PROOF. Let $X^+ = M + A$ and $X^- = N + B$ be the Doob-Meyer decomposition of submartingales X^+ and X^- respectively. That is, M and N are martingales and A and B are previsible increasing processes. Hence

$$A - B = X - M + N$$

is a martingale, from which $A = B$. (See [1] p.109 - 111, V.T36, T38.) Since A is integrable by $X \in \underline{\underline{M}}^1$, the Krickeberg decomposition of X is given by

$$X^{\oplus}_t = M_t + E[A_\infty | \underline{\underline{F}}_t]$$

and

$$X_t^{\ominus} = N_t + E[A_\infty \mid \underline{\underline{F}}_t].$$

Consequently, we obtain

$$X_t^{\oplus} - X_t^+ = E[A_\infty \mid \underline{\underline{F}}_t] - A_t = X_t^{\ominus} - X_t^-.$$

This establishes LEMMA 3. (See [5] p.142, VII.T7.)

We come now the proof of THEOREM 1.

(1)$\Rightarrow$(2) : Let $X \in \underline{\underline{M}}_c^1$. We can choose a sequence $\{T^n\}$ of stopping times such that $T^n \uparrow \infty$, $T^n \leqq n$ and X^{T^n} is bounded. Each martingale $(E[X_{T^n}^+ \mid \underline{\underline{F}}_t])_t$ is continuous by (1), and so it suffices to show that

(8) $\qquad E[X_{T^n}^+ \mid \underline{\underline{F}}_t] \longrightarrow X_t^{\oplus}$ in L^1 as $n \longrightarrow \infty$

for each t, because of LEMMA 1. From LEMMA 3 (See [5] p.138, VI.T20.)

$$E[|X_{T^n\wedge t}^{\oplus} - E[X_{T^n}^+ \mid \underline{\underline{F}}_t]|]$$

$$= E[|E[X_{T^n}^{\oplus} - X_{T^n}^+ \mid \underline{\underline{F}}_t]|]$$

$$= E[X_{T^n}^{\oplus} - X_{T^n}^+] \longrightarrow 0 \quad \text{as} \quad n \to \infty,$$

and obviously

$$X_{T^n\wedge t}^{\oplus} \longrightarrow X_t^{\oplus} \quad \text{in} \quad L^1 \quad \text{as} \quad n \to \infty,$$

from which we obtain (8).

(2)$\Rightarrow$(3) : According to the definition of $\underline{\underline{H}}$ and $\underline{\underline{G}}_t$, $\underline{\underline{H}}$ contains all constant functions and $\sigma(\underline{\underline{H}}) = \underline{\underline{G}}_\infty$. Moreover, $\underline{\underline{H}}$ is a closed subspace of $L^2(\underline{\underline{G}}_\infty)$ by LEMMA 1 and from (2) $f \vee 0 \in \underline{\underline{H}}$ for each $f \in \underline{\underline{H}}$. Consequently $\underline{\underline{H}} = L^2(\underline{\underline{G}}_\infty)$, because of LEMMA 2.

(3)$\Rightarrow$(4) : Let X be a bounded $\underline{\underline{G}}_t$-martingale. According to (3) there exists a continuous $\underline{\underline{F}}_t$-martingale Y with $Y_\infty = X_\infty$. Since Y is a $\underline{\underline{G}}_t$-martingale, we obtain X = Y.

(4) $\Rightarrow$ (5) : It is an immediate consequence of LEMMA 1.

(5) $\Rightarrow$ (1) : Let $X \in \underline{\underline{M}}_c^\infty$. Then X is a $\underline{\underline{G}}_t$-martingale. According to (5) the Krickeberg decomposition for X with respect to $\underline{\underline{G}}_t$-is identical with $X \doteq X^\oplus - X^\ominus$ and so $X^\oplus$ is continuous.

The proof of THEOREM 1 is now complete.

3. PROOF OF THEOREM 2.

Let T be a totally inaccessible $\underline{\underline{F}}_t$-stopping time with $P(T<\infty) > 0$ and X be a continuous $\underline{\underline{F}}_t$-martingale. Put

$$U = \inf \{ t;\ \langle X,X \rangle_t > \langle X^T,X^T \rangle_t \}$$

and

$$V = \sup \{ t;\ \langle X,X \rangle_t < \langle X^T,X^T \rangle_\infty \}.$$

Since $\langle X,X \rangle$ and $\langle X^T,X^T \rangle$ are $\underline{\underline{G}}_t$-adapted by [3] p.92, THEOREM 2, U is a $\underline{\underline{G}}_t$-stopping time and V is $\underline{\underline{G}}_\infty$-measurable. From (5) all $\underline{\underline{G}}_t$-stopping times are accessible with respect to $\underline{\underline{G}}_t$ and so with respect to $\underline{\underline{F}}_t$. (See [1] p.112, V.T41.) This implies that U is an accessible $\underline{\underline{F}}_t$-stopping time, from which $T < U$ a.s. on $\{T<\infty\}$. Consequently we obtain (6).

Before coming to the proof of (7) we will give the following lemma.

LEMMA 4. If all $\underline{\underline{G}}_t$-martingales are $\underline{\underline{F}}_t$-martingales, then for each t $\underline{\underline{G}}_t = \underline{\underline{F}}_t \cap \underline{\underline{G}}_\infty$.

PROOF. It suffices to show that $\underline{\underline{G}}_t \supset \underline{\underline{F}}_t \cap \underline{\underline{G}}_\infty$. Let $\Lambda \in \underline{\underline{F}}_t \cap \underline{\underline{G}}_\infty$. From the assumption $(P(\Lambda|\underline{\underline{G}}_t))_t$ is a $\underline{\underline{F}}_t$-martingale with $P(\Lambda|\underline{\underline{G}}_\infty) = 1_\Lambda$. On the other hand $(P(\Lambda|\underline{\underline{F}}_t))_t$ is a $\underline{\underline{F}}_t$-martingale with $P(\Lambda|\underline{\underline{F}}_\infty) = 1_\Lambda$. Consequently we have

$$1_\Lambda = P(\Lambda|\underline{\underline{F}}_t) = P(\Lambda|\underline{\underline{G}}_t).$$

Hence $\Lambda \in \underline{\underline{G}}_t$. This establishes LEMMA 4.

Finally let us show (7). Suppose that $P(V = T) = 1$. Then T is $\underline{\underline{G}}_\infty$-measurable and from LEMMA 4 T is a $\underline{\underline{G}}_t$-stopping time. Thus T is accessible. This contradicts the fact T is totally inaccessible. This contradiction implies $P(V<T) > 0$ and hence we obtain (7). Thus the proof of THEOREM 2 is now complete.

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REFERENCES

[1] C.Dellacherie, Capacités et processus stochastiques, Ergebnisse der Math. 67, Springer 1972.

[2] R.K.Getoor and M.J.Sharpe, Conformal martingales, Inventions Math. 16 (1972), 271 - 308.

[3] P.A.Meyer, Integrales stochastiques I,II, Séminaire de Probabilités 1, Lecture Notes in Math. 39 (1967) 72 -117.

[4] P.A.Meyer, Processus de Markov, Lecture Notes in Math. 26 (1967).

[5] P.A.Meyer, Probabilités et potentiel, Paris, Hermann 1966.

[6] T.Sekiguchi, Oh the Krickeberg decomposition, Tohoku Math. Journ. to appear.