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Exposé n° 3

RELATIVISTIC HYDRODYNAMICS OF THE DIRAC MATTER

(Exposé de Takehiko TAKABAYASI,[1] les 11.12.1956 et 8.1.1957)

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ABSTRACT.

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PART I.

Despite various related attempts, it has not been attacked in a consequent manner to establish a complete tensor formulation for the Dirac electron field and also thereby to grasp this with a concrete model strictly. This work is concerned with solving these problems. It is shown that the Dirac field can equivalently be represented in a closed form with the set of real field variables :

(1) $$\{ \rho , \theta , u_\mu , w_\mu , k_\mu \} ,$$

which are restricted by the set of subsidiary conditions,

(2) $$u_\mu^2 = -1 , \quad w_\mu^2 = 1 , \quad u_\mu w_\mu = 0 ,$$

and

(3) $$\partial_{[\mu} k_{\nu]} = -\frac{i}{2\varkappa} \varepsilon_{\alpha\beta\gamma\delta} u_\alpha w_\beta (\partial_\mu u_\gamma \partial_\nu u_\delta - \partial_\mu w_\gamma \partial_\nu w_\delta) - \frac{e}{mc^2} F_{\mu\nu} ,$$

where $F_{\mu\nu}$ is the electromagnetic field strength acting on the Dirac field, $\varepsilon_{\alpha\beta\gamma\delta}$ is Levi-Civita's antisymetric symbol, and

$$\varkappa = mc/\hbar , \quad \partial_\mu = \frac{\partial}{\partial x_\mu} , \quad x_4 = ict .$$

The variables (1) directly mean density (ρ) , velocity (u_μ) , momentum (k_μ) , and spin (w_μ) , and one more internal variable θ , respectively.

Formally speaking, the Dirac spinor Ψ and γ-matrices are here superseded completely by the set of two scalar (a scalar ρ and a pseudoscalar θ) and three vector (two vector u_μ , k_μ , and a pseudovector w_μ) variables. The established formulation represents a new kind of relativistic hydrodynamics.

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This yields the hydrodynamical model of the Dirac matter, to which one is led almost inevitably when one tries to depict it into some exact model.

The structure of this formulation is clarified in detail. Equations of motion are brought to compact hydrodynamical forms. Besides them, an important subsidiary condition (3) governs the behaviour of the matter to connect the vorticity of the momentum field to the velocity and spin fields. All of those basic equations of the formulation are deduced from its own Lagrangian. The basic equations are cast also in various different forms, each of which is useful to manifest different aspect of the theory.

Some characteristic features of the hydrodynamics are the fact that the distinctions between proper mass density and rest particle density and also between particle momentum and velocity are primarily specified by the variable θ associated with the possibility of "donkey" like behaviours. The energy-momentum conservation law also has the meaning from the Eulerian equation of flow, showing the occurrence of heat flow and mechanical stress inside the fluid.

The theory also provides us with a new directly physical point of view concerning various transformation properties of the Dirac field, e. g ; those for charge conjugation and time reversal.

The theory is formulated for cases of Dirac matter under external electromagnetic field and also of interacting Dirac and electromagnetic fields. It is manifestly gauge-independent in both cases. In this work, however, the theory is worked out only for the case of c-number Dirac field, in other words, the quantum mechanics of a Dirac electron.

The mathematical apparatus is also systematized to establish how one can manipulate the Dirac field solely with the set of tensor quantities which are related to the Dirac spinor as its bilinear covariants.

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The brief account of this work appeared in

Progr. theor. Phys., 13 (1955) 222 .

Nuovo Cimento (10) 3 (1956) 233-242 .

Phys. Rev., 102 (1956) 297 .

More detailed article will shortly appear in Ann. Inst. H. Poincaré, where references to other papers are also listed.

PART II.

Related with the above work, we also reported the following investigations.

1) We have given the general theory of relativistic hydrodynamics of fluids that carry in general the distributions of intrinsic angular momentum. Especially, we made analyses into the physical meanings of the energy-momentum tensor and angular momentum tensor. We then classified various relativistic hydrodynamics by clarifying the conditions that characterize the specialities of any hydrodynamics.

2) We have pointed out the general situation that in the hydrodynamics representing quantum matters there always appear the flow of heat besides mechanical stress inside the fluid. This situation is illustrated for the case of the hydrodynamics representing the Schrödinger field also : the quantum-mechanical effect must be represented by the occurrences of quantum stress along with "quantum heat flow" there.

3) We analysed the hydrodynamics of Weyssenhof to compare it with the above Dirac hydrodynamics and showed that the former can be reduced essentially to a single equation of motion which manifests that each particle of this fluid moves exactly in the same manner with the spinless charged particle under certain constant hypothetical electromagnetic field.

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As for the details of the theories mentioned in 1) , 2) and 3), separate papers are prepared to be published shortly also.

4) The author would like to add certain new extension of the hydrodynamical theory of the Dirac field ;

We introduce the quantity

$$a_{\mu}^{(\xi)} \quad , \qquad (\xi , \mu = 1 , 2 , 3 , 4)$$

at each point in space, satisfying the constraints :

$$a_{\mu}^{(\xi)} \, a_{\nu}^{(\xi)} = \delta_{\mu\nu} \quad , \tag{4}$$

and identify the velocity and spin variables in (1) such that

$$\begin{cases} i \, v_{\mu} = a_{\mu}^{(4)} & , \\ w_{\mu} = a_{\mu}^{(3)} & . \end{cases} \tag{5}$$

Then the constraints (2) become part of the condition (4). The $a_{\mu}^{(\xi)}$ evidently represents a Loventz transformation defined at each point of space. Thus the

Dirac spinor ψ is represented by the set of variables

$$(6) \qquad \left\{ a_\mu^{(\xi)} , \rho , \theta , \Lambda \right\} ,$$

where Λ is a scalar function. The k_μ variables obeying (3) are now represented in terms of (6) such that

$$(7) \qquad k_\mu = -\frac{e}{mc^2} A_\mu + \partial_\mu \Lambda - \frac{i}{2\varkappa} a_\nu^{(1)} \partial_\mu a_\nu^{(2)} ,$$

where A_μ is the electromagnetic potential.

Now in this representation all of the basic equations of theory are derived from the following Lagrangian density :

$$(8) \qquad \mathcal{L} = -\frac{\hbar c}{2} \rho \left\{ a_\mu^{(3)} \partial_\mu \theta + 2\varkappa \cos\theta + i a_\mu^{(4)} (-2\varkappa \partial_\mu \Lambda + 2\frac{e}{\hbar c} A_\mu + a_\nu^{(1)} \partial_\mu a_\nu^{(2)}) - \right.$$

$$\left. - i \,\varepsilon_{\alpha\beta\gamma\delta}\, a_\alpha^{(4)} a_\beta^{(3)} \partial_\gamma a_\delta^{(4)} \right\} - \frac{\hbar c}{2} \lambda_{\mu\nu} (a_\mu^{(\xi)} a_\nu^{(\xi)} - \delta_{\mu\nu}) .$$

5) Finally, the treatment similar to 4) has also been worked out for the non-relativistic case. There the theory means to depict the Pauli field (two-component spinor) by the model of an assembly of very small rotating bodies continuously distributed in space, and amounts to making the hydrodynamical representation of the Pauli field ([2]) more concrete by giving the model to the particle constituting the fluid. In this case the formal bases of the theory are first the introduction the variable Λ which is conjugate to the density ρ , and second the resolution of the spin variables S_i $(i = 1 , 2 , 3)$ into conjugate pairs $(a_i^{(1)} , a_i^{(2)})$ such that

$$\vec{S} = \frac{\hbar}{2} [\vec{a}^{(1)} \times \vec{a}^{(2)}] , \qquad \vec{a}^{(1)} \perp \vec{a}^{(2)} .$$

The theory realizes new canonical formalism of the Pauli-field keeping to the realistic representation. Thus it will allow us to make the second quantization of the theory in this form.

The details and physical meanings of the theories mentioned in 4) and 5) will be given in forthcoming papers.

([2]) T. TAKABAYASI, Prog. theor. Phys., 12, (1954) 810 ; 14 (1955) 283 .