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REFLECTIONS ON FORMALISM AND REDUCTIONISM IN LOGIC AND COMPUTER SCIENCE

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Many logicians have now turned into applied mathematicians, whose role in Computer Science is increasingly acknowledged even in industrial environments. This fact is gradually changing our understanding of Mathematical Logic as well as its perspectives. In this note, I will try to sketch some philosophical consequences of this cultural and "sociological" change, largely influenced by Computer Science, by a critique of the role of formalism and reductionism in Logic and Computing.

In mathematics, we are mostly used to a dry and schematic style of presentation: numbered definitions and theorems scan the argument. This may add effectiveness and clarity, though nuances may be lost: the little space allowed here requires to stress effectiveness.

Themes - Three main aspects relating Computer Science and the logical foundation of mathematics will be mentioned below, namely

1. the growth of a pragmatic attitude in Logic
2. revitalization and limits of formalism and constructivism
3. the role of space and images.

1. Pragmatism in Logic

1.1 Tool vs. foundation. In Computer Science, Mathematical Logic is no longer viewed as a foundation, but as a tool. There is little interest in setting on firm grounds Cobol or Fortran, say, similarly as Logic aimed at founding Number Theory or Analysis by, possibly complete, axiomatic systems. The actual work is the invention of new programming languages and styles, or algorithms and architectures, by using tools borrowed from Mathematical Logic. This also may originate in attempts to base on clear grounds known constructs, but the ultimate result is usually a novel proposal for computing. Functional and Logic Programming are typical examples for this.

This "engineer's approach" in applied Logic is helping to change the philosophical perspective in pure Logic, as well.

1.2 An analogy: the completeness of mechanical systems. The interests in the foundation of mathematics, in this century, have been a complex blend of technical insights and philosophical views. We must acknowledge first that philosophical a priori, in Logic, may have had a stimulating role, in some cases. Yet ideologies and the blindness of many attitudes have been a major limitation for knowledge, an unusual phenomenon in scientific research. They have been the source of

- wrong conjectures (completeness, decidability, provability of consistency)
- false proofs, by topmost mathematicians (just to mention the main cases: an inductive proof of the consistency of Arithmetic, by Hilbert, refuted by Poincaré in 1904-05; a second attempt, based on a distinction of various levels of induction, debated by Hermann Weyl in the twenties).

The wrong directions taken by the prevailing formalist school may be understood as a continuation of a long lasting attitude in science and philosophy. On the shoulders of last centuries' giants, Newton and Laplace, typically, the positivist perspective believed in perfect and complete descriptions of the world by classical mechanics, namely by sufficiently expressive systems of partial differential equations. Similarly, in Logic, adequate axiomatic systems were supposed to describe completely Analysis or the whole of mathematics. Two levels of descriptions, both exhaustive, or complete in the sense of Logic: one of the world by mathematical equations, the other of mathematics by (finitely many) axioms.

Still, this posivistic vision, in Logic, was not compelled by the times. Hermann Weyl conjectured the incompleteness of number theory and the independence of the axiom of choice in "Das Kontinuum", in 1918. Poincaré rejected purely logical or linguistic descriptions as the only source for mathematics and stressed the role of geometric insight. As I will try to hint below, Poincaré's distinction between analytic work and the intuition of space as well as his approach to the foundation of mathematical knowledge may be today at the base of a renewed foundational work, similarly as his work on the three body problem is at the origin of contemporary mechanics (Lighthill[1986]).

1.3 **Foundation of mathematical knowledge**. Another mathematician should be quoted among those who did not except the reductionist and formalist attitudes, in the first part of this century: Federico Enriquez. Also in Enriquez's philosophical writings, the interest in the interconnections of knowledge and in its historical dynamics suggested more open philosophical perspectives. It may be fair to say that Poincaré, Weyl and Enriquez were interested in the foundation of mathematical knowledge more than in the technical foundation of mathematics.

The difference should be clear: by the first I mean the epistemology of mathematics and the understanding of it "as integral part of the human endeavor toward knowledge" (to put it in Weyl's words). Not a separated transcendence, isolated in a vacuum, but an abstraction emerging from our concrete experience of the real world of relations, symmetries, space and time (Poincaré[1902, 1905], Enriquez [1909], Weyl[1918, 1952]).

In contrast to this broad attitude, the purely technical, internal foundation, as pursued by formalistic and reductive programs in their various forms, somewhat reminds of a drawing of a book on a table and of the belief that the table really supports the book, while they are designed by us by the same technique and the same tools. This cognitive circularity is at the source of the negative results in Logic.

1.4 **The unity of formal systems.** The first step towards a more open attitude comes with the need for a variety of systems and the understanding of their interconnections. The laic attitude inspired by applications and the suggestions coming from geometric intuitions are at the origin of recent inventions of new systems of Logic, where unity is given not by a global, metaphysical, system, but by the possibility of moving from a system to another, by changes in the basic rules and by translations or connecting results. Indeed, Girard's focus on the structural rules in Girard[1987] and his seminal work in Girard[1992] are largely indebted to a pragmatic attitude that views Logic as part of (applied) mathematics, with no special "meta-status." In this perspective, geometric structures and applications suggest formal systems, guide toward relevant changes, propose comparisons, in the common mathematical style where connections and bridges preclude ideological closures within one specific frame. In this sense, its unity is a deep mathematical fact, as much as Klein's unified understanding of Geometry.

2. Formalism and Constructivism in Computer Science (and their limits).

2.1 **Linguistic notations.** The volume in Combinatory Logic by Curry and Feys contains many pages on the renaming of bound variables and related matters (in set-theoretic terms, $\{x \mid P(x)\}$ is the same as $\{y \mid P(y)\}$). I believe that the foundational relevance of these pages, if any, may be summarized in about three lines. The formalist treatment is a typical example of purely symbolic manipulation, where meaning and structures are lost (see Curry's book on a formalist foundation of mathematics for an extremist's view in that direction). This opinion is shared by all working mathematicians who simply ignored the discussion and explain the problem to students in twenty seconds, on the blackboard. Still, it happened that variable binding is a crucial issue in computer programming. Thus, the discussion in Curry et al. has been largely and duly developed in functional programming and it is at the core of detailed treatments in implementations.

This example is just a small, but typical one, of the revitalization of formalism and constructivism due to Computer Science. It happens that computers proceed as our founding fathers of Logic described the parody of mathematics: linguistic definitions and formal deductions, with no meaning as a guidance. Meaningless, but effective constructions of programs, more than unifying insights and concepts. Mathematical invariants are lost, but denotations are very precise. This requires technically difficult insights into pure calculi of symbols and, sometimes, brand new mathematics. However, the branching of methods and results, due to translations and meaning, which are at the core of knowledge, may be lost within extremely hard, but closed, games of symbols. This is part of everyday's experience on the hacker's side even in theory of computing.

2.2 **Denotational semantics**. Fortunately, though, even programming has been affected by meaning. In the last twenty and odd years, various approaches to the semantics of programming languages embedded programming into the broader universe of mathematics. Here is the main merit of the Scott-Strachey approach, as well as of the algebraic or other proposals, most of which are unifiable in the elegant frame of Category Theory (this is partly summarized in Asperti&Longo[1991]). In some cases, the meaning of formal systems for computing, over geometric or algebraic structures, suggested variants or extensions of existing languages. More often, obscure syntactic constructs, evident at most to the authors, have been clarified and, possibly, modified. As a matter of fact, in the last decade, computer manuals have slowly begun to be readable, as they are moving towards a more mathematical style, that is towards rigor, generality and meaning at once. We are not there yet, as most hackers think in terms of pure symbol pushing and are supported in this attitude by the formalist tradition in Logic. Many still do not appreciate from mathematics that the understanding and, thus, the design of a strictly constructive, but complicated system may also derive from highly non constructive, but conceptually simple, intellectual experiences.

2.3 **Resources and memory**. Brouwer, the founding father of intuitionism, explicitly considers human memory as "perfect and unlimited," for the purposes of his foundational proposal (Troelstra[1990]). This is implicitly at the base of the formalist approach as well. Indeed, computer memories are perfect and, by a faithful abstraction, unlimited. This has nothing to do with human memory, and mathematics is done by humans. One component of mathematical abstraction, as emerging from our "endeavour towards knowledge," may precisely derive from the need to organize language and space in least forms, for the purposes of memory saving. A "principle of minima" may partly guide our high level organization of concepts, Sambin[1990]. Moreover, imperfection of storage is an essential part of approximate recognition, of analogy building.

For the aim of founding mathematical knowledge, we need exactly to understand the emergence of abstraction, the formation of conceptual bridges, of methodological contaminations between different areas of mathematical thinking. There may be a need for psychology and neurophysiology in this approach. Good: the three mathematicians quoted in 1.3 have been often accused of "psychologism", of "wavering between different approaches", in their foundational remarks. The proposed "one-way" alternatives lead to the deadlock where formalism and reductionism brought us in understanding mathematics (and the world). Moreover, so much happened in this century in other areas of knowledge, that we should start to take them into account.

2.4 **Top down vs. bottom-up**. There is no doubt that formalism and reductionism have been at the base of Computer Science as it is today and of its amazing progress. In particular, top-down deductions and constructive procedures set the basis for the Turing and Von Neuman machines as well as for all currently designed languages, algorithms and architectures. Yet, there is a growing need to go beyond top-down descriptions of the world, even in Computer Science. The recent failures of strong Artificial Intelligence are the analogue of incompleteness and independence results in Logic: most phenomena in perception and reasoning escape the stepwise-deductive approach. Partly as a consequence of these failures, there is an increasing interest in bottom-up approaches. Relevant mathematics is being developed in the study of the way images, for example, organize space by singularities, or how the continuum becomes discrete and reassembles itself, in vision or general perception, in a way which leads from quantitative perception to qualitative understanding, Petitot[1992].

3. Space and Images

3.1 Denotations and Geometry. In the practice of mathematics, formal notations and meaning are hardly distinguished. Indeed, one may even have symbolic representations where, besides the geometric meaning, there exists a further connection to Geometry, at the notational level. Relevant examples of this are given by Feyman's and Penrose's calculi or by Girard's proof nets.

In Feyman's calculus, planar combinations of geometric figures allows computations representing subatomic phenomena. Penrose's extends familiar tensorial calculi over many dimensional vector spaces in a very powerful way: bidimensional connections between indexes explicitly use properties of the plane to develop computations. A more recent example may be found in Girard's Linear Logic. In this system, formal deductions are developed by drawing planar links between formulae in a proof tree. Proofs (and cuts) are carried on by an explicit use of the geometric representation, by modifying the links. As the proof-theoretic calculus is essentially complex, according to recent complexity results, the use of the geometric representations comes in as an essential tool for the formal computation.

In a sense, all these calculi derive from (physical) space or Geometry and, after an algebraic or syntactic description, end up in geometric representations, possibly unrelated to the original one. As a matter of fact, even Linear Logic originated in Geometry, as it was suggested by the distributive or linear maps over coherent spaces (Girard[1987]), and ends up into a Geometry of proofs.

3.2 Geometric insight. I would like to mention here the possible relations of the novel mathematical approaches to vision mentioned in 2.4, and similar ones in other forms of perception, to the wise blend of linguistic, or analytic, and geometric experiences required by the practice and the foundation of mathematics.

It should be clear that, in mathematics, synthetic explanations may provide an understanding and a foundation as relevant as stepwise reductionist descriptions. The drawing on a blackboard may give as much certainty as the search for least axioms for predicative Analysis. The point now is to understand what is behind the drawing, which intellectual experiences give to it so much expressiveness and certainty. The point is to turn this practice of human communication, by vision and geometric insight, into a fully or better understood part of knowledge. This is where our endeavor towards general knowledge cannot be separated from the foundation of mathematics. Mathematics is just a topmost human experience in language and space perception, unique as for generality and objectivity, but part of our relation to the world.

A few examples may be borrowed from neurophysiology (see Ninio[1991], Maffei[1992]). It seems that only human beings perform interpolations (vertices of a triangle or of a square seen as complete figures, sets of stars as a constellation...). Apparently this is done by minimal lines which complete incomplete images. We seem to interpolate by splines, when needed. More: there are neurons which recognise (send an impulse) only in presence of certain angles, or others which react only to horizontal or vertical lines. This is recomposed in intellectual constructions which are at the base of our everyday vision and of the so called optic illusions (which are just attempted reconstructions of images). And, why not, at the base of our geometric generalizations.

But how? How can we make this "composition of basic mental images" as part of a new foundation of mathematical knowledge, in the same way as formal, linguistic axioms, have been describing part of the analytic developments in mathematics?

I can only mention the problem, for the time being, and stress what is really missing, the possible source of incompleteness: the lack of Geometry and images in foundational studies. A modern rediscovery of these aspects may be at the core of an understanding of image recognition which goes together with an appreciation of geometric abstraction in mathematics.

In a sense we should enrich the insufficient attempts to deduce all of mathematics by linguistic axioms, by adding, at least, the knowledge we have today of space perception and of the process of image formation. This may help to focus the way in which mathematics emerges, surely by compositions of elementary components (the lines and triangles I mentioned before), but also by "synthesis" and reorganization of space, as mentioned in 2.4. In this, a renewed Artificial Intelligence, far away from the prevailing formalist one, may be a novel contribution of Computer Science to the foundation of mathematical knowledge, and conversely. The difficult point is to be able to move, in foundational studies and everyday's work, from local, quantitative and analytic approaches to global, qualitative and geometric perspectives and still preserve the crucial (informal) rigor of mathematics.

3.3 **The continuum and minima: more about reductionism.** In "Das

Kontinuum" Weyl[1918], Hermann Weyl raises the issue of the continuum of Analysis vs. the continuum of time. The understanding of the latter is based on the simultaneous perception of past, present and future. In this irreducible phenomenological intuition of time it is not possible to isolate the temporal point, in contrast to the analytic description where this can be done by reduction to linguistic abstractions, that is to symbols and (derivable) properties. Weyl, a mathematician working also in relativity theory, expresses his unsatisfaction and raises a major point for mathematical knowledge: the convenient analytic unity of space and time does not correspond to our fundamental experience (see also Petitot[1992]). This problem has not been sufficiently studied since then, as we were mostly concerned by formalist reductions and the search for complete and (self-)consistent Set Theories, as a basis for Analysis. These formal theories have not been able to tell us anything even about the cardinality of an arbitrary set of reals (independence of the Continuum Hypothesis), let alone the profound mismatch between time and the analytic description of space, as given by the real line.

This need of ours to "fill up the gaps", possibly by continuity, may probably go together with the principles of minima, mentioned in 2.3 and 3.2 as a possible description of some aspects of abstraction (memory optimization and the formation of images, respectively). These principles are usually very complex in mathematics and, when referring to them, we depart from reductionism. Yet another relevant mathematical experience then, to be added to the continuum of time, which seems to escape reductionism. Reductions are surely a relevant part of scientific explanations, however they are far from proposing complete methodologies or providing the only possible foundation of knowledge.

In conclusion, we need to focus on alternative approaches to formalism and reductionism both in applied as well as in theoretical approaches to cognition. In 2.4 and 3.2, the role is mentioned of current inverse paradigms with respect to the prevailing top-down, deductive formalizations: bottom-up descriptions, for example, which may give a complementary account of perception and conceptual abstraction. What really matters now is to extend, not to keep reducing our tools. Our rational paradigms must be made to comprehend the mathematical, indeed human, intuition of space and time. In other words, we need to lower the amount of magic and mystery in these forms of intuition, and bring them into the light of an expanded rationality.

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