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# SKEW FIELDS OF NONCOMMUTATIVE RATIONAL FUNCTIONS, AFTER AMITSUR (PRELIMINARY VERSION)

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We give here a generalized and simplified version of Amitsur's construction and classification of skew fields of noncommuting rational functions [1]. We show that, if $D$ is a sfield (= skew field), $E$ an oversfield with infinite center, and $n$ a positive integer, then the $n$ coordinate functions on the space $E^n$, together with the constant functions from $D$, generate a "sfield of almost-everywhere-defined functions". In contrast to the situation for commutative fields, the structure of this sfield of rational functions may depend on $E$ as well as $D$ ; but, in a large class of cases, the family of sfields of functions so obtained is very easily classified, and contains a "maximal member", of which all other such sfields are specializations.

Our approach leads to a kind of "noncommutative algebraic geometry".

Several conjectures and problems are also raised. Elsewhere we shall develop this material, starting from more general concepts, and obtaining in many places stronger results.

1. - Our construction will depend on the already known construction of a more modest sort of "sfield of rational functions" :

Let $E$ be a sfield, and $E[t]$ the "polynomial ring" gotten by adjoining a <u>commuting</u> indeterminate $t$. $E[t]$ will be a right (and left) Ore ring, and hence has a sfield of fractions $E(t)$, elements of which can be written $PQ^{-1}$ $(P , Q \in E[t]$, $Q \neq 0)$. Further, because one can take "right greatest common divisors" in $E[t]$, each member of $E(t)$ has an essentially unique expression in "lowest terms".

Note that any formal expression gotten from the symbol $t$ and the elements of $E$ by formal operations of addition, multiplication and (multiplicative) inverse, either reduces to a unique element of $E(t)$, or fails to define such an element, because at some step, one takes the inverse of an expression which corresponds to zero in $E(t)$. (E. g., $a + (bt - tb)^{-1}$, for $a , b \in E$, does not reduce to a member of $E(t)$.)

For any element $\tau$ in $\mathcal{C}(E)$, the center of $E$, we can define a substitution homomorphism of $E[t]$ to $E$, sending $P(t)$ to $P(\tau)$. From this, we can get a specialization map of $E(t)$ into $E$ sending $t$ to $\tau$, defined for all elements $PQ^{-1}$ (assumed written in lowest terms) such that $Q(\tau) \neq 0$.

A non-zero element of $E[t]$ can assume the value zero for only finitely many elements $\tau \in \mathcal{C}(E)$, by the same reasoning as for commutative polynomials. (Note that $t - \tau$ will lie in $\mathcal{C}(E[t])$.) Hence a non-zero element of $E(t)$ can specialize to 0 at only finitely many $\tau \in \mathcal{C}(E)$.

2. - Let $f(t)$ be any expression formed from the elements of $E$ and the symbol $t$ by addition, multiplication and inverses. Then $f$ induces a function on a subset of $\mathcal{C}(E)$, by simple substitution and evaluation wherever possible ; we shall say that $f$ is "defined" on this set of points. In particular, a necessary condition for $f$ to be defined at some point $\tau \in \mathcal{C}(E)$ is that $f$ reduce to an element $\varphi \in E(t)$ which can be specialized to $\tau$ ; if $f$ is defined at $\tau$, its value will then equal the value to which $\varphi$ specializes. (But the domain of definition of $f$ may be smaller than that of $\varphi$. For instance, $t^{-1} - t^{-1}$ is defined on $\mathcal{C}(E) - \{0\}$, but reduces in $E(t)$ to 0, which is defined everywhere.)

LEMMA 1. - If an expression $f(t)$ is defined at some $\tau \in \mathcal{C}(E)$, then it is defined at all but finitely many points of $\mathcal{C}(E)$.

Proof. - We use induction on the expression $f$. If the statement is valid for two expressions $f$ and $g$, it is clearly true for their formal sum and product. Now suppose the formal inverse of an expression, $f(t)^{-1}$, is defined at some $\tau \in \mathcal{C}(E)$. Inductively, assume $f(t)$ is defined except on a finite set of points $S_1 \subseteq \mathcal{C}(E)$. Then its values, where it is defined, correspond to the specializations of some $\varphi \in \mathcal{C}(E)$. Now $\varphi$ must be non-zero, because $f(t)^{-1}$ is defined at $t = \tau$ ; hence it will specialize to zero only on a finite set $S_2$. Then $f(t)^{-1}$ will be defined except on $S_1 \cup S_2$.

3. - Now let $D \subseteq E$ be skew fields, and $X_1 , \ldots , X_n$ indeterminates, thought of as representing elements of $E$, not necessarily central. Any formal expression $f(X_1 , \ldots , X_n)$ in sums, products, and inverses of elements of $D$ and $X_1 , \ldots , X_n$, will yield a function on some subset of $E^n$, which we will call the "domain of definition of $f$". We shall call $f$ nondegenerate (on $E^n$), if this set is nonempty.

Let us now assume $\mathcal{C}(E)$ to be infinite. (We conjecture, however, that the next result and all its consequences hold with only $E$ assumed infinite. They are false, of course, if $E$ is finite. AMITSUR raises this question at [1], p. 358.)

LEMMA 2. - If $f$ and $g$ are nondegenerate expressions, their domains of definition have some point in common.

Proof. - Let $f$ be defined at $p = (p_1, \dots, p_n) \in E^n$, and $g$ at $q = (q_1, \dots, q_n)$. Writing $tp + (1 - t)q$ for

$$(tp_1 + (1 - t)q_1, \dots, tp_n + (1 - t)q_n) ,$$

we see that $f(tp + (1 - t)q)$ and $g(tp + (1 - t)q)$ will be expressions in $t$, defined for $t = 1$ and $t = 0$ respectively. Hence each is defined for all but finitely many values in $\mathcal{C}(E)$, hence both will be defined at some $\tau \in \mathcal{C}(E)$ ; from which we get a common point of definition for $f$ and $g$.

Let us write $f \sim g$, if $f$ and $g$ are nondegenerate expressions whose values agree at every point where both are defined.

COROLLARY.

(a) If $f$ and $g$ are nondegenerate, so are $f + g$, $fg$, and, if $f \not\sim 0$, so is $f^{-1}$.

(b) $f \sim g \sim h \implies f \sim h$.

(c) If $f \sim f'$ and $g \sim g'$, then $f + g \sim f' + g'$, $fg \sim f'g'$, and, if $f \not\sim 0$, $f^{-1} \sim f'^{-1}$.

Proof. - Assertion (a) follows immediately from lemma 2. To show (b), suppose $f \not\sim h$ ; then $(f - h)^{-1}$ is nondegenerate, hence has a point of definition $p$ in common with $g$. Then $f(p) \neq h(p)$, hence, either $f(p) \neq g(p)$, or $g(p) \neq h(p)$ ; contradiction. Assertion (c) follows from the definition, once we know that $f + g$, etc., are nondegenerate.

It follows that :

THEOREM 3. - The quotient of the set of nondegenerate expressions by the equivalence relation $\sim$ acquires a structure of skew field, under operations induced by the formal operations $+ , . , ( )^{-1}$ on these expressions. This may also be regarded as the skew field of functions from $E^n$ to $E$ (defined up to changes in domain of definition) generated by the coordinate functions and the constant functions from the subsfield $D$.

We shall call this skew field $D_{E^n}\langle\langle X\rangle\rangle$, where $X$ stands for $X_1, \dots, X_n$.

Note that, to every $f \in D_{E^n}\langle\langle X\rangle\rangle$, we can associate a function defined on the union of the domains of definition of all expressions for $f$. One would like to know whether there is always an expression for f having this whole set for domain of definition, a "universal" expression for the rational function f .

If D and E are fields ( E infinite), then $D_{E^n}\langle\langle X\rangle\rangle$ will be the ordinary field of rational functions, D(X) , independent of E . In this case, the "universal expressions" problem has an affirmative solution, given by the expression for f as a quotient, in lowest terms, of polynomials.

4. – The above construction can be seen in a much more general context. Let us replace $E^n$ by an arbitrary set S , let us associate to each $s \in S$ a sfield $E_s$, and let F be a family of functions f , each defined on a subset $S_f \subseteq S$ , and sending each $s \in S_f$ to an element of $E_s$ .

By the pre-sfield of functions $\widetilde{F}$ generated by F , we shall mean the closure of $F \cup \{1\}$ under sums $f+g$ , negatives $-f$ , products $fg$ , and inverses $f^{-1}$ , these being defined to have domains of definition $S_1 = S$ , $S_{f+g} = S_f \cap S_g$ , $S_{-f} = S_f$ , $S_{fg} = S_f \cap S_g$ , and $S_{f^{-1}} = \{s \in S_f \mid f(s) \neq 0\}$ .

The sets $S_f$ $(f \in F)$ can be taken as a basis of open sets for a topology on S . Let us call S irreducible with respect to this topology, or with respect to the set of functions $\widetilde{F}$ , if it is not a union of two proper closed subsets under this topology. Then lemma 2 says essentially that $E^n$ is irreducible with respect to the pre-sfield of functions generated by D and the coordinate functions. By the same argument used to deduce theorem 1 from lemma 2, we can see that, for any S irreducible with respect to an $\widetilde{F}$ as above, we can form a sfield $\langle\langle F\rangle\rangle_S$ out of the equivalence classes of nondegenerate functions in $\widetilde{F}$ .

If a subspace $T \subseteq S$ is irreducible (with respect to the relative topology, which is the same as the topology induced on it by $\widetilde{F}|T$ ), we obtain a specialization of $\langle\langle F\rangle\rangle_S$ onto $\langle\langle F\rangle\rangle_T$ . (We shall rather consistently abuse our notation in this manner, writing F for $F|T$ when the restricted domain is indicated elsewhere.) One easily verifies that the irreducibility of a set depends only on its closure, and that the sfields $\langle\langle F\rangle\rangle_T$ and $\langle\langle F\rangle\rangle_U$ ( U , $T \subseteq S$ , irreducible) will be isomorphic as specializations of $\langle\langle F\rangle\rangle_S$ if, and only if, the closures of T and U in S are equal.

5. – We proved the irreducibility of $E^n$ by showing, in lemma 1, that the space $\mathcal{C}(E)$ is irreducible (with respect to the pre-sfield generated by the constants and the identity function) if it is infinite, and then using the fact that any two points of $E^n$ are connected by an image of this space. (Incidentally, $\mathcal{C}(E)$ is also closed in E , being the intersection of the domains of non-definition of the expressions $(at - ta)^{-1}$ , $a \in E$ .) This argument can also be generalized,

but we shall only sketch this generalization, without proof, since the application we shall make in section 6 will be transparently analogous to the proof of lemma 2 :

Let us define a pre-map from the triple $(S, \{E_s\}, \widetilde{F})$ to $(T, \{E_t\}, \widetilde{G})$ ( $S$ a set, $E_s$ a sfield for each $s \in S$, $\widetilde{F}$ a pre-sfield of functions on $S$ ; $T$, $\{E_t\}$, $\widetilde{G}$ similarly) to consist of a function $m$ from a subset $S_m \subseteq S$ to $T$, and, for every $s \in S_m$, a specialization of sfields : $m_s : E_{m(s)} \to E_s$, such that, if $g \in \widetilde{G}$, the function $m_s \circ g \circ m$ (defined at $s$ if, and only if, $m$ is defined at $s$, $g$ is defined at $m(s)$, and $m_s$ is defined on $g(m(s))$ ) lies in $\widetilde{F}$.

Now consider such a triple $(T, \{E_t\}, \widetilde{G})$, and suppose that, for any two points $t, t' \in T$, there exists a pre-map $m$, from a triple $(S, \{E_s\}, \widetilde{F})$ with irreducible $S$, to $(T, \{E_t\}, \widetilde{G})$, such that $t, t' \in m(S)$. Then $T$ itself is easily shown to be irreducible.

6. - Let us return to our sfields of rational functions, $D_{E^n}\langle X_1, \ldots, X_n\rangle$. The method we used to show $E^n$ irreducible can also be applied to a large class of subspaces of $E^n$ : Let us call a subset $S \subseteq E^n$ "flat" (with respect to $\mathcal{C}(E)$ ), if, for all $p, q \in S$, $S$ also contains $\tau p + (1-\tau)q$ for infinitely many $\tau \in \mathcal{C}(E)$. (Note that a closed flat subset will contain $\tau p + (1-\tau)q$ for all $\tau \in \mathcal{C}(E)$.) Exactly like lemma 2, we get :

LEMMA 4. - Any flat subset of $E^n$ is irreducible.

Hence, the closure of any flat subset will also be irreducible. Here is an interesting case of a flat set whose closure is not flat : Let $D$ be an infinite field, lying in the center of a noncommutative sfield $E$, and let us form $D_{E^2}\langle X_1, X_2\rangle$. The subset $D^2 \subseteq E^2$ is clearly flat. $D^2$ contains the "line" $D \times \{0\}$, and the closure of this line is $E \times \{0\}$ (for a single element of $E$ will commute with $D$, and hence satisfy any relation with coefficients in $D$ satisfied by the elements of $D$ ). So the closure of $D^2$ contains $E \times \{0\}$, and similarly, $\{0\} \times E$. But, if we take $x$ and $y \in E$ which do not commute, the closure of $D^2$ will not contain any points $(x_1, x_2) = \tau(x, 0) + (1-\tau)(0, y)$ $(\tau \in \mathcal{C}(E) - \{0, 1\})$, for such a point does not satisfy $X_1 X_2 - X_2 X_1 = 0$.

The key to this example is : $D$ is dense in $E$ (with respect to expressions with coefficients in $D$ ), but $D^2$ is not similarly dense in $E^2$ :

Examples of flat closed subspaces of a space $E^n$ are all the spaces defined by families of "linear" relations, $\sum_i^n \sum_j^{m_i} a_{ij} X_i b_{ij} = c$ $(a_{ij}, b_{ij}, c \in D)$. Each such set $S$ will be irreducible by lemma 4, and hence will yield a skew field $D_S\langle\langle X\rangle\rangle$, in indeterminates $X_1, \ldots, X_n$ constrained to satisfy the corresponding equations. For example, the set $C(E)^m \times E^{n-m}$ will give a sfield of rational functions in $m$ central and $n - m$ noncommuting indeterminates.

But, for most closed sets $S \subseteq E^n$, it is not clear how to determine whether $S$ is irreducible. E. g., is the set determined by the equation $X_1 X_2 - X_2 X_1 = 0$ irreducible for arbitrary $D$ and $E$ ? It would also be useful to know when a closed set is, at least, a finite union of irreducible closed subsets (as is always true in the commutative case !).

7. - By a polynomial in $X_1, \ldots, X_n$ over $D$, we shall mean an expression formed from these symbols and the elements of $D$ by the operations of addition and multiplication only. If $P$ is such a polynomial, the set of points of $E^n$ satisfying $P = 0$, will be the domain of non-definition of the expression $P^{-1}$, and hence will be closed. An intersection of such a family of closed sets will be called a polynomially defined closed set. We shall now prove a weak but useful lemma to the effect that certain closed sets are polynomially defined ; but we first need two observations :

If $S$ is a nonempty irreducible subset of $E^n$, and $p$ is a point not in the closure of $S$, this means that some expression $f$ in our coordinates is note defined anywhere on $S$, but is defined at $p$. Since $S$ is irreducible, the degeneracy of $f$ on $S$ cannot arise from adding or multiplying together (at some step in the definition of $S$ ) two nondegenerate expressions with disjoint domains of definition on $S$ ; hence it must result, from inverting an expression $g$, that is nondegenerate but equals zero. Thus, there is a $g$ that is nondegenerate on $S$, and zero at all points of $S$ where it is defined, but is non-zero at $p$.

Secondly, any element of $E(t)$ which is defined at $t = 0$ can be expanded as a formal power series in $t$. (More generally, $E(t)$ can be embedded in the ring $E((t))$ of formal Laurent series in $t$, over $E$.) The basic formula for taking inverses is : $(a + B)^{-1} = a^{-1}(1 + Ba^{-1})^{-1} = \sum_0^\infty a^{-1}(-Ba^{-1})^n$ ( $a$ = the constant term, $B$ divisible by $t$ ). Note that the coefficients in this expression do not involve inverses of any element, but the constant term $a$ !

LEMMA 5. - *If $S \subseteq D^n$ is flat with respect to $C(D)$, then its closure in $E^n$ is polynomially defined.*

*Proof.* - Let $p$ not lie in the closure of $S$. We must find a polynomial over $D$ in the coordinate functions, that is zero on $S$, but non-zero at $p$.

We know that there will exist a rational expression $f$ nondegenerate on $S$ and zero there, but nonzero at $p$ [1]. Let $q$ be a point of $S$ where $f$ is defined. For any $x \in E^n$, consider the expression $f((1-t)q + tx)$. This will reduce to a well-defined member of $E(t)$, because it is defined for $t = 0$. If $x \in S$, this expression must reduce to zero in $E(t)$, because of the flatness of $S$, while for $x = p$, it will be nonzero, because it is (defined and) nonzero at $t = 1$.

Now, in obtaining a formal power series expansion from the expression $f((1-t)q + tx)$, whenever we must take the inverse of a sub-expression $h(t)$, the constant term $h(0)$ will be non-zero, and will not involve the coordinates of $x$. From this, one can deduce that the coefficients of the final expansion, considered as functions of the coordinates of $x$, will be polynomials. Their coefficients will come from $D$, because $q \in S \subseteq D^n$. These polynomials must all be zero on $S$, but at least one of them will be non-zero at $p$ [2].

Q. E. D.

(An example of a closed set that apparently is not polynomially defined, at least if $E$ is sufficiently noncommutative, is the domain of non-definition of $(X_1^{-1} + X_2^{-1} + X_3^{-1})^{-1}$. Actually, this set is not irreducible : it is the union of the domains of non-definition of $X_1^{-1}$, $X_2^{-1}$, and $X_3^{-1}$, and a fourth set : the intersection of the domains of non-definition of $(1 + X_1 X_2^{-1} + X_1 X_3^{-1})^{-1}$, $(X_2 X_1^{-1} + 1 + X_2 X_3^{-1})^{-1}$, $(X_3 X_1^{-1} + X_3 X_2^{-1} + 1)^{-1}$, which we can show is irreducible and we conjecture is not polynomially defined.)

LEMMA 6. - Let $D$ be a sfield with infinite center, and $E$ an over-sfield satisfying every polynomial identity in $n$ variables with coefficients in $D$ that is satisfied in $D$. Then $D_{E^n}\langle X_1, \dots, X_n\rangle \simeq D_{D^n}\langle X_1, \dots, X_n\rangle$. Hence, if $C$ is any sub-sfield of $D$, $C_{E^n}\langle X_1, \dots, X_n\rangle \simeq C_{D^n}\langle X_1, \dots, X_n\rangle$.

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[1] This argument requires that we know $S$ irreducible, but we have only proved this when $S$ is flat with respect to $\mathcal{C}(E)$ which may not lie in $S(E)$. But the flatness of $S$ with respect to $\mathcal{C}(D)$ implies $S$ irreducible in $D^n$ ; and the topology of $D^n$ is just the relative topology induced from $E$.
Perhaps our definition of flat needs improving. The most general condition from which we can deduce lemma 4 is : for every $p, p' \in S$, finite set $A \subseteq D$, and positive integer $r$, there exist $\tau_1, \dots, \tau_r$ commuting with the coordinates of $p$ and $p'$ and with the members of $A$, such that $\tau p + (1-\tau)p' \in S$.

[2] The use of formal power series was not strictly necessary (we could have shown that all coefficients in the numerator of $f((1-t)q + tx)$ can be taken to be polynomials). But it is more convenient to be able to take advantage of familiar facts abt power series.

Proof. - By lemma 5, the closure of $D^n$ in $E^n$ is polynomially defined ; but every polynomial relation holding on $D^n$ holds on $E^n$, so $E^n$ is this closure, i. e., $D^n$ is dense in $E^n$, from which the first result follows. The second statement holds, because $C_{E^n}\langle\langle X\rangle\rangle$ is simply the sub-sfield of $D_{E^n}\langle\langle X\rangle\rangle$ generated by the coordinate functions and the constants from $C$. Alternatively, since $D^n$ is dense in $E^n$ with respect to the pre-sfield generated by the coordinate functions and the constants in $D$, it will certainly be dense with respect to the smaller pre-sfield gotten by restricting our constants to $C$.

8. - Following AMITSUR, we shall now attempt to classify our rational function fields $D_{E^n}\langle\langle X\rangle\rangle$. For given base sfield $D$ and integer $n$, this means that we want to classify over-sfields $E \supseteq D$ according to the rational identities in $n$ variables, with coefficients in $D$ that their elements satisfy. Lemma 6 allows us to partially reduce this problem to that of polynomial identities.

It is known that, if a sfield (more generally, a simple ring) $E$ is finite-dimensional over its center, the dimension will be of the form $d^2$ $(d \geq 1)$, and the sfield will satisfy an identity in $2d$ variables with integral coefficients :

$$\sum_{\pi \in S_{2d}} \pm X_{\pi(1)} \cdots X_{\pi(2d)} = 0 \quad ;$$

which is not satisfied by any sfield of larger dimension over its center. Hence, the number $(E:\mathcal{C}(E))$ (an integral square or $\infty$) will be an invariant of the rational identities satisfied by $E$ over $D$ (at least if we allow enough variables).

Another such invariant is the sub-field $\mathcal{C}(E) \cap D \subseteq \mathcal{C}(D)$.

We shall find that, when $\mathcal{C}(E) \cap D = \mathcal{C}(D)$, the number $(E:\mathcal{C}(E)) = d^2$ precisely classifies the structure of $D_{E^n}\langle\langle X\rangle\rangle$ (except for the case of $n = 1$ and $D$ a field, where $D_E\langle\langle X_1\rangle\rangle$ is the commutative rational function field $D(X_1)$ <u>independent</u> of $d$). On the other hand, we have examples, which we will not give here, of a sfield $D$ and over-sfields $E$, $E'$, with $(E:\mathcal{C}(E)) = (E':\mathcal{C}(E')) < \infty$, and $D \cap \mathcal{C}(E) = D \cap \mathcal{C}(E')$, such that $D_{E^n}\langle\langle X\rangle\rangle \not\cong D_{E'^n}\langle\langle X\rangle\rangle$. Nonetheless, we believe that, for $(E:\mathcal{C}(E)) < \infty$, a complete classification of the sfields $D_{E^n}\langle\langle X\rangle\rangle$ can be obtained with the help of Galois theory ; we shall attempt to do this elsewhere. For the case $(E:\mathcal{C}(E)) = \infty$, we conjecture that $D_{E^n}\langle\langle X\rangle\rangle$ depends only on the field $D \cap \mathcal{C}(E) \subseteq \mathcal{C}(D)$.

9. - Let us call an over-sfield $E'$ of a sfield $E$ regular, if $\mathcal{C}(E) \subseteq \mathcal{C}(E')$ ; and let us begin our analysis of rational identities with the case $(E:\mathcal{C}(E)) = \infty$ .

AMITSUR proves, in [2], that a sfield $E$ infinite-dimensional over its center satisfies no polynomial identities with coefficients in $E$ , except those saying that all elements commute with elements of the center. If we combine this with lemma 6 (taking $E$ for "D" , any regular over-sfield $E'$ for "E" , and any subsfield $D$ for "C" ), we get :

LEMMA 7. - Let $D$ be a sfield, $E$ any over-sfield with infinite center and infinite-dimensional over this center, and $E'$ any regular over-sfield of $E$ . Then, for any $n \geq 0$ , $D_{E^n}\langle\langle X_1 , \dots , X_n\rangle\rangle \simeq D_{E'^n}\langle\langle X_1 , \dots , X_n\rangle\rangle$ .

Thus, the conjecture just made would follow, if we could show that any two such $E_1$ and $E_2$ , with $\mathcal{C}(E_1) \cap D = \mathcal{C}(E_2) \cap D$ , may be joined by a chain of ascending and descending regular inclusions of such sfields (3). We can do this in the case of sfields regular over $D$ , by means of a construction that functorially embeds an arbitrary sfield in a regular over-sfield of the desired sort :

Given any sfield $E$ , let us adjoin a family of commuting indeterminates $u_\alpha$ indexed by the group $\mathbb{Q}/\mathbb{Z}$ . Let us then adjoin to the sfield $E(u_\alpha)_{\alpha\in\mathbb{Q}/\mathbb{Z}}$ an indeterminate $v$ , and all rational powers thereof, $v^r$ $(r \in \mathbb{Q})$ , so that these commute with $E$ , but noncommute with the $u$'s by the formula $u_\alpha v^r = v^r u_{\alpha+\bar{r}}$ ( $\bar{r}$ = residue of $r$ (mod $\mathbb{Z}$) ). Call the resulting sfield $E! = E(u_\alpha , v^r)$ . Elements of the center, to commute with elements of $E$ , must lie in $\mathcal{C}(E)(u_\alpha , v^r)$ ; to commute with all $v^r$ , they must then lie in $\mathcal{C}(E)(v^r)$ , and to commute with the $u_\alpha'$ s , they must in fact lie in $\mathcal{C}(E)(v)$ . This is easily seen to be the center of $E!$ , and it is infinite, contains $\mathcal{C}(E)$ , and $E!$ is infinite-dimensional over it, as desired.

The above construction is not the simplest possible, but it has the following property, which we shall find of use much later : $E!$ is the direct limit of the sub-sfields $E!_d$ gotten by restricting the exponent of $v$ to the subgroup $\frac{1}{d}\mathbb{Z}$ . (We may also restrict $\alpha$ to $\left(\frac{1}{d}\mathbb{Z}\right)/\mathbb{Z}$ , if we are feeling parsimonious. In any

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(3) This is, in fact, equivalent to our conjecture, for, if the conjecture is true, such a chain will be given by $E \subseteq E_{E^n}\langle\langle X\rangle\rangle \supseteq D_{E^n}\langle\langle X\rangle\rangle \simeq D_{E'^n}\langle\langle X\rangle\rangle \subseteq E'_{E'^n}\langle\langle X\rangle\rangle \supseteq E'$ (unless $D$ has finite center, in which case we adjoin a commuting indeterminate $t$ to all the intermediate stages, to assure that the middle state has infinite center).

case, the direct limit is with respect to the set of positive integers, ordered by divisibility.) We find that the center of $E!_d$ is the sub-field of $\mathcal{C}(E)(u_\alpha)(v)$ fixed under the operation of $v^{1/d}$, which has order $d$. The field just named, $\mathcal{C}(E)(u_\alpha)(v)$, has index $d$ in $\mathcal{C}(E)(u_\alpha)(v^{1/d})$, which has index $(E:\mathcal{C}(E))$ in $E!_d$. We conclude that $(E!_d:\mathcal{C}(E!_d)) = d^2(E:\mathcal{C}(E))$.

Using this construction of $E!$, we can prove :

THEOREM 8. - <u>Let</u> $D$ <u>be a sfield ; then there exist regular over-sfields</u> $E \supseteq D$ <u>with infinite center, and infinite-dimensional over their centers ; and, for each integer</u> $n$, <u>all of these yield the same rational function sfield</u> $D_{E^n}\langle\langle X_1, \ldots, X_n\rangle\rangle$. <u>We shall call this sfield</u> $D\langle\langle X_1, \ldots, X_n\rangle\rangle_\infty$ ; <u>the elements</u> $X_1, \ldots, X_n$ <u>can be specialized to any elements</u> $x_1, \ldots, x_n$ <u>of</u> any <u>regular over-sfield</u> $E$ <u>of</u> $D$.

<u>Proof</u>. - $D!$ is a sfield satisfying the conditions of the first phrase ; and, given any other such sfield $E$, lemma 7, applied to the inclusions $E \subseteq E! \supseteq D!$, tells us that $D_{E^n}\langle\langle X\rangle\rangle \simeq D_{D!^n}\langle\langle X\rangle\rangle$.

Given <u>any</u> regular over-sfield $E$ over $D$, we will have $D\langle\langle X\rangle\rangle_\infty \simeq D_{E!^n}\langle\langle X\rangle\rangle$, which we can specialize to the point $(x_1, \ldots, x_n) \in E^n \subseteq E!^n$.

The next two sections, which consider geometric and algebraic ramifications of the above theorem, are not used in subsequent sections, and may be skipped.

<u>10</u>. - Suppose $D$ is a sfield, and $E$, $E'$ two over-sfields. The coordinate functions on the disjoint union $E^n \cup E'^n$ (which take points of $E^n$ to members of $E$, and points of $E'^n$ to members of $E'$), and the constant functions from $D$, will generate a pre-sfield of functions, and so induce a topology on $E^n \cup E'^n$. Then the condition "all rational identities in $n$ indeterminates, holding in $E$ with coefficients in $D$, hold in $E'$ as well" is easily seen equivalent to the condition " $E^n$ is dense in $E^n \cup E'^n$ ". We shall then say that $E^n$ <u>dominates</u> $E'^n$ over $D$. $E^n \cup E'^n$ will be irreducible, if, and only if, one of $E^n$ or $E'^n$ (say $E^n$) dominates the other and is itself irreducible ; an equivalent statement is that $E^n$ is irreducible, and the indeterminates of $D_{E^n}\langle\langle X\rangle\rangle$ can be specialized over $D$ to any $n$ elements of $E'$. Dominance is easily seen to be reflexive and transitive. If $E^n$ and $E'^n$ dominate each other over $D$, they will be called <u>equivalent</u> over $D$.

Thus, theorem 8 can be formulated as saying that all the affine spaces $E^n$, for $E$ as in the first sentence, are equivalent over $D$, that the set of these is non-empty, and that they dominate <u>any</u> $E^n$ where $E$ is regular over $D$.

If two affine spaces $E^n$ and $E'^n$ are equivalent over $D$, their lattices of closed subsets need not be isomorphic. (Thus, over the rational numbers $\underset{\sim}{Q}$, the rational and real affine lines are equivalent, but the latter contains the closed set $(\sqrt{2}, -\sqrt{2})$ which has no equivalent in the former. $\underset{\sim}{Q}$ is dense in $\underset{\sim}{R}$, but does not meet every closed subset.) However, we can form a "universal" space for these considerations : Let $D\langle X_1, \ldots, X_n\rangle$ designate the $\mathcal{C}(D)$-algebra freely generated over $D$ by $n$ indeterminates, and let Sfield-Spec $D\langle X_1, \ldots, X_n\rangle$ be a space consisting of one representative from each isomorphism class of pairs $(E, f)$ : $E$ a sfield, $f$ a map of $D\langle X_1, \ldots, X_n\rangle$ into $E$ whose image generates $E$. Each element $u \in D\langle X_1, \ldots, X_n\rangle$ induces a function on this space, sending $(E, f)$ to $f(u) \in E$ ; and these generate a pre-sfield of functions. Each space $E'^n$ ( $E'$, any regular over-sfield of $E$ ) can be mapped naturally into Sfield-Spec $D\langle X\rangle$ by sending the point $(x_1, \ldots, x_n)$ to $(E_0, f)$, where $E_0$ is the sub-sfield of $E$ generated by $x_1, \ldots, x_n$, and $f : D\langle X\rangle \twoheadrightarrow E_0$ sends $X_i$ to $x_i$. Any closed set of $E^n$ can be seen to be the inverse image of a closed subset of Sfield-Spec $D\langle X\rangle$.

Theorem 8 can be shown equivalent to the statement that Sfield-Spec $D\langle X\rangle$ is irreducible, and for any regular $E$ with infinite center and infinite-dimensional over its center, the image of $E^n$ in this space is dense.

If $R$ is any ring, $A$ any irreducible closed subspace of Sfield-Spec $R$ (defined as above), $E$ the associated sfield, and $f$ the natural map of $R$ into $E$, then (the point of Sfield-Spec $R$ is isomorphic to) $(E, f)$ will be the generic point of the set $A$ (the unique point having $A$ as its closure). Thus, the generic point of the whole space Sfield-Spec $D\langle X\rangle$ is the one corresponding to the sfield $D\langle X\rangle_\infty$.

The operation "Sfield-Spec" may be compared with the operation of forming the reduced spectrum of a commutative ring in ordinary algebraic geometry.

11. - It would be desirable to know whether every relation [4]

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[4] Relations holding in $D\langle X\rangle_\infty$, or indeed, any sfield, can be expressed in two forms : " $f = 0$ " or " $g$ is undefined". The second form has the advantage that it is possible to ask whether it holds for any expression $g$ in elements of our sfield ; while we can say whether $f = 0$ only when we know $f$ is defined (i. e., only when we have verified that all subexpressions of $f$, whose inverse is used in $f$, are not zero). It is for this reason that we used the second form, in defining "closed sets", for instance. But we here revert to the first form, because of its greater familiarity. The reader should, in any case, be aware of how the information conveyed by the two types of "relations" is equivalent.

$$f(X_1 , \dots , X_n) = 0 \quad ,$$

holding in $D\langle\langle X_1 , \dots , X_n\rangle\rangle_\infty$ , can be "deduced algebraically" from the existence of the inverses involved in the expression of $f$ .

Let us look at this question in a more general light. Suppose we **wish** to construct some kind of natural enveloping sfield for a ring $R$ . The naive approach would be to take any non-zero non-unit $a_1 \in R$ , and adjoin an inverse (that is, adjoin to $R$ an indeterminate $Y$ , and divide out by the two-sided ideal generated by $1 - Ya_1$ and $1 - a_1 Y$ ). In the resulting ring, $R\langle a_1^{-1}\rangle$ , we again choose a non-zero non-invertible element (if there is any), and adjoin its inverse. We continue, by transfinite induction, until we get a sfield (or until our construction suddenly "degenerates", if one of the ideals we divide out by turns out to contain the element 1 !). To avoid the latter contingency, let us make a more sophisticated analysis :

Let us define $k_0(R)$ to be the two-sided ideal $0$ ; $k_1(R) \subseteq R$ to be the two-sided ideal of $R$ generated by all elements $a \in R$ such that $R\langle a^{-1}\rangle$ is "degenerate", i. e., satisfies $1 \in k_0(R\langle a^{-1}\rangle)$ ; and, generally, let us define $k_{i+1}(R)$ recursively as the two-sided ideal of $R$ generated by all elements $a$ such that $1 \in k_i(R\langle a^{-1}\rangle)$ . One can show (by induction on $i$ ) that, for all $i$ and all rings $R$ , $k_i(R)$ is a set of elements which can be shown, by a finite calculation, to go to zero, under any map of $R$ into a sfield. Let us define $k(R)$ to be the union of the ascending chain $k_0(R) \subseteq k_1(R) \subseteq \dots$ .

If $1 \in k(R)$ , the problem of mapping $R$ into any nondegenerate sfield, clearly has no solution. In the contrary case, it is not hard to see, from the definition of $k(R)$ , that, for any $a \notin k(R)$ , the ring $R\langle a^{-1}\rangle$ will also satisfy $1 \notin k(R\langle a^{-1}\rangle)$ . Further, a direct limit of such rings will have the same property. Hence we may repeatedly adjoin inverses of elements not in $k$ of our ring, until we reach a stage where this ideal contains all noninvertible elements ; at this point, there is nothing we can do but divide out by this ideal ; the result will be a sfield. We thus see that $k(R)$ consists precisely of the elements that go to zero under all maps into sfield. It is not hard to show that the sfields that can be so constructed correspond to the generic points of the maximal irreducible closed subsets of Sfield-Spec $R$ .

Given $R$ with $1 \notin k(R)$ , we can ask the question whether sfields constructed by this method are unique. This is clearly equivalent to asking whether Sfield-Spec $R$ is irreducible ; and the answer, we have seen, is "yes" for the $R$ we are interested in, the free $\mathcal{C}(D)$-algebra over $D$ , $D\langle X_1 , \dots , X_n\rangle$ . Another question is whether the intermediate sfields $R\langle a_1^{-1} , \dots , a_i^{-1} , \dots\rangle$ all have the property

$k(R') = 0$ . If this is so, we can say that a rational expression $f$ is zero in all sfields into which we can map $R$ , if, and only if, this can be deduced from the existence of those inverses $a_1^{-1}, \dots, a_m^{-1}$ which it involves (for that is just to say that it is zero in $R\langle a_1^{-1}, \dots, a_m^{-1}\rangle$ ). Note that, for general $R$ , it is also equivalent to saying that the "naive" method of constructing enveloping sfields really works for $R$ .

We may note that the three complicated identities given on the first page of [1]:

$$x - [x^{-1} + (y^{-1} - x)^{-1}]^{-1} - xyx = 0 \quad ,$$

$$y - [(1 + y)\ x(1 + y)^{-1} - yxy^{-1}]^{-1}\ [x - (1 + y)\ x(1 + y)^{-1}] = 0 \quad ,$$

$$y^{-1}(x^{-1} + y^{-1})x^{-1} - (x + y)^{-1} = 0 \quad ,$$

can all be reduced to polynomial tautologies, by multiplying on the right or left by elements whose inverses appear, simplifying, and repeating the process. Hence these identities can indeed be deduced from the existence of the inverses they involve. But, whether this is generally true, we cannot say.

Another independent question of interest is whether there is any algorithm for determining whether a given expression is zero in $D\langle X\rangle_\infty$ .

12. - We shall now consider the case of sfields $E$ finite-dimensional over their centers. Here, we will be able to apply the concepts of classical algebraic geometry, considering $E$ as an affine variety $C^{d^2}$ over its center $C$ , and rational functions on $E^n$ as rational maps from $C^{d^2 n}$ to $C^{d^2}$ .

If $E$ is any simple ring (in particular, a sfield) of dimension $d^2 < \infty$ over its center, it is known that there exists a finite extension field $K$ of $\mathcal{C}(E)$ , such that $E \otimes_{\mathcal{C}(E)} K$ is isomorphic over $K$ to $m_d(K)$ , the ring of $d \times d$ matrices over $K$ . To make use of this fact, we shall have to allow ourselves to make constructions of the sort $D_{E^n}\langle\langle X\rangle\rangle$ , in cases where $E$ is not a sfield, e. g., where $E$ is a matrix ring. The first step, of constructing a pre-sfield of functions on $E^n$ , and topologizing this set by using "domains of definition", goes over unchanged (except that the domain of definition of $f^{-1}$ will be the set where $f$ is defined and invertible). However, the arguments by which we concluded (for $E$ , a sfield with infinite center) that $E^n$ was irreducible, and that our equivalence classes of nondegenerate functions formed a sfield (specifically : that we could take inverses) involved the fact that, if a function was non-zero at a point, its inverse was defined there. To get a sfield of rational functions using a more general ring E (over some base sfield D , as usual), it is necessary to prove in some manner that :

(1) If two rational functions on E with coefficients in D are nondegenerate, they have a common point of definition, and

(2) If a rational function of this sort is non-zero at any point, it is invertible at some point.

A situation in which these conditions are immediate is if E is rational-identity-equivalent over D to some sfield E' , or indeed, to any ring over D for which these conditions hold (meaning that any rational function that is nondegenerate, or is identically zero, over one, is so as well over the other).

LEMMA 9. - Let A be a finite-dimensional algebra with unit over a field K , and let L , L' be extension fields of K of infinite cardinality. Then $A \otimes L$ , $A \otimes L'$ , are rational-identity-equivalent over A ; hence also over any subring $D \subseteq A$ .

Proof. - It follows immediately from elementary algebraic geometry ; we must merely note that the multiplicative inverse of $a \in A$ will be a rational function of a over K , computable with the help of the determinant over K of the map multiplication-by-a .

We shall also need :

LEMMA 10. - Let D be any sfield, and d , d' positive integers. Then the matrix ring $m_d(D)$ can be mapped homomorphically over D into $m_{d'}(D)$ , if, and only if, $d|d'$ , and the map is then unique up to a D-automorphism of $m_{d'}(D)$ .

Proof. - This may be restated : $D^{d'}$ can be made an $m_d(D)$-module, if, and only if, $d|d'$ , and then in essentially only one way. This is immediate from the theory of modules over matrix rings.

(In our application, the D of lemma 10 will be a field.)

LEMMA 11. - Let D be a sfield of finite dimension $d^2$ over its center, and let d' be any positive integer. Then, all simple regular over-rings $E \supseteq D$ , with infinite center, and of dimension $d'^2 d^2$ over this center, are rational-identity-equivalent over D .

Proof. - By lemma 9, all over-rings of the form $m_{d'}(D \otimes_{C(D)} L)$ (equivalently : $m_{d'}(D) \otimes L$ ), for L an infinite over-field of C(D) , will be equivalent to each other, so it will suffice to show any E equivalent over D to such a ring.

Given $E$ , we can choose an extension field $K_0$ of $C(E)$ , such that

$$E \otimes_{C(E)} K_0 \simeq m_{d'd}(K_0) \quad ,$$

and an extension $K$ of $K_0$ , such that

$$D \otimes_{C(D)} K \simeq m_d(K) \quad .$$

Then, upon tensoring with $K$ , the inclusion of $D$ in $E$ becomes isomorphic to a map of $m_d(K)$ into $m_{d'd}(K)$ ; the inclusion of $D \otimes K$ in $m_{d'}(D \otimes K)$ is also such a homomorphism. By lemma 10, these maps are identical up to a $K$-automorphism of the range ring, hence $E \otimes_{C(E)} K$ and $m_{d'}(D \otimes K)$ are isomorphic <u>over</u> $D$ , hence $E$ and $m_{d'}(D \otimes K)$ are rational-identity-equivalent over $D$ , by lemma 9.

THEOREM 12. - <u>Given</u> $D$ , $d'$ <u>as in lemma</u> 11, <u>and a positive integer</u> $n$ , <u>all rings</u> $E$ <u>of the class described permit construction of a sfield</u> $D_{E^n}\langle\langle X_1 , \ldots , X_n\rangle\rangle$, <u>and all these sfields are equal ; this sfield will be called</u> $D\langle\langle X_1 , \ldots , X_n\rangle\rangle_{d'd}$ .

<u>Proof</u>. - Since all the rings referred to are rational-identity-equivalent, to prove the first statement, it suffices to give one example which is a sfield : $D'_d$ (see note (3)) is such.

13. - It remains to determine, for given $D$ (finite-dimensional over its center) and $n$ , whether all the sfields $D\langle\langle X_1 , \ldots , X_n\rangle\rangle_d$ ( $d$ ranging over multiples of $(D:C(D))$ and $\infty$ ) are distinct, and which are specializations of which. We shall begin with the first question.

If $D$ is a field and $n = 1$ , $D\langle\langle X_1\rangle\rangle_d$ , for any $d$ , will be commutative ; it is easy to see that it will thus be the rational-function field $D\langle X_1\rangle$ , independent of $d$ . In all other cases, we shall show that $D\langle\langle X\rangle\rangle_d$ is of dimension $d^2$ over its center, so that they are all distinct.

$D\langle\langle X\rangle\rangle_d$ will be of dimension $\leq d^2$ over its center, because, we recall, this is equivalent to a certain polynomial $P_d(y_1 , \ldots , y_{2d})$ being identically zero in this sfield, which will be true, because it is so in the rings $E$ from which the sfield was constructed. (We define $P_\infty = 0$ .) What we must prove, then, is that $P_{d'}$ is not identically zero in $D\langle\langle X\rangle\rangle_d$ for $d' < d$ . (Note that this is obvious, if $n \geq 2d'$ .)

LEMMA 13. - <u>Let</u> $K$ <u>be an algebraically closed field</u>, $d$ <u>a positive integer, and</u> $\alpha$ <u>any noncentral element of</u> $m_d(K)$ . <u>Then there exists</u> $\beta \in m_d(K)$ <u>such that</u> $\alpha$ <u>and</u> $\beta$ <u>generate</u> $m_d(K)$ <u>as a</u> $K$-<u>algebra</u>.

Proof. - We can assume $\alpha$ in Jordan canonical form. Let $\beta$ be the matrix which cyclically permutes the elements of our basis.

If $\alpha$ has more than one eigenvalue, the subalgebra it generates will contain a projection onto a non-zero proper sub-space of $K^d$ : a diagonal matrix $\gamma$ , with some zeroes, and some ones on the diagonal. If there is only one eigenvalue, we can get a nilpotent matrix, having $d-1$ or fewer ones just above the diagonal, zeroes elsewhere, and, multiplying by $\beta$ , we get a $\gamma$ of the same sort as before. Conjugating $\gamma$ by various powers of $\beta$ , and multiplying together the results, we can get a matrix with only a single diagonal entry " 1 ". Multiplying this on the right and on the left by all combinations of powers of $\beta$ , we get the whole K-basis $\{e_{ij}\}$ of $m_d(K)$ .

LEMMA 14. - Let $K$ be an infinite field, $E$ a central simple K-algebra of dimension $d^2 < \infty$ , and $\alpha$ a noncentral element of $E$ . Then there exist $\beta \in E$ , and polynomials $f_1(X , Y) , \dots , f_{2d-2}(X , Y)$ over $K$ such that

$$P_{d-1}(f_1(\alpha , \beta) , \dots , f_{2d-2}(\alpha , \beta)) \neq 0 \quad .$$

Proof. - Straightforward, from lemma 13 and methods we have used before.

THEOREM 15. - Let $D$ be a sfield, $d$ a multiple of $(D:\mathcal{C}(D))$ , or $\infty$ , and $n$ a positive integer. Then, if either $D$ is noncommutative or $n > 1$ , $D\langle X_1 , \dots , X_n\rangle_d$ is of dimension $d^2$ over its center.

Proof. - First, assume $D$ finite-dimensional over its center ; $(D:\mathcal{C}(D)) = d'^2$ ; and $d < \infty$ .

If $D$ is noncommutative, its center will be an infinite field $K$ . Let $\alpha \in D - K$ . Then $\alpha$ will also be a noncentral element of $E = m_{d/d'}(D)$ . Choose $\beta$ and $f_1 , \dots , f_{2d-2}$ as in lemma 14. Then the function

$$P_{d-1}(f_1(\alpha , X_1) , \dots , f_{2d-2}(\alpha , X_1)) \in D\langle X_1 , \dots , X_n\rangle_d$$

will be non-zero at the point $(\beta , 0 , \dots , 0) \in (m_{d/d'}(D))^n$ , hence non-zero. So $P_{d-1}$ is not identically zero on $D\langle X\rangle_d$ .

Q. E. D.

If $n > 1$ , we choose $\alpha , \beta \in E = m_{d/d'}(D)$ and $f_1 , \dots , f_{2d-2}$ as in lemma 14, and look at $P_{d-1}(f_1(X_1 , X_2) , \dots , f_{2d-2}(X_1 , X_2))$ at $(\alpha , \beta , 0 , \dots)$ , and get the same result.

If $D$ is still finite-dimensional over its center, but $d = \infty$ , we note that $D\langle\langle X\rangle\rangle_\infty$ will not satisfy any equation $P_d = 0$ $(d < \infty)$ , because it can be specialized to a $D\langle\langle X\rangle\rangle_{d'}$ , $d' > d$ , which does not satisfy this equation.

If $D$ is infinite-dimensional over its center, the result is trivial.

14. - THEOREM 16. - Let $D$ be a sfield of dimension $d^2 < \infty$ over its center, let $e$ , $e'$ and $n$ be positive integers, and suppose $d > 1$ or $n > 1$ . Then, a necessary condition for there to exist a specialization of $D\langle\langle X_1 , \ldots , X_n\rangle\rangle_{de}$ into $D\langle\langle X_1 , \ldots , X_n\rangle\rangle_{de'}$ (respecting $D$ and the $X_i$ ) is that $e$ be $> e'$ ; and a sufficient condition is that $e$ be divisible by $e'$ .

Proof. - The necessity of the first condition is clear : in the contrary case, we have just constructed a polynomial in $X$ , that goes to 0 in the first sfield, but not in the second. To see the sufficiency of the second condition, let $E$ and $E'$ be regular over-rings of $D$ , having infinite centers, and with

$$(E:\mathcal{C}(E)) = (de)^2 , \qquad (E':\mathcal{C}(E')) = (de')^2 .$$

Tensoring with appropriate base extensions, $D$ , $E$ and $E'$ become $m_d(K)$ , $m_{de}(K)$ and $m_{de'}(K)$ , where the maps of the first matrix ring into the other two can be taken to be those induced by writing $m_{de}(K)$ (resp. $m_{de'}(K)$ ) as $m_d(m_e(K))$ (resp. $m_d(m_{e'}(K))$ ). Now, it is clear that, if $e$ is divisible by $e'$ , we can similarly embed $m_{de'}(K)$ in $m_{de}(K)$ , hence every rational identity satisfied by the latter ring over $m_d(K)$ is also satisfied by the former. Hence, in particular, every rational identity satisfied by $E$ over $D$ is satisfied by $E'$ .

Q. E. D.

In [1], AMITSUR claims, in effect, that the first condition is not only necessary but sufficient. However, his argument is based on theorem 1 of [3], and the proof of this theorem uses maps of matrix rings which do not preserve the identity element. Hence that theorem must be interpreted as referring only to polynomial relations not involving the unit, and in particular cannot be applied to those relations by which the inverse of an element is defined.

It can be deduced from theorem 1 of [3], or, more directly, from the fact that $m_a(K)$ embeds in $m_b(K)$ as a ring without unit whenever $a \leq b$ , that, in the situation of the above theorem, $e > e'$ implies that all polynomial identities holding in $E$ with coefficients in $D$ hold in $E'$ as well. It might appear that we could extend this to rational relations with the help of lemma 5, using $D!_e$ for the $D$ of that lemma, and $D!_{ee'}$ (which contains $D!_{e'}$ ) for the $E$ . Unfortunately, the polynomials to which the lemma refers would then have coefficients in $D!_e$ ; and $D!_{e'}$ (as a sub-sfield of $E$ ), in general, does not satisfy all such polynomials that $D!_e$ does.

It would be interesting to study the structure of $(D\langle\langle X\rangle\rangle_{dd'})$ $(d^2 = (D: (D)))$. Is it pure transcendental over $(D)$ ?

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