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Séminaire de probabilités (Strasbourg), tome 5 (1971), p. 76
<http://www.numdam.org/item?id=SPS_1971__5__76_0>

Université de Strasbourg
Séminaire de Probabilités 1969/70

A SIMPLE PROOF OF DOOB'S CONVERGENCE THEOREM

by K.L. CHUNG

Doob's version of the fundamental convergence theorem of potential theory asserts that <u>if</u> (f_n) <u>is a decreasing sequence of excessive function and</u> f <u>is the supermedian function</u> $\inf_n f_n$, <u>then the set where</u> f <u>differs from</u> $\hat{f}$ (<u>its regularized function</u>) <u>is semi-polar</u>. Many beautiful proofs of this result are available in the literature. Here is a trivial one.

By truncation we can assume f_1 is bounded. Set $A=\{f\geqq\hat{f}+\varepsilon\}$, $\varepsilon>0$. Since f is fine u.s.c., $\hat{f}$ fine continuous, A is finely closed. Calling P_A the usual réduite operator, we have

$$f\geqq P_Af \geqq P_A(\hat{f}+\varepsilon) = P_A\hat{f}+P_A\varepsilon$$

(the first $\geqq$ follows from $f_n\geqq P_Af_n$, the second $\geqq$ from the fact that the measures $P_A(x,.)$ are carried by A). Applying the semi-group

$$P_tf \geqq P_tP_A\hat{f} + P_tP_A\varepsilon$$

As $t\to 0$, we get

$$\hat{f} \geqq P_A\hat{f} + P_A\varepsilon$$

At a point x regular for A, this implies the absurd inequality $\hat{f}(x)\geqq \hat{f}(x)+\varepsilon$. Thus A has no regular point, and $\{f>\hat{f}\} = \cup \{f\geqq\hat{f}+ \frac{1}{n}\}$ is semi-polar.

The same proof shows that if f <u>is nearly Borel</u> , <u>positive and such that</u> $f\geqq P_Kf$ <u>for every compact set</u> K (it is well known that f then is supermedian), <u>then</u> $\{f>\hat{f}\}$ <u>is semi-polar</u>. One must just take care to apply the above reasoning, not to A (which isn't known to be finely closed) but to $K\subset A$, compact. If x is regular for A one chooses an increasing sequence K_n of compact subsets of A such that $T_{K_n}\downarrow 0$ P^x-a.s. and gets the same contradiction as above.

As a matter of fact the latter version, given very recently by Getoor and Rao, is the one that I stumbled into. On going over the above proof with Sieveking he pointed out the even shorter cut to the original Doob version.