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The Tragedy of Grassmann

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The Tragedy of Grassmann

I. GRASSMANN'S LIFE

In the whole gallery of prominent mathematicians who, since the time of the Greeks, have left their mark on science, Hermann Grassmann certainly stands out as the most exceptional in many respects. When compared with other mathematicians, his career is an uninterrupted succession of oddities: unusual were his studies; unusual his mathematical style; highly unusual his own belated realization of his powers as a mathematician; unusual and unfortunate the total lack of understanding of his ideas, not only during his lifetime but long after his death; deplorable the neglect which compelled him to remain all his life professor in a high-school ("Gymnasiallehrer") when far lesser men occupied University positions; and the shroud of ignorance and uncertainty still surrounds his life and works in the minds of most mathematicians of our time, even when they put his original ideas to daily use.

Against this succession of failures should be pitted all the gifts which he had received from nature: his indomitable patience and energy, his incredible capacity for work of all kinds and at top speed, his encyclopaedic curiosity and ability to become proficient in any subject of his choice, and of course, last but not least, the startling originality of his mathematical ideas. One is tempted to repeat for him the fairy story which A. Weil recently imagined regarding Grassmann's younger contemporary Eisenstein,† in which, after beneficient fairies have bestowed on the cradle of the newborn infant all the gifts in their possession, a malevolent witch nullifies everything; and

† A. Weil, *Bull. Amer. Math. Soc.* **82** (1976), 658.

even if Grassmann's fairly long life gives him some advantage over Eisenstein (he died at 68), the fate of his ideas among contemporaries and posterity has been incomparably worse.†

Hermann Grassmann was born in 1809 in Stettin (East Prussia), the third of the 12 children of Justus Grassmann, who was professor at the Gymnasium of that city. It is important here to put some emphasis on what were the duties of a Gymnasium professor in Germany at that time, and in fact all through Grassmann's lifetime: he was supposed to lecture on all subjects, from religion to biology through latin, mathematics, physics and chemistry, and at all levels, at a rate of 18 to 30 hours a week; and of course there were a lot of additional duties such as examinations, administrative work, etc. Hermann naturally was a pupil in his father's school, where soon he developed a lifelong interest in philology, and a desire to become a Lutheran minister; he therefore went to Berlin after his school years, to study theology and philology for 6 semesters at the University. Back in Stettin in 1830, he immediately started a career as professor, at first in an institution different from his father's school. Despite a few attempts to obtain a University position, after he had published his main mathematical papers, he was to remain "Gymnasiallehrer" all his life, in various schools of Stettin (except for a short period in Berlin in 1835), finally succeeding his father after the latter's death in 1852.

For a long time he clung to his idea of becoming a minister, submitting himself to several aptitude tests until around 1840, when his awakening interest in mathematics led him to abandon the idea altogether. He married rather late in life, but had 11 children, of whom 7 survived him, and in spite of his many-sided interests and duties, he managed to be a devoted father and family man; many evenings in his home were spent in reading aloud and musical activities, for music was one of his permanent interests: he directed a chorale in his school, and went as far as harmonizing folk songs, of which he was very fond. But music was only a small item in the fantastic range of his activities: among his publications are papers on the theory of colors, on the theory of sound, on a new heliostat of his conception, plus a reading primer and a book on elementary arithmetic; and one wonders how he found time to regularly attend the meetings of his Freemason lodge, or, for some years, to be quite active in a society founded for the purpose of bringing the Gospel to the Chinese! And of course we have left aside the two interests which dominated his life after 1840: mathematics, and (after he had become disappointed over the poor reception of his ideas) philology,

† How potent the curse was can be judged by two recent facts: Grassmann's name is not mentioned in the recent Japanese *Encyclopedic Dictionary of Mathematics*; and the editors of the monumental *Dictionary of Scientific Biography*, which often devotes several pages to nonentities, apparently forgot to include Grassmann's name, until somebody made them aware of their blunder, which they had to repair in a *Supplement* to the Dictionary!

especially sanskrit and Indo-European languages, in which he rapidly became an expert and attained a recognition which had been refused to his mathematical works.

II. THE GREAT IDEAS

The invention of cartesian coordinates had brought about great progress in Analysis, Mechanics and Geometry itself during the eighteenth century. But at the end of that period, more and more people deplored the long (and mostly awkward) computations often needed to prove by the method of coordinates (which was given the name "analytic geometry" at the end of the century) geometrical results of a very simple nature. But a return to the traditional "euclidean" methods was not very appealing either to the contemporaries, due to the necessity of considering many different cases in a problem, according to the respective positions of the geometric objects one was considering: it was for instance unpleasant to have to write in different ways the relation between the segments having extremities at three points A, B, C on a line, according to the position of C relatively to the segment AB. There was therefore, around 1800, a general dissatisfaction with Geometry as it was then practiced, and a longing for some "third way" which could deal directly with geometric objects without recourse to irrelevant "coordinates", but would also be free from the euclidean shackles.

These general trends were to lead to two different developments: on one hand the invention of complex projective geometry, with the enormous simplification and elegance it brought to "synthetic" geometry, and which aroused general enthusiasm during the whole nineteenth century; on the other hand the conception of *vectors*, first in the plane and ordinary space, and then in n dimensions, which later would give rise to our linear algebra, but had much less immediate success. Gauss of course had been familiar with the concept of vector in the plane ever since he had used the geometric representation of complex numbers around 1796, but he published nothing on the subject until 1831; in 1832, Bellavitis independently arrived at the same concept, and Möbius had been using an equivalent notion, adapted to affine geometry in 2 and 3 dimensions, since 1827; but nothing else appeared on the subject until 1843.

Grassmann had not learnt much mathematics beyond the high-school curriculum when, after his return from Berlin in 1830, he began to take interest in Geometry, in connection with the examinations he had to take to improve his position as teacher. He soon came to feel the same dissatisfaction about its methods, although it is unlikely that he was much influenced by others; he did not become aware of the existence of Möbius's Barycentric

Calculus until around 1841, and only read Bellavitis and Gauss much later; it is therefore very likely that he came to the concept of vector entirely by himself around 1832. During the next few years he does not seem to have tried to develop this idea, being occupied by his study of theology and a new interest in cristallography. What brought him back to mathematics was an examination which he had to prepare in 1839 in order to obtain a promotion as "Hauptlehrer" in a new school. He had to submit a paper on the theory of tides, for which he went to Laplace's "Mécanique céleste", and from there was led back to Lagrange's "Mécanique analytique", where he realized that his calculus of vectors greatly simplified the exposition. This was the spark which determined him to deepen and develop his ideas into a complete system, and in less than 2 years he had finished the manuscript of *Die Ausdehnungslehre*, which was published in 1844.

This first edition has been reprinted in Grassmann's *Complete Mathematical Works* [G], published with many notes and comments by F. Engel and E. Study from 1894 to 1911, but unfortunately there has been no recent detailed study or evaluation of that extraordinary book, which certainly would deserve a monograph. With our streamlined linear and multilinear algebra of today, it is not too hard to understand precisely what notions Grassmann had in mind, but even for a modern mathematician the book makes heavy reading. The main reason is that, even by the standards of the time, it was not a book of mathematics; there are no definitions in the usual sense of the word, and very few genuine proofs. What Grassmann does is to describe his *vision* of new objects, in a manner quite similar to Riemann's famous papers on Riemann surfaces and n-dimensional multiplicities 10 years later, but in a language still more abstract.

The vision certainly is impressive, especially when contrasted against the background of what was understood by algebra and geometry in contemporary mathematics; very few people were familiar with the concept of vector, and that was limited to 3 dimensions, the only operations on vectors being addition and multiplication by a scalar. And here, all at once, one was propelled into a new world of arbitrary dimension, where besides vectors a whole panoply of unheard of geometric objects, the multivectors, had to be combined by non-commutative operations†; no wonder that even for

† Ever since the beginning of the nineteenth century, non-commutativity had become familiar in some contexts, such as composition of functions. It is a remarkable coincidence that Hamilton's discovery of quaternions occurred precisely during the time Grassmann's book was being printed; this of course led him to consider for any integer $n > 3$ the set of all n-tuples of real numbers as a generalization of ordinary space, but essentially from an algebraic point of view. But neither he nor Cayley (who independently started at the same time to use some geometrical language when dealing with n-tuples of numbers) ever came to the concept of an intrinsically defined vector space. One should also mention that in 1845, de Saint-Venant arrived independently at the idea of exterior product of vectors, but only in 3 dimensions.

mathematicians who, like Möbius, were best prepared to understand these generalizations, *Die Ausdehnungslehre* remained a book of seven seals.

Grassmann's father seems to have had more than a passing interest for the foundations of geometry; he published a textbook in which, in particular, he reflected on the way a segment on a line could be conceived as "generated" by a moving point, and a rectangle as "generated" by a moving segment, parallel to one side, and having one of its extremities moving on the other side; he considered this "generation" as a "geometric product" of the two sides, involving more than the usual computation of the area as the arithmetic product of their lengths ([G], I_2, p. 507). This idea struck deep roots in Grassmann's mind; the passage from a rectangle to a parallelogram, and then to 3 dimensions, was easy enough, but could one continue this process of "generation"? He decided it could be done, provided one replaced the usual geometric "intuition" by a more abstract one, in which unspecified "generating elements" would be susceptible of "continuous variation" according to a certain "law", giving birth to what he called an *extensive form of the first level* ("Ausdehnungsgebilde erster Stufe") conceived as the set ("Gesamtheit") of the elements deduced from the "generating elements" by its variation. If there were p "independent laws" of variation, the set of elements deduced by all these different variations (combined in all possible ways) from a "generating element" would similarly be an *extensive form of the p-th level*, and should be considered as the *product* of the p forms of the first level corresponding to each of the "laws". We are clearly witnessing here what is probably one of the first attempts to do mathematics on objects which have no existence (nor even approximate "representatives" as the objects of classical geometry) outside of the mind (objects "durch das Denken *gewordenen*" as Grassmann says, or "Gedankendinge" as Hankel will say 20 years later).

Grassmann repeatedly insists (e.g. [G], I_1, p. 46) on the fact that classical geometry merely gives examples of his "new science" for dimensions $\leqslant 3$, but that it really is irrelevant to its development. Nevertheless, it is clear that he was guided by geometric intuition when he defined algebraic operations on his "Ausdehnungsgrösse" (to which we shall from now on give their modern name of *p-vectors* and which we will write in modern notation). For instance, when he studies the exterior product $a \wedge b$ of two vectors, he imposes on it the distributive law $a \wedge (b_1 + b_2) = a \wedge b_1 + a \wedge b_2$ which he justifies by the diagram of parallelograms (Figure 1) for which he writes $(abfe) + (efdc) = (abdc)$, showing what he had in mind by addition of bivectors ([G], I_1, p. 84). Similarly, the relation $(a+b) \wedge b_1 = a \wedge b_1$, when the vectors b and b_1 are colinear, is pictured by another diagram of parallelograms (Figure 2). From these two relations, he deduces ([G], I_1, p. 87) that $b \wedge b_1 = 0$ for colinear vectors, and then, using distributivity, he concludes from $(a+b) \wedge (a+b) = 0$ the fundamental anticommutativity $b \wedge a = -a \wedge b$.

Such is the starting point from which Grassmann endeavors to develop simultaneously the theory of n-dimensional vector spaces and exterior algebra in what we now would call an *intrinsic* way, without any mention of coordinates, in a rambling and unsystematic fashion, interspersed with many

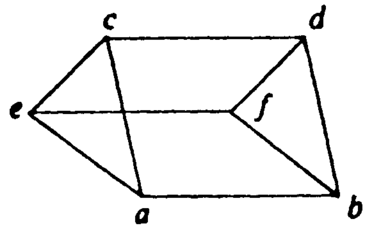

FIGURE 1

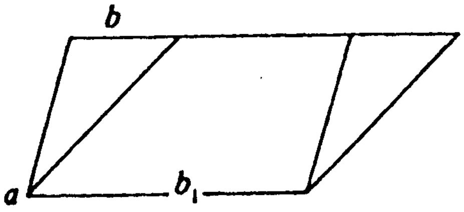

FIGURE 2

application to Geometry, Mechanics and Physics. We can check that he keeps a clear intuition of what he is trying to do, as for instance when he writes for the first time the relation

$$\dim V + \dim W = \dim (V \cap W) + \dim (V + W) \tag{1}$$

(without of course using that notation) for two vector subspaces V, W of a vector space ([G], I_1, p. 209); he also has the notion of *basis* for a finite dimensional space. But one must acknowledge that, even disregarding the vagueness of definitions and proofs, what we have in this first edition is a bare embryo of the exterior algebra of today. In sharp contrast with Hamilton the algebraist,† Grassmann is primarily interested in n-dimensional geometry, and not in algebra: the only p-vectors which he really studies are the "pure" or "decomposable" ones, exterior products of p linearly independent vectors, spanning a p-dimensional vector subspace in an n-dimensional vector space; these subspaces are clearly the objects he wants to deal with, to which the decomposable p-vectors "up to a scalar factor" should correspond as algebraic objects, just as vectors "up to a scalar factor" correspond to 1-dimensional subspaces. He realizes of course that, just as vectors, p-vectors must be added, and he knows that for $2 \leqslant p \leqslant n-2$, the sum of two decomposable p-vectors needs not be decomposable; but he dismisses the difficulty by saying that in these cases the concept of addition is "purely formal" ([G], I_1, p. 108), and he does not even bother to compute the number of linearly independent p-vectors for arbitrary n and p!

Furthermore, the way in which Grassmann presents the applications of his "new science" smacks a little bit of "sales talk". As Engel honestly points out,

† It is quite symptomatic that Hamilton, who spent 20 years of his life looking for applications of his quaternions to all kinds of problems, published lots of papers dealing with applications of quaternions to "geometry", but by that he always meant geometry in 3 dimensions, not in 4 dimensions [H]!

they really are only simplifications of notations, which certainly are valuable, but do not bring genuinely new results. For instance, Grassmann realizes that a system of n linear equations in n unknowns can be written as a single equation between vectors

$$x_1 p_1 + x_2 p_2 + \ldots + x_n p_n = p_0 \quad (2)$$

where the x_j are the unknown scalars and the p_j are vectors; by multiplying on both sides by the $(n-1)$-vector $p_1 \wedge \ldots \wedge \hat{p}_j \wedge \ldots \wedge p_n$, he gets the formulas

$$x_j(p_1 \wedge p_2 \wedge \ldots \wedge p_n) = p_1 \wedge \ldots \wedge p_{j-1} \wedge p_0 \wedge p_{j+1} \wedge \ldots \wedge p_n \quad (3)$$

for $1 \leqslant j \leqslant n$, which are of course the Cramer formulas, written in a much more pleasant form.† At the end of the nineteenth century, not much more could be credited to exterior algebra, and Engel concluded his biography on a melancholy note, relegating Grassmann to the rank of unheeded prophet of what was then the fashion among mathematicians (and has remained as popular with present day physicists), the horrible "Vector analysis", which we now see as a complete perversion of Grassmann's best ideas.‡ It is only in our time that Grassmann's faith in the value of his ideas has been completely vindicated by genuine applications, which have made exterior algebra an indispensable tool of modern mathematics in ways he could not have foreseen: first of all E. Cartan's calculus of differential forms, which is now the basis of Differential geometry and of the theory of Lie groups; and second the definition of the Grassmannians as projective algebraic varieties, and the realization that their algebraic and topological structures hold the key to many results of differential topology and algebraic geometry.§

III. THE MISSING DUALITY

Having practically no contact with other mathematicians of his time, Grassmann was expecting a reception for his book quite different from the one he got. The only one who made some effort to understand at least some

† In 1853, Cauchy, in a series of Compte-rendus Notes, developed (without any geometric connotation) a similar symbolism, which he called "Clefs algébriques". Although de Saint-Venant, in a subsequent Compte-rendus Note, observed the analogy of Grassmann's book with Cauchy's Notes, it is unlikely that Cauchy had ever read Grassmann.

‡ It is limited to 3 dimensions, replaces bivectors by the awful "vector product" and trivectors by the no less awful "mixed product", notions linked to the *euclidean structure* and which have no decent algebraic properties!

§ The idea to consider the Grassmannian for $n = 4$, $p = 2$ as a quadric hypersurface in $\mathbf{P}_5(\mathbf{C})$ is due to Klein (1872); it was extended to all Grassmannians by Severi in 1915. The first study of the topology of the Grassmannians was made by C. Ehresmann in his thesis (1934).

of his ideas was Möbius,† with whom he started a correspondence which lasted (with long interruptions) for about 20 years. But Gauss was obviously repelled by a style so different from his own; and Kummer, who was called upon to write a report on the scientific achievements of Grassmann, was very unfavorably impressed by the obvious defects of *Die Ausdehnungslehre*: he clearly thought he had to deal with a man who generalized for generality's sake (a species which already existed at that time, although it was not so widespread as it is today), and he had against such people the usual reaction of a mathematician who knows what hard work means.

Grassmann had planned a second volume of his *Ausdehnungslehre*, but he did not publish it; its substance was incorporated (in a shorter form) in a manuscript which he sent in 1846 to the Leipzig Academy, to enter the competition for a prize proposed on the theme of Leibniz's attempts for forge a "geometrical analysis" (it is likely that the theme had been suggested by Möbius, in order to have the prize given to Grassmann). In that paper, Grassmann gave a summary of the results published in his book of 1844, and in addition, in order to treat problems of Euclidean geometry, he introduced a new notion (which he had announced in the introduction to his book): this was the *scalar product* of two vectors which appears there for the first time for spaces of arbitrary dimension.‡ Even more interesting are Grassmann's attempts to define, in the same paper, an "interior product" of a bivector and a vector, which he was to expand later.

After he had been made aware (by Möbius) in 1853 of Cauchy's work on his "clefs algébriques", Grassmann began to contemplate a new edition of his book, which he finally published (at his own cost) in 1862. This new *Ausdehnungslehre* is, at least externally, very different from the first edition. Belatedly realizing that he had no chance of being heard if he persisted in his way of writing mathematics (as Möbius had warned him), Grassmann reluctantly abandoned his idea of defining vector spaces and exterior algebra intrinsically, and starts as Hamilton (although independently) by defining a vector space of dimension n as the set of linear combinations of n "ursprüngliche Einheiten" $e_1, \ldots, e_n$ assumed to be linearly independent. We certainly can understand his reluctance, but we clearly see that, in the absence of the set theoretical language which enables us to give the intrinsic definitions which he vainly sought, he had no other alternative open to him. But where he stands quite apart of Hamilton and Cayley is in his constant striving to obtain geometric notions and results independent of any choice of

† Möbius took the trouble to write an Appendix to Grassmann's prize paper on "geometrical analysis" (see below) to make its reading easier, but he did not go beyond the notions of scalar product of vectors in 3-dimensional space.

‡ In dimensions ≤ 4, this notion naturally derived from Hamilton's theory of quaternions, as the scalar part of the product of two quaternions; but it is unlikely that Grassmann read Hamilton until well after 1860.

basis: the "Austauschsatz" (exchange theorem) which one generally attributes to Steinitz (1910) is stated and proved quite correctly without using determinants ([G], I_2, p. 19), from which he deduces (as we now do) the invariance of dimension, as well as a clear proof of relation (1) ([G], I_2, p. 21).

Of course, in Grassmann's plan, these are only preliminaries to the bulk of his book, the definition and study of his various "products". But if for a moment we bypass this part, we are amazed to discover that at the end of the book, as "applications" of his methods, he introduces "intrinsic" conceptions which have only become familiar to most mathematicians in a very recent past. Remember that at that time the general concept of a mapping of a set into a set has not yet been formulated; but Grassmann knows how to define linear and multilinear mappings without using coordinates. For instance, he writes $[l \mid a]$ the linear form which we would now write $x \mapsto (x \mid a)$ (scalar product); the letter l stands for a place to be filled by a variable vector, and he calls that a "Lückenausdruck". Similarly, he defines an endomorphism of an n-dimensional vector space E by considering a basis $a_1, \ldots, a_n$ of E, and the images $b_1, \ldots, b_n$ of the a_j's by the endomorphism, which he writes as a "quotient"

$$Q = \frac{b_1, \ldots, b_n}{a_1, \ldots, a_n}$$

or as a linear combination $\Sigma_{j=1}^{n} b_j[l \mid a_j]$ of "Lückenausdrücke" (what we now write $\Sigma_{j=1}^{n} b_j \otimes a_j^*$, using the basis (a_j^*) dual to (a_j)). He also defines what we now write $\wedge Q$ for an endomorphism Q, and finally he is the first to show that any endomorphism of a complex vector space can be expressed by a triangular matrix with respect to a convenient basis ([G], I_2, p. 256).

A little further, we see that Grassmann already had the idea of a vector space whose elements would be functions, and that he writes f for a function, instead of $f(x)$ as everybody did at the time.† When he deals with differential calculus, and studies a differentiable mapping f of a finite dimensional vector space E into itself, he considers, as we do today, the tangent mapping $f'(x)$ as an endomorphism of E (whereas all his contemporaries only dealt with Jacobian determinants). Finally, at the end of the second *Ausdehnungslehre*, he applies his method to the Pfaff problem of classification of differential forms, going beyond the results Clebsch obtained at the same time, and arriving at a conclusion which is substantially equivalent to what is usually called Darboux's theorem ([G], I_2, p. 493).

It is almost unbelievable that such an original work should have been entirely ignored by contemporary mathematicians; Hankel was the first to "discover" it in 1867; after 1870, several well-known mathematicians, such

† He also vehemently insists on the fact that a function can only have *one* value for a given value of the variable ([G], I_2, p. 224).

as Klein and Lie, began to mention Grassmann, but, as Engel points out, rather from hearsay than from having really read his works. The usual obdurate conservatism of most mathematicians, who will gladly wade through oceans of computation rather than relinquish the techniques they learnt at the University, is certainly to blame to a large extent: they had been taught to work with determinants, they would go on working with determinants for another 60 years! However, it must be acknowledged that Grassmann himself has to shoulder part of the blame; had he limited himself to a short exposition of his exterior product, with the applications mentioned above, he might have had better success. But the bulk of the second *Ausdehnungslehre* describes at length the various "products" which he thought necessary to introduce in addition to the exterior calculus; this is both the triumph and the failure of Grassmann, namely a lot of wonderful ideas rendered ineffective by lack of the necessary technique, and ending up in a frightful mess.

We already have stressed the fact that Grassmann was entirely led by geometric insights, and that the objects he wanted to introduce in algebra were the vector subspaces of a finite dimensional space E. He had early observed that when V and W are two such subspaces such that $V \cap W = \{0\}$, if they are spanned by decomposable multivectors a and b, their "join" $V + W$ is spanned by the exterior product $a \wedge b$. But what should be done if $V \cap W \neq \{0\}$, for then $a \wedge b = 0$, although $V + W$ still exists; and how can one obtain a multivector spanning $V \cap W$ in a similar way? Grassmann never solved the first problem, but he had a bright idea for the second, at least in the special case in which $V + W = E$. He had a clear notion of the extension of projective duality to n-dimensional spaces (at a time when everybody considered "geometry" to be limited to 3 dimensions). Suppose some duality has been defined in E, and that V' and W' respectively correspond to V and W by that duality; then, if $V + W = E$, one has $V' \cap W' = \{0\}$, and $V \cap W$ corresponds by duality to $V' + W'$. So all Grassmann needed was a computational device, attaching to a decomposable p-vector spanning a subspace V, a decomposable $(n-p)$-vector spanning V'. Now, for Grassmann as for all his contemporaries (and all mathematicians until quite recently), "euclidean space" $\mathbf{R}^n$ was always conceived as equipped with all its structures, of which "euclidean distance" was one (it is only for us, accustomed as we are to the "dissociation" of structures, that there are infinitely many euclidean structures (isomorphic but distinct) on a vector space). The "natural" duality was therefore the one which assigned to V the orthogonal subspace V', and so, to a q-vector a spanning V, Grassmann associated its "complement" ("Ergänzung"), written $|a$, and which he defined algebraically in the following way. Start with an orthonormal basis $e_1, \ldots, e_n$ of E; for each q with $1 \leqslant q \leqslant n$, and each subset $H: i_1 < i_2 < \ldots$

$< i_q$ of q elements in $\{1, 2, \ldots, n\}$, let $e_H = e_{i_1} \wedge e_{i_2} \wedge \ldots \wedge e_{i_q}$, so that the $\binom{n}{q}$ elements e_H form a basis for $\overset{q}{\wedge} E$. Take the complement H' of H in $\{1, 2, \ldots, n\}$, similarly ordered; then $e_H \wedge e_{H'} = \rho(H, H')e_1 \wedge e_2 \wedge \ldots \wedge e_n$ with $\rho(H, H') = \pm 1$; Grassmann defined the complement $|e_H$ as equal to $\rho(H, H')e_{H'}$, and then $|a$ by linearity for all q-vectors. If now a is a p-vector, b a q-vector, and $p+q \geqslant n$, Grassmann defines the "regressive" product $a \vee b$ as equal to $|((|a) \wedge (|b))$; if a and b are decomposable and span V and W respectively, then $a \vee b = 0$ if $V + W \neq E$, but $a \vee b$ spans $V \cap W$ if $V + W = E$. It is easy to see that the regressive product is invariant under unimodular linear transformations, but the notion of "Ergänzung" is only invariant under orthogonal transformations.

One could also form the "interior product" $a \wedge (|b)$, a $(p+n-q)$-vector. However, for $p = q = 1$, Grassmann wanted that product to be, not an n-vector, but a *number*, namely the scalar product which he had earlier defined; this is probably the reason why he unfortunately *identified* n-vectors and scalars, thus paving himself the way for the degenerate "Vector analysis" which uninspired hacks concocted later out of his and Hamilton's ideas!

This is the beginning of the debacle. Faced with three different "products", Grassmann tries to combine them, and of course gets nowhere, except in very particular cases: trivial examples show that in general no properties of associativity or commutativity are to be expected.

And so it was that for almost 80 years the wonderful opportunities which Grassmann's exterior algebra might have opened remained unsuspected. When E. Cartan, around 1900, developed the calculus of differential forms, he very wisely only used the exterior product, to the exclusion of the other ones; but as his own work was little understood until 1930, it did not do much to popularize Grassmann's ideas. They had to wait until a proper understanding of linear algebra, and in particular a reasonable theory of *duality* of vector spaces, had given them their full power.

To conclude, it is perhaps not superfluous to summarize the main results of exterior algebra as we now conceive it, and to show that it fully lives up to the expectations which Grassmann had envisioned when he invented its fundamental notions.

To a finite dimensional vector space E over a commutative field K, of dimension n, are associated *two* exterior algebras $\wedge E$ and $\wedge E^*$, over E and its dual E^* respectively. They are in natural duality (but are *not* to be identified!) by a bilinear form $\langle z, z^* \rangle$ defined as follows: one has $\langle z, z^* \rangle = 0$ if $z \in \overset{p}{\wedge} E$, $z^* \in \overset{q}{\wedge} E^*$ with $p \neq q$, and

$$\langle x_1 \wedge x_2 \wedge \ldots \wedge x_p, x_1^* \wedge x_2^* \wedge \ldots \wedge x_p^* \rangle = \det((\langle x_i, x_j^* \rangle)) \qquad (4)$$

for vectors $x_i \in E$, $x_j^* \in E^*$ and $0 \leqslant p \leqslant n$. An *interior product* is defined, *not* between multivectors of $\wedge E$, but between a q-vector $z_q \in \overset{q}{\wedge} E$ and a $(p+q)$-vector $u_{p+q}^* \in \overset{p+q}{\wedge} E^*$, for all $p \geqslant 0$ and $q \geqslant 0$; it is a p-vector $z_q \lrcorner u_{p+q}^*$ *in* $\overset{p}{\wedge} E^*$, defined by the condition that, for any p-vector $v_p \in \overset{p}{\wedge} E$,

$$\langle v_p, z_q \lrcorner u_{p+q}^* \rangle = \langle v_p \wedge z_q, u_{p+q}^* \rangle; \tag{5}$$

in other words, the mapping $u_{p+q}^* \mapsto z_q \lrcorner u_{p+q}^*$ of $\overset{p+q}{\wedge} E^*$ into $\overset{p}{\wedge} E^*$ is the *transposed* mapping of the linear mapping $v_p \mapsto v_p \wedge z_q$ of $\overset{p}{\wedge} E$ into $\overset{p+q}{\wedge} E$. In particular, for dual bases (e_H) and (e_H^*) of $\wedge E$ and $\wedge E^*$, one has

$$\begin{aligned} e_H \lrcorner e_L^* &= 0 && \text{if } H \not\subset L \\ e_H \lrcorner e_L^* &= \rho_{L-H,H} e_{L-H}^* && \text{if } H \subset L \end{aligned} \tag{6}$$

where $\rho_{L-H,H}$ is the scalar factor equal to ± 1 such that $e_{L-H} \wedge e_H = \rho_{L-H,H} e_L$. One of course defines similarly an interior product $u_q^* \lrcorner z_{p+q}$ of a q-vector $u_q^* \in \overset{q}{\wedge} E^*$ and a $(p+q)$-vector $z_{p+q} \in \overset{p+q}{\wedge} E$, which is a p-vector in $\overset{p}{\wedge} E$.

These notions being *intrinsically* defined,† what replaces Grassmann's "Ergänzung" is a linear mapping $\phi : z \mapsto z \lrcorner e^*$ of $\wedge E$ onto $\wedge E^*$ (and *not* onto itself!), where e^* is an n-vector $\neq 0$ in $\overset{n}{\wedge} E^*$, which is therefore well defined up to a scalar factor. If a p-vector $z = x_1 \wedge x_2 \wedge \ldots \wedge x_p$ is decomposable and $\neq 0$, and $V(z)$ is the p-dimensional subspace of E spanned by the x_i's, then $\phi(z)$ is a decomposable $(n-p)$-vector such that $V(\phi(z))$ is the *subspace of* E^* *orthogonal to* $V(z)$; this follows at once from (6) when one takes the basis (e_j) of E such that $e_H = z$.

The *Grassmannian* $G_{n,p}(E)$ is the subset of the projective space $P(\overset{p}{\wedge} E)$ which is the image of the set of decomposable p-vectors $\neq 0$ in $\overset{p}{\wedge} E$, and is therefore in 1–1 correspondence with the set of all *vector subspaces of dimension* p in E. It is an *algebraic* submanifold of $P(\overset{p}{\wedge} E)$; to prove this, one has to write down the necessary and sufficient conditions which must be satisfied by the coordinates ζ_H of a p-vector $z = \Sigma_H \zeta_H e_H$ (summation over

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† The most important interior products correspond to $q = 1$, because for $x \in E$ the mapping $u^* \mapsto x \lrcorner u^*$, written $i(x)$ or i_x, is an *antiderivation* of degree -1 of the graded algebra $\wedge E^*$; furthermore, one has $i(x) \circ i(x) = 0$, and any interior product $u^* \mapsto z \lrcorner u^*$ can always be written as linear combination of products $i(x_1) \circ i(x_2) \circ \ldots \circ i(x_q)$. In the work of E. Cartan, these operators appear as "derivations"; for a p-form $F = f \,.\, du_1 \wedge du_2 \wedge \ldots \wedge du_p$, $\partial F / \partial u_k$ is the $(p-1)$-form $f \,.\, du_1 \wedge \ldots \wedge \widehat{du_k} \wedge \ldots \wedge du_p$. The algebraists of the nineteenth century dimly perceived that the polynomials $\Sigma_\alpha a_\alpha x^\alpha$ and the differential operators $\Sigma_\alpha b_\alpha D^\alpha$ were in duality, and used that fact, in particular in the theory of invariants; this was their closest approach to the fact that a vector space and its dual consist of different objects.

the $\binom{n}{p}$ subsets H of the set $\{1, 2, \ldots, n\}$) in order that z be decomposable (or 0), and show that these relations are polynomial equations between the ζ_H. This is a problem which Grassmann only tackled (for n and p arbitrary†) in the last year of his life, in a fairly complicated way ([G], II_1, pp. 283–294); Engel and Study gave simpler methods in their comments included in Grassmann's complete works, and Study's solution can be presented in the following way. In general, for *any* p-vector $z_p \in \overset{p}{\wedge} E$, there is a *smallest* vector subspace $M(z_p) \subset E$ such that $z_p \in \overset{p}{\wedge} M(z_p)$; Study's remark is that $M(z_p)$ is the subspace generated by the vectors $u^*_{p-1} \lrcorner z_p$, as u^*_{p-1} runs through $\overset{p-1}{\wedge} E^*$. To prove this, choose a basis of E containing a basis of $M(z_p)$; using (6) (with the roles of E and E^* exchanged), it is clear that for any $u^*_{p-1} \in \overset{p-1}{\wedge} E^*$, one has $u^*_{p-1} \lrcorner z_p \in M(z_p)$. Conversely, we show that if M' is any vector subspace of $M(z_p)$, of codimension 1, it is not possible that $u^*_{p-1} \lrcorner z_p \in M'$ for *all* $u^*_{p-1} \in \overset{p-1}{\wedge} E^*$. Indeed, let $x \in M(z_p)$ be such that $x \notin M'$, and take a basis of $M(z_p)$ consisting of a basis of M' and the vector x; then one may write $z_p = z'_p + x \wedge z''_{p-1}$, where $z'_p \in \overset{p}{\wedge} M'$ and $z''_{p-1} \in \overset{p-1}{\wedge} M'$, and the definition of $M(z_p)$ implies that $z''_{p-1} \neq 0$. By duality, there exists $u^*_{p-1} \in \overset{p-1}{\wedge} E^*$ such that $\langle z''_{p-1}, u^*_{p-1} \rangle = \lambda \neq 0$, and it follows from (6) that $u^*_{p-1} \lrcorner z_p = \pm \lambda x + x'$ with $x' \in M'$, which ends the proof.

From this the conditions of Study expressing that z_p is decomposable follow immediately:

$$z_p \wedge (u^*_{p-1} \lrcorner z_p) = 0 \quad \textit{for all} \quad u^*_{p-1} \in \overset{p-1}{\wedge} E^*. \tag{7}$$

Indeed, this expresses that $z_p \wedge x = 0$ for *all* $x \in M(z_p)$, and this is only possible for a p-vector z_p when $M(z_p)$ has dimension p (take a basis of $M(z_p)$ and express z_p in that basis).

The same argument shows that if z_p is a decomposable p-vector, z'_q a decomposable q-vector, with $p < q$, the relation $V(z_p) \subset V(z'_q)$ is equivalent to

$$z'_q \wedge (u^*_{p-1} \lrcorner z_p) = 0 \quad \textit{for all} \quad u^*_{p-1} \in \overset{p-1}{\wedge} E^*. \tag{8}$$

Of course, it is enough to write (7) or (8) for the $(p-1)$-vectors forming a

† For $n = 4, p = 2$, decomposable bivectors correspond to projective lines in projective 3-dimensional space. Cayley is probably the first to have mentioned in 1862 that systems of 6 numbers not all 0 and satisfying

$$p_{01}p_{23} + p_{02}p_{31} + p_{03}p_{12} = 0$$

determine projective lines, a point of view which was developed systematically by Plücker in 1865; but neither Cayley nor Plücker mentioned Grassmann.

basis of $\overset{p-1}{\wedge} E^*$; one thus obtains a finite number of *quadratic relations* (7) between the coordinates of z_p, which define the Grassmannian as an algebraic variety in $P_{\binom{n}{p}-1}$, and a finite number of *bilinear relations* (8) expressing that a subspace is contained in another one.

More generally, one can express by algebraic relations the fact that the intersection $V(z_p) \cap V(z'_q)$ has dimension $\geqslant s$; a similar argument proves that this is equivalent to

$$z'_q \wedge (u^*_{s-1} \lrcorner\, z_p) = 0 \quad \textit{for all} \quad u^*_{s-1} \in \overset{s-1}{\wedge} E^*. \tag{9}$$

Finally, we may derive from this an expression for a decomposable multivector spanning the sum $V(z_p) + V(z'_q)$ of two *arbitrary* subspaces of E. From (9) it follows that the dimension s of $V(z_p) \cap V(z'_q)$ is the *smallest* integer k such that the products $z'_q \wedge (u_k \lrcorner\, z_p)$ *are not all* 0 when u^* runs through $\overset{k}{\wedge} E^*_k$; and then all $(p+q-s)$-vectors $z'_q \wedge (u^*_s \lrcorner\, z_p)$, for $u^*_s \in \overset{s}{\wedge} E^*$, are scalar multiples of a single decomposable $(p+q-s)$-vector $\neq 0$ spanning $V(z_p) + V(z'_q)$.

The gist of these results is that *incidence relations* between vector subspaces of E may be expressed by relations between coordinates of these subspaces in their respective Grassmannians, which certainly was Grassmann's ultimate (if unformulated) goal.

References

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