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On The Construction of Kernels

by

R. K. Getoor*

1. Introduction

In a number of recent papers (see, for example, [1], [2], [4], [5] and [7]) the authors found it necessary to regularize certain Radon-Nikodym type densities by means of kernels. This is straightforward if the underlying space is nice enough. However, when the underlying space is somewhat more complicated certain difficulties arise. Therefore it seems as though it might be worthwhile to formulate explicitly general conditions under which such a construction is possible. The main results are Propositions 4.1 and 4.5 in Section 4. In Section 5 these results are applied to the construction of densities for continuous additive functionals of Markov processes. In Section 6, following Mokobodzki [8], we apply these results to the disintegration of measures and the existence of regular conditional probabilities. Only in Section 5 is an acquaintance with the theory of Markov processes assumed. Section 6 assumes no such acquaintance and is independent of Section 5.

This paper is purely expository and contains no new results. All of the results described here are classical and are well known.

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2. Notation and Definitions.

Let $(E, \mathcal{E})$ be a measurable space and let $\underline{\underline{E}}$ denote the Banach space of bounded real valued $\mathcal{E}$ measurable functions on E under the supremum norm. Let $\mathfrak{n} \subset \mathcal{E}$ be closed under countable unions and be hereditary in the sense that if $N \in \mathfrak{n}$ and $B \subset N$, $B \in \mathcal{E}$, then $B \in \mathfrak{n}$. In addition we assume that $E \notin \mathfrak{n}$. The sets in $\mathfrak{n}$ are called "negligible" and a property $p(x)$ depending on $x \in E$ is said to hold almost everywhere $(\mathfrak{n})$, abbreviated a.e. $(\mathfrak{n})$, if the set of x for which $p(x)$ does not hold is contained in $\mathfrak{n}$. When no confusion is possible we shall write simply a.e. rather than a.e. $(\mathfrak{n})$.

A measure will mean a positive finite measure unless explicitly stated otherwise. If $(E, \mathcal{E})$ is a measurable space and μ a measure on $(E, \mathcal{E})$, then $\mathcal{E}^{\mu}$ denotes the completion of $\mathcal{E}$ with respect to μ and $\mathcal{E}^{*} = \bigcap_{\mu} \mathcal{E}^{\mu}$ is the σ-algebra of universally measurable sets over $\mathcal{E}$. Here the intersection is over all measures μ on $(E, \mathcal{E})$. A measure μ on $(E, \mathcal{E})$ has a unique extension to $(E, \mathcal{E}^{*})$ which we again denote by μ. Of course, $\underline{\underline{E}}^{*}$ then denotes the Banach space of bounded real valued $\mathcal{E}^{*}$ measurable functions.

In a topological space a Borel set is an element of the smallest σ-algebra containing the open sets and a universally measurable set is one that is universally measurable over the Borel sets.

A topological space Ω is said to be a U-space provided it is homeomorphic to a universally measurable subspace of a compact metric space $\hat{\Omega}$. We shall always identify Ω with a universally measurable subspace of $\hat{\Omega}$ when Ω is a U-space. A topological space Ω is a Lusin space provided it is homeomorphic to a Borel subspace of a compact metric space $\hat{\Omega}$. This definition differs from the one given in [3], but it follows readily from the results in [3] that it is an equivalent definition. Again we shall always identify a Lusin space Ω with a Borel subspace of $\hat{\Omega}$. Clearly every Lusin space is a U-space. It follows from the Choquet capacitability theorem that if Ω is a Souslin subspace or the complement of a Souslin subspace of a compact metric space $\hat{\Omega}$, then Ω is a U-space. We refer the reader to [3] for the definition of a Souslin space.

Let Ω be a universally measurable subspace of a compact metric space $\hat{\Omega}$. Then $\mathfrak{F}$ (resp. $\hat{\mathfrak{F}}$) denotes the σ-algebra of Borel subsets of Ω (resp. $\hat{\Omega}$), and $\mathfrak{F}^*$ (resp. $\hat{\mathfrak{F}}^*$) denotes the σ-algebra of universally measurable sets over $(\Omega, \mathfrak{F})$ (resp. $(\hat{\Omega}, \hat{\mathfrak{F}})$). By hypothesis $\Omega \in \hat{\mathfrak{F}}^*$. It is easy to see that $A \in \mathfrak{F}$ if and only if there exists $\hat{A} \in \hat{\mathfrak{F}}$ such that $A = \hat{A} \cap \Omega$, that is, $\mathfrak{F} = \hat{\mathfrak{F}}_\Omega$ where $\hat{\mathfrak{F}}_\Omega$ is the trace of $\hat{\mathfrak{F}}$ on Ω defined by

$$(2.1) \qquad \hat{\mathfrak{F}}_\Omega = \{A \subset \Omega\colon\ A = \Omega \cap \hat{A} \ \text{ for some } \ \hat{A} \in \hat{\mathfrak{F}}\}\ .$$

For emphasis we repeat that, in general, Ω is <u>not</u> an element of $\hat{\mathfrak{F}}$. It follows from this that a real valued function, f, on Ω is Borel, i.e. $\mathfrak{F}$-measurable, if and only if there exists a real function $\hat{f}$ on $\hat{\Omega}$ that is Borel, i.e. $\hat{\mathfrak{F}}$-measurable, such that $f = \hat{f}|_\Omega$. It follows immediately from these facts that $\mathfrak{F}^* = \hat{\mathfrak{F}}^*_\Omega$, but since $\Omega \in \hat{\mathfrak{F}}^*$ we have that $A \in \mathfrak{F}^*$ if and only if $A \subset \Omega$ and $A \in \hat{\mathfrak{F}}^*$.

Let $\underline{\underline{F}}$ (resp. $\underline{\underline{F}}^*$) denote the space of bounded $\mathfrak{F}$ (resp. $\mathfrak{F}^*$) measurable functions on Ω. If $\underline{\underline{V}}$ is a vector subspace of $\underline{\underline{F}}^*$, then a map $T: \underline{\underline{V}} \to \underline{\underline{E}}$ is almost linear and almost positive on $\underline{\underline{V}}$ provided

$$T(\alpha f + \beta g) = \alpha Tf + \beta Tg \quad \text{a.e.} \tag{2.2}$$

for $f, g \in \underline{\underline{V}}$ and real α, β;

$$f \in \underline{\underline{V}},\ f \geq 0 \text{ implies } Tf \geq 0 \quad \text{a.e.}. \tag{2.3}$$

Recall that a.e. means except on a subset of η. If $1 \in \underline{\underline{V}}$ it is straightforward to check that

$$|Tf| \leq \|f\|\, T1 \quad \text{a.e.} \tag{2.4}$$

for all $f \in \underline{\underline{V}}$.

Finally if $(Y, \mathcal{G})$ is a measurable space, a kernel K from $(E, \mathcal{E})$ to $(Y, \mathcal{G})$ is a function $K(x, A)$ defined for $x \in E$ and $A \in \mathcal{G}$ such that $x \to K(x, A)$ is $\mathcal{E}$ measurable for each $A \in \mathcal{G}$, and $A \to K(x, A)$ is a measure on $\mathcal{G}$ for each $x \in E$. The kernel K is bounded if $\sup\{K(x, Y): x \in E\} < \infty$. If K is a bounded kernel from $(E, \mathcal{E})$ to $(Y, \mathcal{G})$, then

$$f \to Kf = K(\cdot,\ f) = \int K(\cdot,\ dy)\ f(y) \tag{2.5}$$

defines a bounded, positive, linear map from $\underline{\underline{A}}$ to $\underline{\underline{E}}$ such that if $(f_n) \subset \underline{\underline{A}}$ with $0 \leq f_n \uparrow f \in \underline{\underline{A}}$, then $Kf_n \uparrow Kf$. Moreover, it is immediate that any such map from $\underline{\underline{A}}$ to $\underline{\underline{E}}$ is given by a bounded kernel K as in (2.5).

3. The Compact Metric Case.

In this section we assume that Ω is a compact metric space and $\mathfrak{F}$ is the σ-algebra of Borel subsets of Ω. Let $\underline{\underline{C}} = \underline{\underline{C}}(\Omega)$ denote the space of real valued continuous functions on Ω. Let $(E, \mathcal{E})$ and $\mathfrak{n}$ be as in Section 2.

(3.1) **Proposition.** *Let $T: \underline{\underline{C}} \to \underline{\underline{E}}$ be almost linear and almost positive. Then there exists a bounded kernel, K, from $(E, \mathcal{E})$ to $(\Omega, \mathfrak{F})$ such that $Tf = Kf$ a.e. for each $f \in \underline{\underline{C}}$.*

Proof. Let $\underline{\underline{H}} \subset \underline{\underline{C}}$ be a countable vector space over the rationals $\underline{\underline{Q}}$ which contains 1 and is dense in $\underline{\underline{C}}$. Let $\underline{\underline{H}}^+ = \{h \in \underline{\underline{H}}: h \geq 0\}$ and let $M = \|T1\|$. Define $t(x, h) = Th(x)$ for $h \in \underline{\underline{H}}$. If $\alpha, \beta \in \underline{\underline{Q}}$ and $f, g \in \underline{\underline{H}}$, let

$$N(\alpha, \beta, f, g) = \{x: t(x, \alpha f + \beta g) \neq \alpha t(x, f) + \beta t(x, g)\} .$$

For $h \in \underline{\underline{H}}$ and $h^+ \in \underline{\underline{H}}^+$ let

$$N(h) = \{x: |t(x, h)| > M\|h\|\}; \; N(h^+) = \{x: t(x, h^+) < 0\} .$$

It is immediate from (2.2), (2.3), and (2.4) that each of the above sets is in $\mathfrak{n}$. Let N be the union over all $\alpha, \beta \in \underline{\underline{Q}}$, $f, g, h \in \underline{\underline{H}}$, and $h^+ \in \underline{\underline{H}}^+$ of the sets $N(\alpha, \beta, f, g)$, $N(h)$, and $N(h^+)$. Then $N \in \mathfrak{n}$

and we define

$$k(x, h) = t(x, h) \quad \text{for } x \notin N,\ h \in \underline{\underline{H}}$$
$$= 0 \qquad\qquad \text{for } x \in N,\ h \in \underline{\underline{H}}\ .$$

Then for each $x \in E$, $h \to k(x, h)$ is a positive, rational linear functional on $\underline{\underline{H}}$ that is bounded by M. In addition $Th = k(\cdot, h)$ a.e. for each $h \in \underline{\underline{H}}$, and $x \to k(x, h)$ is in $\underline{\underline{E}}$ for each $h \in H$. Since $\underline{\underline{H}}$ is dense in $\underline{\underline{C}}$ (in the uniform norm) one can extend $h \to k(x, h)$ by continuity to $\underline{\underline{C}}$ for each x. Denoting this extension by $k(x, f)$ again, it is clear that for each $x \in E$, $f \to k(x, f)$ is a positive linear functional on $\underline{\underline{C}}$ that is bounded by M such that $Tf = k(\cdot, f)$ a.e. and $x \to k(x, f)$ is in $\underline{\underline{E}}$ for each $f \in \underline{\underline{C}}$. Consequently for each $x \in E$, there exists a measure $K(x, \cdot)$ on $\mathfrak{F}$ such that $k(x, f) = \int K(x, dy) f(y)$ for each $f \in \underline{\underline{C}}$. Since $x \to Kf(x) = k(x, f)$ is $\mathcal{E}$ measurable for each $f \in \underline{\underline{C}}$, it follows from the monotone class theorem that K is a kernel from $(E, \mathcal{E})$ to $(\Omega, \mathfrak{F})$. Clearly K is bounded since $K(x, \Omega) = k(x, 1) \le M$ for all x. This completes the proof of (3.1).

The following corollary is, perhaps, of more importance than (3.1) itself. The assumptions on $(\Omega, \mathfrak{F})$ in the first sentence of this section are still in force and, as in Section 2, $\underline{\underline{F}}$ denotes the bounded $\mathfrak{F}$ measurable functions on Ω.

(3.2) Corollary. Let $T: \underline{\underline{F}} \to \underline{\underline{E}}$ be almost linear and almost positive and satisfy

(3.3) if $(f_n) \subset \underline{\underline{F}}$ and $0 \le f_n \uparrow f \in \underline{\underline{F}}$, then

$Tf_n \uparrow Tf$ a.e. .

Then there exists a bounded kernel K from $(E, \mathcal{E})$ to $(\Omega, \mathfrak{F})$ such that $Tf = Kf$ a.e. for each $f \in \underline{\underline{F}}$.

Proof. If we restrict T to $\underline{\underline{C}}$, then using (3.1) we can find a bounded kernel K from $(E, \mathcal{E})$ to $(\Omega, \mathfrak{F})$ such that $Tf = Kf$ a.e. for all $f \in \underline{\underline{C}}$. But if $\underline{\underline{V}}$ denotes set of $f \in \underline{\underline{F}}$ for which $Tf = Kf$ a.e., then $\underline{\underline{V}}$ is a vector space containing $\underline{\underline{C}}$ and by (3.3) has the property that if $(f_n) \in \underline{\underline{V}}$ and $0 \le f_n \uparrow f$ with f bounded, then $f \in \underline{\underline{V}}$. Consequently $\underline{\underline{V}} = \underline{\underline{F}}$.

(3.4) Remark. In (3.1) and (3.2) the proof shows that $K(x, \Omega) \le \|T1\|$ for all $x \in E$.

4. The General Case.

This section contains two extensions of Corollary 3.2; the first when Ω is a Lusin space and the second when Ω is s U-space. If Ω is Lusin Corollary 3.2 remains true as stated, but if Ω is a U-space we need to make an additional assumption. The result for Lusin spacesis contained in Proposition 4.1 and that for U-spaces in Proposition 4.5. The notation is the same as in the previous sections. In particular $\underline{\underline{F}}$ denotes the bounded Borel (i.e. $\mathfrak{F}$ measurable) functions on Ω, while $(E, \mathcal{E})$, $\underline{\underline{E}}$, and η have the same meanings as in Sections 2 and 3.

(4.1) Proposition. *Let Ω be a Lusin space and let* $T: \underline{\underline{F}} \to \underline{\underline{E}}$ *be almost linear and almost positive and satisfy* (3.3). *Then there exists a bounded kernel* K *from* $(E, \mathcal{E})$ *to* $(\Omega, \mathfrak{F})$ *such that* $Tf = Kf$ a.e. *for each* $f \in \underline{\underline{F}}$.

Proof. Let $\hat{\Omega}$ be a compact metric space in which Ω is a Borel set. Let $\hat{\mathfrak{F}}$ be the σ-algebra of Borel subsets of $\hat{\Omega}$ and $\hat{\underline{\underline{F}}}$ the bounded Borel functions on $\hat{\Omega}$. Then $f \in \underline{\underline{F}}$ if and only if $\hat{f} \in \hat{\underline{\underline{F}}}$ where $\hat{f} = f$ on Ω and $\hat{f} = 0$ on $\hat{\Omega} - \Omega$. Define $\hat{T}: \hat{\underline{\underline{F}}} \to \underline{\underline{E}}$ by

$$\hat{T}\hat{f} = T(\hat{f}|_{\Omega}) . \tag{4.2}$$

Since $\hat{f} \to \hat{f}|_{\Omega}$ is positive, linear, and preserves pointwise limits, it is clear that $\hat{T}$ satisfies the assumptions of Corollary 3.2 relative to $\hat{\Omega}$. Thus there exists a kernel $\hat{K}$ from $(E, \mathcal{E})$ to $(\hat{\Omega}, \hat{\mathfrak{F}})$ such that $\hat{T}\hat{f} = \hat{K}\hat{f}$ a.e. for all $\hat{f} \in \hat{\underline{\underline{F}}}$. But we may identify $\underline{\underline{F}}$ with those elements of $\hat{\underline{\underline{F}}}$ which vanish off Ω. Consequently $Tf = \hat{K}f$ a.e. for all $f \in \underline{\underline{F}}$. If $\hat{f} = 1_{\hat{\Omega}-\Omega}$, then a.e.

$$\hat{K}(\cdot, \hat{\Omega} - \Omega) = \hat{T}(1_{\hat{\Omega}-\Omega}) = T(0) = 0 .$$

Thus $N = \{x: \hat{K}(x, \hat{\Omega} - \Omega) \neq 0\}$ is in $\mathfrak{n}$. If we define $K(x, \cdot) = \hat{K}(x, \cdot)$ for $x \notin N$ and $K(x, \cdot) = 0$ for $x \in N$, then K is a kernel from $(E, \mathcal{E})$ to $(\Omega, \mathfrak{F})$ such that $Tf = Kf$ a.e. for all $f \in \underline{\underline{F}}$, proving (4.1).

In order to treat U-spaces we need to assume that $\mathfrak{n}$ has a special form. We assume that there exists a family M of measures on $(E, \mathcal{E})$ such that

$$(4.3) \qquad \mathfrak{n} = \{B \in \mathcal{E}: m(B) = 0 \text{ for all } m \in M\} .$$

There is considerable leeway in the choice of M for a given $\mathfrak{n}$. For example, if $M = (m_i)$ is countable we may replace M by the single measure

$$m = \sum_i 2^{-i}(m_i(E))^{-1} m_i$$

without changing $\mathfrak{n}$. For $\mathfrak{n}$ of the form (4.3) we define

$$(4.4) \qquad \mathfrak{n}^* = \{B \in \mathcal{E}^* : m(B) = 0 \text{ for all } m \in M\} .$$

(4.5) Proposition. *Let Ω be a U-space and let $\mathfrak{n}$ be as above. Let $T: \underline{\underline{F}} \to \underline{\underline{E}}$ be almost linear and almost positive and satisfy (3.3). Then there exists a bounded kernel K from $(E, \mathcal{E}^*)$ to $(\Omega, \mathfrak{F})$ such that $Tf = Kf$ a.e. $(\mathfrak{n}^*)$ for each $f \in \underline{\underline{F}}$. If, in addition, M consists of a single measure, K may be chosen to be a kernel from $(E, \mathcal{E})$ to $(\Omega, \mathfrak{F})$. In this case $Tf = Kf$ a.e. $(\mathfrak{n})$ for each $f \in \underline{\underline{F}}$.*

Proof. Recall from Section 2 that if $\hat{f} \in \hat{\underline{\underline{F}}}$ then $\hat{f}|_{\Omega} \in \underline{\underline{F}}$. If $\hat{f} \in \hat{\underline{\underline{F}}}$ define $\hat{T}\hat{f} = T(\hat{f}|_{\Omega})$. As in the proof of (4.1), $\hat{T}$ satisfies the hypotheses of Corollary 3.2 relative to $\hat{\Omega}$. Consequently there exists a bounded kernel $\hat{K}$ from $(E, \mathcal{E})$ to $(\hat{\Omega}, \hat{\mathfrak{F}})$ such that $\hat{T}\hat{f} = \hat{K}\hat{f}$ a.e. $(\mathfrak{n})$ for all $\hat{f} \in \hat{\underline{\underline{F}}}$. As usual the unique extension of each $\hat{K}(x, \cdot)$ to $\hat{\mathfrak{F}}^*$ is again denoted by $\hat{K}(x, \cdot)$. It is immediate that $x \to \hat{K}(x, \hat{f})$ is $\mathcal{E}^*$ measurable for each $\hat{f} \in \hat{\underline{\underline{F}}}^*$. (Given $\hat{f} \in \hat{\underline{\underline{F}}}^*$ and μ a measure on $\mathcal{E}$, choose $\hat{f}_1, \hat{f}_2 \in \hat{\underline{\underline{F}}}$ with $\hat{f}_1 \le \hat{f} \le \hat{f}_2$ and $\nu(\hat{f}_1) = \nu(\hat{f}_2)$ where $\nu(\cdot) = \int \mu(dx)\, \hat{K}(x, \cdot)$ is a measure on $\hat{\mathfrak{F}}$.) If $f \in \underline{\underline{F}}$, then $\hat{f} = f$ on Ω and $\hat{f} = 0$ on $\hat{\Omega} - \Omega$ is in $\hat{\underline{\underline{F}}}^*$ and so $\hat{K}f = \hat{K}\hat{f}$ exists and is in $\underline{\underline{E}}^*$. We claim that $Tf = \hat{K}f$ a.e. $(\mathfrak{n}^*)$.

To this end fix $m \in M$. Then $\nu(\hat{f}) = m(\hat{T}\hat{f})$ defines a measure on $\hat{\underline{\underline{F}}}$ that agrees with $\hat{f} \to \int m(dx)\, \hat{K}(x, \hat{f})$ on $\hat{\underline{\underline{F}}}$. Consequently the unique extension of ν to $\hat{\underline{\underline{F}}}^*$ is given by $\nu(\cdot) = \int m(dx)\ \hat{K}(x, \cdot)$. Given $f \in \underline{\underline{F}}$ let $\hat{f} = f$ on Ω and $\hat{f} = 0$ on $\hat{\Omega} - \Omega$. Then $\hat{f} \in \hat{\underline{\underline{F}}}^*$ and so there exist $\hat{f}_1, \hat{f}_2 \in \hat{\underline{\underline{F}}}$ with $\hat{f}_1 \le \hat{f} \le \hat{f}_2$ and $\nu(\hat{f}_1) = \nu(\hat{f}_2)$. Since $f_1 = \hat{f}_1|_\Omega$ and $f_2 = \hat{f}_2|_\Omega$ are in $\underline{\underline{F}}$ and satisfy $f_1 \le f \le f_2$ it follows that $Tf_1 \le Tf \le Tf_2$ a.e. $(\mathfrak{n})$. But by definition $\hat{T}\hat{f}_i = Tf_i$, while $\hat{T}\hat{f}_i = \hat{K}\hat{f}_i$ a.e. $(\mathfrak{n})$ by construction of $\hat{K}$, $i = 1, 2$. Finally $\hat{K}\hat{f}_1 \le \hat{K}\hat{f} \le \hat{K}\hat{f}_2$ and by the definition of ν, $\hat{K}\hat{f}_1 = \hat{K}\hat{f}_2$ a.e. (m). Combining these facts with $\hat{K}\hat{f} = \hat{K}f$ we see that $Tf = \hat{K}f$ a.e. (m). Since $\{Tf \ne \hat{K}f\} \in \mathcal{E}^*$ and $m \in M$ is arbitrary, $Tf = \hat{K}f$ a.e. $(\mathfrak{n}^*)$ for each $f \in \underline{\underline{F}}$. Consequently a.e. $(\mathfrak{n}^*)$ one has $\hat{K}1_{\hat{\Omega}} = \hat{T}1_{\hat{\Omega}} = T1_\Omega = \hat{K}1_\Omega$, or

$$N^* = \{x\colon \hat{K}(x, \hat{\Omega}) \ne K(x, \Omega)\} \in \mathfrak{n}^* . \tag{4.6}$$

Thus if we define $K(x, \cdot) = \hat{K}(x, \cdot)$ for $x \notin N^*$ and $K(x, \cdot) = 0$ for $x \in N^*$, then K is a kernel from $(E, \mathcal{E}^*)$ to $(\Omega, \mathcal{F})$ and $Tf = Kf$ a.e. $(\mathfrak{n}^*)$ for all $f \in \underline{\underline{F}}$. This establishes the first assertion in (4.5).

If M consists of a single measure m, then there exists $N \in \mathcal{E}$ such that $N^* \subset N$ and $m(N) = m(N^*) = 0$ where N^* is defined in (4.6). In this case if we define $K(x, \cdot) = \hat{K}(x, \cdot)$ for $x \notin N$ and $K(x, \cdot) = 0$ for $x \in N$, then $K(x, \cdot)$ is a measure on $(\Omega, \mathcal{F})$ for each x, and $\{x\colon \hat{K}(x, \cdot) \ne K(x, \cdot)\} = N$. If $f \in \underline{\underline{F}}$ then there exists $\hat{f}$ in $\hat{\underline{\underline{F}}}$

such that $\hat{f}|_{\Omega} = f$. If $x \notin N$, $K(x, \cdot) = \hat{K}(x, \cdot)$ is carried by Ω and so $K(x, f) = \hat{K}(x, \hat{f})$, while if $x \in N$, $K(x, f) = 0$. Since $\hat{K}\hat{f} \in \underline{\underline{E}}$ and $N \in \mathfrak{n} \subset \mathcal{E}$, it follows that Kf is in $\underline{\underline{E}}$. That is, K is a kernel from $(E, \mathcal{E})$ to $(\Omega, \mathcal{F})$ Clearly $Kf = Tf$ a.e. $(\mathfrak{n})$ in this case. This completes the proof of (4.5).

Remarks. In certain applications one would like to construct a kernel K from $(E, \mathcal{E})$ to $(\Omega, \mathcal{F})$ rather than from $(E, \mathcal{E}^{*})$, for general families M. I have not succeeded in doing this, and to the best of my knowledge it remains an open question. Also of interest is whether or not Proposition 4.1 is valid as stated when Ω is a U-space. In the actual applications that I have in mind (see the next section) Ω is the complement in $\hat{\Omega}$ of a Souslin subspace of $\hat{\Omega}$, but I do not see how to make use of this added information.

5. Continuous Additive Functionals.

Let E be a Lusin space and $\mathcal{E}$ the Borel sets of E. Let $(P_t)_{t\geq 0}$ be a semigroup of Markov kernels from $(E, \mathcal{E}^*)$ to $(E, \mathcal{E})$ that satisfies the hypotheses of the right, that is, HD1 and HD2 of [9]. Let

$$X = (\Omega, \mathcal{F}^0, \mathcal{F}^0_t, X_t, P^\mu, \ldots)$$

be the canonical right continuous realization of (P_t). We refer the reader to [9] for the basic properties of "right" processes. It is known (see [6] p. 235) that there exists a compact metric space $\hat{\Omega}$ containing Ω such that $\hat{\Omega} - \Omega$ is Souslin and $\mathcal{F}^0$ is the σ-algebra of Borel subsets of Ω. In particular $(\Omega, \mathcal{F}^0)$ is a U-space. However, Ω is not a Lusin space in general.

Let $\mathcal{A}_+$ denote the collection of continuous, adapted additive functionals A such that $E^x(A_t) < \infty$ for all $x \in E$ and $t < \infty$. Let $\mathcal{A}^1_+$ be those elements of $\mathcal{A}_+$ having bounded one-potentials, and set $\mathcal{A} = \mathcal{A}_+ - \mathcal{A}_+$ and $\mathcal{A}^1 = \mathcal{A}^1_+ - \mathcal{A}^1_+$. For simplicity we shall deal with $\mathcal{A}^1$ but one could just as well consider $\mathcal{A}$. We shall need the following result of Benveniste and Jacod [2]. If $A, B \in \mathcal{A}^1_+$ and $A << B$, then $A = f * B$ where $f \geq 0$ is $\mathcal{E}^1$ measurable. Here $\mathcal{E}^1$ is the smallest σ-algebra on E relative to which all α-excessive functions, $\alpha \geq 0$, are measurable, and $f * B$ is the functional

$$(f * B)_t = \int_0^t f(X_s)\, dB_s \ .$$

Clearly f is determined up to a set of B potential zero, that is, a set $\Gamma \in \mathcal{E}^1$ with $U_B^1(x, \Gamma) = 0$ for all $x \in E$. If $\mathcal{E}^n$ is the σ-algebra of nearly Borel subsets of E, then $\mathcal{E} \subset \mathcal{E}^1 \subset \mathcal{E}^n \subset \mathcal{E}^*$ and $(\mathcal{E}^1)^* = \mathcal{E}^*$.

Let $(W, \mathcal{G})$ be a U-space and suppose that we are given a map from $\underline{\underline{G}}$ to $\mathcal{A}^1$, $g \to A(g)$ satisfying

(5.1) $$g \geq 0 \text{ implies } A(g) \in \mathcal{A}_+^1 \ ;$$

(5.2) $$A(\alpha g + \beta h) = \alpha A(g) + \beta A(h) \quad \text{for } g, h \in \underline{\underline{G}}$$
and α, β real;

(5.3) Given $(g_n) \in \underline{\underline{G}}$ with $0 \leq g_n \uparrow g \in \underline{\underline{G}}$ then $U_{A(g_n)}^1 f \uparrow U_{A(g)}^1 f$ for all $f \in \underline{\underline{E}}^+$.

In the usual applications $(W, \mathcal{G})$ is either $(\Omega, \mathcal{F}^0)$ or $(E, \mathcal{E})$. Let $A = A(1) \in \mathcal{A}_+^1$ and let $\mathfrak{h}$ consist of all $\Gamma \in \mathcal{E}^1$ of A-potential zero. Then relative to the measurable space $(E, \mathcal{E}^1)$, $\mathfrak{h}$ is a collection of negligible sets of the form (4.3). It follows readily from (5.1) and (5.2) that for each $g \in \underline{\underline{G}}^+$, $A(g)$ is absolutely continuous with respect to A, and hence $A(g) = (Tg) * A$ where $Tg \geq 0$ is $\mathcal{E}^1$ measurable and one may suppose that $|Tg| \leq \|g\|$, $T1 = 1$. If $g \in \underline{\underline{G}}$ and we set $Tg = T(g^+) - T(g^-)$, then again $A(g) = (Tg) * A$. It is immediate that $T: \underline{\underline{G}} \to \underline{\underline{E}}^1$ is almost linear and almost positive relative to $\mathfrak{n}$. Finally it follows from (5.3) that T satisfies condition (3.3).

If $(W, \mathcal{G})$ is Lusin, then by (4.1) there exists a kernel, N, from $(E, \mathcal{E}^1)$ to $(W, \mathcal{G})$ that is bounded by one such that

$$A(g) = N(\cdot, g) * A \quad \text{for all} \quad g \in \underline{\underline{G}} . \tag{5.4}$$

If one only knows that $(W, \mathcal{G})$ is a U-space, then by (4.5) there exists a kernel N from $(E, \mathcal{E}^*)$ to $(W, \mathcal{G})$ such that (5.4) holds. If, however, all of the measures $U^1_A(x, \cdot)$ are absolutely continuous with respect to a single measure m, for example if X posseses a reference measure, then even when $(W, \mathcal{G})$ is only a U-space there exists a kernel N from $(E, \mathcal{E}^1)$ to $(W, \mathcal{G})$ such that (5.4) holds.

Of course, if (P_t) is Borel, then $\mathcal{E}^1$ may be replaced by $\mathcal{E}$ in all of the above statements.

6. Disintegration of Measures and Regular Conditional Probabilities.

Let $(\Omega, \mathfrak{F})$ be a U-space and $(E, \mathcal{E})$ a measurable space. We fix a measure ν on $(\Omega, \mathfrak{F})$ and a measurable map $\varphi\colon \Omega \to E$. Let $\mu = \varphi(\nu)$ be the image of ν under φ. Thus μ is a measure on $(E, \mathcal{E})$ and

(6.1)
$$\int g d\mu = \int (g \circ \varphi)\, d\nu$$

for all $g \in \underline{\underline{E}}$. If $f \in \underline{\underline{F}}$, $f\nu$ denotes the signed measure $A \to \int_A f d\nu$ on $(\Omega, \mathfrak{F})$. Let $\mu^f = \varphi(f\nu)$. Then μ^f is a signed measure on $(E, \mathcal{E})$, and clearly $\mu^f << \mu$. For each $f \in \underline{\underline{F}}$, let Tf be a density for μ^f with respect μ. We may assume that $Tf \in \underline{\underline{E}}$ for each $f \in \underline{\underline{F}}$ and that $T1 = 1$. If $\mathfrak{n} = \{B \in \mathcal{E}\colon \mu(B) = 0\}$, then it is evident that $T\colon \underline{\underline{F}} \to \underline{\underline{E}}$ is almost linear and almost positive relative to $\mathfrak{n}$ and that it satisfies condition (3.3). Consequently there exists a kernel K from $(E, \mathcal{E})$ to $(\Omega, \mathfrak{F})$, bounded by 1, such that $\mu^f = (Kf)\mu$ for all $f \in \underline{\underline{F}}$. Since $K1 = 1$ a.e., by replacing $K(x, \cdot)$ by unit mass at a fixed point ω_0 if x is not in $\{x\colon K(x, \Omega) = 1\}$, we may assume that $K(x, \Omega) = 1$ for all x in E. Combining this with $\mu^f = \varphi(f\nu)$ and (6.1) we find

(6.2)
$$\int K(x, f)\, h(x)\, \mu(dx) = \mu^f(h) = \int (h \circ \varphi)\, f\, d\nu$$

for all $f \in \underline{\underline{F}}$ and $h \in \underline{\underline{E}}$. When $h = 1$ this becomes

$$(6.3) \qquad \nu(\cdot) = \int K(x, \cdot)\ \mu(dx)\ .$$

In order to obtain the existence of regular conditional probabilities we specialize (6.3) as follows. Let $\mathcal{G}$ be a sub-σ-algebra of $\mathfrak{F}$. Take $(E, \mathcal{E}) = (\Omega, \mathcal{G})$ and φ the identity map from $(\Omega, \mathfrak{F})$ to $(\Omega, \mathcal{G})$. Then $\mu = \varphi(\nu)$ is just the restriction of ν to $\mathcal{G}$ and K is a kernel from $(\Omega, \mathcal{G})$ to $(\Omega, \mathfrak{F})$. This means that $\omega \to K(\omega, \Lambda)$ is $\mathcal{G}$ measurable for each $\Lambda \in \mathfrak{F}$, while (6.2) states that

$$\int K(\omega, \Lambda)\ h(\omega)\ \nu(d\omega) = \int_\Lambda h(\omega)\ \nu(d\omega)\ .$$

for all $\Lambda \in \mathfrak{F}$ and $h \in \underline{\underline{G}}$. Consequently $K(\omega, \cdot)$ is a regular conditional probability on $(\Omega, \mathfrak{F}, \nu)$ given $\mathcal{G}$.

Let us return to the general situation of (6.2) and (6.3). If we assume a bit more about $(E, \mathcal{E})$, then we can obtain more information about the kernel K. Let $\Delta = \{(x, y): x = y\}$ be the diagonal in $E \times E$ and assume that

$$(6.4) \qquad \Delta \in \mathcal{E} \otimes \mathcal{E}$$

where $\mathcal{E} \otimes \mathcal{E}$ is the usual product σ-algebra on $E \times E$. Now it follows from (6.2) that if $F(x, \omega)$ is a bounded $\mathcal{E} \otimes \mathfrak{F}$ measurable function on $E \times \Omega$, then

$$(6.5) \qquad \int \mu(dx) \int F(x, \omega)\, K(x, d\omega) = \int F[\varphi(\omega), \omega]\, \nu(d\omega) .$$

Under assumption (6.4), $F(x, \omega) = 1_\Delta(x, \varphi(\omega))$ is $\mathcal{E} \otimes \mathfrak{F}$ measurable, and with this F, (6.5) becomes

$$(6.6) \qquad \int K(x, \varphi^{-1}(x))\, \mu(dx) = \nu(1) = \int K(x, 1)\, \mu(dx) ,$$

where $\varphi^{-1}(x) = \{\omega\colon \varphi(\omega) = x\}$. Since $K(x, \varphi^{-1}(x)) = \int F(x, \omega)\, K(x, d\omega)$, it is clear that $x \to K(x, \varphi^{-1}(x))$ is $\mathcal{E}$ measurable, and it follows from (6.6) that $K(x, \cdot)$ is carried by $\varphi^{-1}(x)$ a.e. . Of course, by setting $K(x, \cdot) = 0$ on the set of x for which $K(x, \cdot)$ is not carried by $\varphi^{-1}(x)$ we obtain a kernel that is carried by $\varphi^{-1}(x)$ for all x. But this destroys the fact that $K(x, \cdot)$ is a probability for each x.

We close this section by indicating conditions under which one can choose a kernel K satisfying (6.2) and such that for all x in E both $K(x, \Omega) = 1$ and $K(x, \cdot)$ is carried by $\varphi^{-1}(x)$. We assume that E is a separable metric space and $\mathcal{E}$ the σ-algebra of Borel subsets of E. This guarantees that (6.4) holds. We assume that Ω is a Polish space and that $\varphi\colon \Omega \to E$ is a continuous surjection with the property that $\varphi(A) \in \mathcal{E}$ whenever A is closed in Ω. (If Ω is σ-compact (Polish), then every continuous surjection of Ω on E has this property.) Under these assumptions there exists a Borel cross section for φ (see Ch. IX, Sec. 6.8 of [3']). That is, there exists a Borel set $\Omega_0 \subset \Omega$ such that φ restricted to Ω_0 is a bijection of Ω_0 on E. Let $\psi = (\varphi|_{\Omega_0})^{-1}$ so that ψ is a bijection of E on Ω_0. But ψ is Borel from E to Ω_0, because if $A \subset \Omega_0$, $A \in \mathcal{F}$, then $\psi^{-1}(A) = \varphi(A) \in \mathcal{E}$ since φ is a continuous bijection of the Lusin space Ω_0 on E (see Ch. IX, Sec. 6.7 of [3]).

Armed with these facts it is easy to construct the desired kernel under the above assumptions. Let K be a kernel from $(E, \mathcal{E})$ to $(\Omega, \mathcal{F})$ satisfying (6.2) with $K(x, \Omega) = 1$ for all x and with $K(x, \cdot)$ carried by $\varphi^{-1}(x)$ a.e.. Let Γ be the set of those x such that $K(x, \cdot)$ is not carried by $\varphi^{-1}(x)$. Then $\Gamma \in \mathcal{E}$ and $\mu(\Gamma) = 0$. Defining $N(x, \cdot) = K(x, \cdot)$ for $x \notin \Gamma$ and $N(x, \cdot)$ to be unit mass at $\psi(x)$ for $x \in \Gamma$, it is evident that N is a kernel from $(E, \mathcal{E})$ to $(\Omega, \mathcal{F})$ with the desired properties.

References

1. J. Azema, Le retournement du temps II. To appear.

2. A. Benveniste et J. Jacod, Systèmes de Lévy des processus de Markov. Invent. Math. 21 (1973), 183-198.

3. N. Bourbaki, General Topology, Part 2. Hermann (1966). Paris.

4. R. K. Getoor and M. J. Sharpe, Last exit decompositions and distributions. Indiana Univ. Math. Journ. 23 (1963), 377-404.

5. B. Maisonneuve, Lévy exit systems. To appear in Ann. Prob.

6. P. A. Meyer, Le retournement du temps, d'apres Chung et Walsh. Strasbourg Sem. V. Lecture Notes in Math. 191. Springer (1971). Berlin.

7. P. A. Meyer, Ensembles aleatoires Markoviens homogenes II. To appear in Strasbourg Sem. VIII.

8. G. Mokobodzki, Relèvement Borélien compatible avec une classe d'ensembles négligeables. Application à la désintégration des mesures. Unpublished.

9. J. B. Walsh et P. A. Meyer, Quelques applications des résolvantes de Ray. Invent. Math. 14 (1971), 143-166.