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KAI LAI CHUNG

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A SIMPLE PROOF OF DOOB'S CONVERGENCE THEOREM

by K.L. CHUNG

Doob's version of the fundamental convergence theorem of potential theory asserts that if  $(f_n)$  is a decreasing sequence of excessive function and  $f$  is the supermedian function  $\inf_n f_n$ , then the set where  $f$  differs from  $\hat{f}$  ( its regularized function ) is semi-polar. Many beautiful proofs of this result are available in the literature. Here is a trivial one.

By truncation we can assume  $f_1$  is bounded. Set  $A = \{f \geq \hat{f} + \varepsilon\}$ ,  $\varepsilon > 0$ . Since  $f$  is fine u.s.c.,  $\hat{f}$  fine continuous,  $A$  is finely closed. Calling  $P_A$  the usual réduite operator, we have

$$f \geq P_A f \geq P_A(\hat{f} + \varepsilon) = P_A \hat{f} + P_A \varepsilon$$

( the first  $\geq$  follows from  $f_n \geq P_A f_n$ , the second  $\geq$  from the fact that the measures  $P_A(x, \cdot)$  are carried by  $A$  ). Applying the semi-group

$$P_t f \geq P_t P_A \hat{f} + P_t P_A \varepsilon$$

As  $t \rightarrow 0$ , we get

$$\hat{f} \geq P_A \hat{f} + P_A \varepsilon$$

At a point  $x$  regular for  $A$ , this implies the absurd inequality  $\hat{f}(x) \geq \hat{f}(x) + \varepsilon$ . Thus  $A$  has no regular point, and  $\{f > \hat{f}\} = \cup \{f \geq \hat{f} + \frac{1}{n}\}$  is semi-polar.

The same proof shows that if  $f$  is nearly Borel, positive and such that  $f \geq P_K f$  for every compact set  $K$  ( it is well known that  $f$  then is supermedian ), then  $\{f > \hat{f}\}$  is semi-polar. One must just take care to apply the above reasoning, not to  $A$  ( which isn't known to be finely closed ) but to  $K \subset A$ , compact. If  $x$  is regular for  $A$  one chooses an increasing sequence  $K_n$  of compact subsets of  $A$  such that  $T_{K_n} \downarrow 0$   $P^X$ -a.s. and gets the same contradiction as above.

As a matter of fact the latter version, given very recently by Gettoor and Rao, is the one that I stumbled into. On going over the above proof with Sieveking he pointed out the even shorter cut to the original Doob version.