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Remarks concerning the canonical formulation of field equations

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REMARKS CONCERNING THE CANONICAL FORMULATION OF FIELD EQUATIONS

par C. LANCZOS

[Summation over repeated indices is assumed].

1. The method of surplus variables.

The field equations of general relativity become hopelessly complicated if we deviate from the simple Lagrangean $L = R_{\mu\nu} g^{\mu\nu}$, chosen by EINSTEIN in his relativistic investigations. The logically appealing possibility of an action principle with the action density

$$L = \frac{1}{2} (R_{\mu\nu} R^{\mu\nu} + \beta R^2) \tag{1}$$

leads to differential equations of the fourth order in the $g_{\mu\nu}$ which in their original form cannot be tackled successfully. Great simplifications are obtained however, by the introduction of properly chosen surplus variables. The use of such variables permits us to reduce an arbitrarily complicated system of field equations to a normal form, comparable to the "canonical form" of the Hamiltonian equations of dynamics, if only one single variable is involved. This transformation is based on two leading principles :

a. Elimination of algebraic variables. - Let w be one of the dynamical variables which appears in L without its derivatives :

$$A = \int L(w)\, dt \tag{2}$$

The variations of w gives

$$\delta A = \int \frac{\partial L}{\partial w} \delta w \, dt$$

and the variational principle $\delta A = 0$ yields the condition

$$\frac{\partial L}{\partial w} = 0 \tag{3}$$

Since we get no "boundary term", the condition (3) guarantees the vanishing of δA without demanding the vanishing of w at the limits.

If we now eliminate w from the relation (3) in terms of the other variables, this elimination means that the relation which holds for the actual motion, is assumed for the varied motion as well. Generally this elimination is not permitted because it restricts the free variation of w in a way which may not satisfy the condition that the variation must vanish at the two endpoints, (or more generally on the boundary of the domain). Since, however, this condition is not demanded in the case of an algebraic variable, the preliminary elimination of an algebraic variable on the basis of the equation (3) is always permitted.

b. The momenta as Lagrangean multipliers. - Let D be an arbitrarily chosen differential operator, operating on u. Let the Lagrangean of the action principle be some function of Du. We will now replace Du by the algebraic variable w, considering the equation $Du - w = 0$ as an auxiliary condition of the given variational problem. By this artifice we have not changed anything on the original problem but we have now $L(w)$ instead of $L(Du)$, with the auxiliary condition

$$Du - w = 0 \tag{4}$$

Applying the method of the Lagrangean multiplier we obtain the new Lagrangean

$$L' = L(w) + p(Du - w) \tag{5}$$

The new L' is variationally completely equivalent to the original L but it has the added advantage that it is linear in Du, while the earlier L may have contained Du in an arbitrarily involved fashion. On the other hand, we have added to u the two new variables w and p. But w is a purely algebraic variable and can thus be eliminated, according to (a): this means that we solve the equation

$$\frac{\partial L'}{\partial w} = \frac{\partial L}{\partial w} - p = 0 \tag{6}$$

for w and substitute the w thus obtained in L'. The new L' is a function of u and p but w is no longer present. The number of variables has increased to 2 but not 3.

The method described is essentially equivalent to "Ampère's transformation", but has the advantage of greater flexibility. It also brings the role of the momenta as Lagrangean multipliers in direct evidence.

2. Linearization of the quadratic action principle.

Consider the action principle (1). We introduce $R_{ik} = w_{ik}$ as an algebraic variable, obtaining the new Lagrangean

$$L' = \frac{1}{2}(w_{ik}\, w^{ik} + \beta\, w^2) + p^{ik}(R_{ik} - w_{ik}) \tag{7}$$

We eliminate w_{ik} with the help of the equation

$$\frac{\partial L'}{\partial w_{ik}} = w^{ik} + \beta w g^{ik} - p^{ik} = 0$$

which gives

$$w^{ik} = p^{ik} + \sigma\, p g^{ik}$$

$$p^{ik} + \sigma\, p g^{ik} + \beta(1 + 4\sigma)\, p g^{ik} - p^{ik} = 0$$

$$\sigma + \beta(1 + 4\sigma) = 0$$

$$\sigma = -\frac{\beta}{1+4\beta}$$

We thus obtain

$$L' = p^{ik} R_{ik} + \frac{1}{2} p^{ik} w_{ik} - p^{ik} w_{ik} = p^{ik} R_{ik} - \frac{1}{2}(p^{ik} p_{ik} + \sigma p^2) \tag{8}$$

In the new form of the action principle we have the double number of variables, viz. g_{ik} and p^{ik}, but the complicated differential operator R_{ik} appears now in linear form, just as in Einstein's action principle. In fact, we obtain Einstein's linear action principle if we restrict p^{ik} by the condition

$$p^{ik} = \varepsilon\, g^{ik}$$

Then our action principle becomes

$$L = \varepsilon\,(R - 2\varepsilon\,(1 + 4\sigma))$$

and we are back at the action principle of Einstein's "cosmological equations".

3. Relations to Weyl's theory.

The linearized action principle leads to certain consequences which one would hardly recognize without the method of the surplus variables. Taking advantage of the vanishing divergence of the "metrical matter tensor" $R_{ik} - \frac{1}{2} R\, g_{ik}$, we can change $p_{ik} R^{ik}$ to

$$[p_{ik} + \frac{1}{2}(\Phi_{i;k} + \Phi_{k;i} - \Phi^{\alpha}{}_{;\alpha}\ g_{ik})]\ R^{ik}$$

where Φ_i is an arbitrary vector of the field [1]. Integration by parts shows that we have merely added a pure divergence which is variationally zero ; indeed the added term becomes

$$\frac{\partial}{\partial X_\alpha}\ \sqrt{g}\,(R^{i\alpha}\ \Phi_i - \frac{1}{2}\,R\,\Phi^{\alpha}) - \Phi_\alpha\underbrace{(R^{\alpha\mu} - \frac{1}{2}\,Rg^{\alpha\mu})_{;\mu}}_{0}\ \sqrt{g}$$

Now, if we introduce the modified p_{ik} as a new p_{ik} and perform the proper mathematical transformations, we obtain the new Lagrangean

$$(9)\quad L = p^{ik}\ R^{\star}_{ik} - \frac{1}{2}(p^{ik}\ p_{ik} + \sigma\,p^2) - \frac{1}{8}\,F_{ik}\ F^{ik} - \frac{1}{2}(1 + \sigma)(\Phi^{\alpha}{}_{;\alpha} - \frac{1}{2}\,\Phi^{\alpha}\,\Phi_\alpha)^2$$

where $F_{ik} = \Phi_{i,k} - \Phi_{k,i}$ is the usual "electromagnetic field strength", while $R^{\star}_{ik}$ is a "modified Riemann tensor" :

$$(10)\quad R^{\star}_{ik} = R_{ik} - \frac{1}{2}(\Phi_{i;k} + \Phi_{k;i} + \Phi_i\,\Phi_k) + \frac{1}{2}[(1 + 2\sigma)\,\Phi^{\alpha}{}_{;\alpha} - \sigma\,\Phi^{\alpha}\,\Phi_\alpha]g_{ik}$$

While this result came about by a legitimate canonical transformation, WEYL assumed a form of geometry which is more general than Riemann's geometry. WEYL deduced the modified $R^{\star}_{ik}$ on the basis of geometrical considerations. His $R^{\star}_{ik}$ and the present $R^{\star}_{ik}$ differ in quantities of second order. For the special choice $\sigma = -1$ of the parameter σ the two kinds of $R^{\star}_{ik}$ coincide exactly. Hence it seems unnecessary to enlarge the Riemannian basis of geometry for the sake of the "vector potential" since the quadratic action principle yields this quantity automatically, as a natural consequence of the given transformation possibilities, without any enlargement of the basic geometrical structure. The vectorial function Φ_i appears here as a "function of integration" which occurs in the solution of the basic field equations. A closer investigation reveals that the interpretation of Φ_i as the "vector potential" is untenable and is to be replaced by the interpretation "electric current".

4. The canonical form of variational equations.

All variational equations of a single variable t can be reduced to Hamilton's canonical equations :

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(1) " ; " denotes covariant differentiation.

(11) $$\dot{q}_i = \frac{\partial H}{\partial p_i} \qquad \dot{p}_i = -\frac{\partial H}{\partial q_i}$$

which are derived from the action principle

(12) $$L = p_\mu \; \dot{g}_\mu - H \; .$$

We want to generalize the canonical system to the case of more than one independent variable, in fact to the case of an arbitrary number n of independent variables.

This can always be accomplished by the proper introduction of surplus variables. The only difference compared with the one-dimensional case is that the simple "pairing" of q_i and p_i variables is no longer possible. Generally we have n independent variables $X_1, X_2, \ldots, X_n$, m dependent variables $\varphi_1, \varphi_2, \ldots, \varphi_m$, and M momenta $p_1, p_2, \ldots, p_M$. The canonical Lagrangean takes the form

(13) $$L = p_\mu \, \beta^\alpha_{\mu\nu} \frac{\partial \varphi_\nu}{\partial X_\alpha} - H$$

where the $\beta^\alpha_{\mu\nu}$ form a system of $n \times m \times M$ numerical coefficients, determined by the specific structure of the given variational problem. The first term of L shall be called "the canonical matrix form" of the canonical Lagrangean L.

The canonical equations become

(14) $$\beta^\alpha_{\mu\nu} \frac{\partial \varphi_\nu}{\partial X_\alpha} = \frac{\partial H}{\partial p_\mu}$$
$$-\beta^\alpha_{\mu\nu} \frac{\partial p_\mu}{\partial X_\alpha} = \frac{\partial H}{\partial \varphi_\nu}$$

The right side is purely algebraic in the variables φ_i, p_j. The linearization of the equations in the derivatives is once more accomplished.

5. The canonical system in uniform formulation.

The discrepancy of dimensions between the φ_i and the p_j can be eliminated by including the entire set of variables (φ_i, p_j) into one single set of $m + M = N$ variables φ_i. This can be done by putting $p_j = \varphi_{m+j}$. The "phase space" is then replaced by the single N-dimensional "configuration space" of the uniformized variables φ_i. Let us do thet first for the one-dimensional case. We write for p_i the new q-variable q_{n+i}, thus obtaining the single system of $2n$ variables q_i $(i = 1, 2, \ldots, 2n)$. Now the canonical integrand becomes

$$L = q_i \ \beta_{ik} \ \dot{q}_k - H(q_\alpha \ , t)$$

where β_{ik} is a numerical matrix. The first term is variationally aquivalent to

$$q_i \ \beta_{ik} \ \dot{q}_k - \frac{1}{2} \frac{d}{dt}(q_i \ \beta_{ik} \ q_k) = \frac{1}{2}(q_i \ \beta_{ik} \ \dot{q}_k - \dot{q}_i \ \beta_{ik} \ q_k) = \frac{1}{2}(\beta_{ik} - \beta_{ki}) \ q_i \ \dot{q}_k$$

Since $\frac{1}{2}(\beta_{ik} - \beta_{ki})$ is anti-symmetric in i , k , we can put

(15) $$L = \frac{1}{2} q_i \ \beta_{ik} \ \dot{q}_k - H$$

with the understanding that β_{ik} is an anti-symmetric matrix $\beta_{ki} = -\beta_{ik}$. The Hamiltonian system is characterized by the matrix

$$\beta = \left[\begin{array}{c|c} 0 & I \\ \hline -I & 0 \end{array}\right]$$

The canonical system is now included in the single equation

$$\beta_{i\alpha} \dot{q}_\alpha = \frac{\partial H}{\partial q_i}$$

The generalizatbion to partial differential equations occurs in the form

(16) $$L = \frac{1}{2} \ \varphi_i \ \beta^{\alpha}_{i\mu} \ \frac{\partial \varphi_\mu}{\partial X_\alpha} - H$$

$(\beta^{\alpha}_{ik} = - \beta^{\alpha}_{ki})$. The canonical equations become

(17) $$\beta^{\alpha}_{i\mu} \ \frac{\partial \varphi_\mu}{\partial X_\alpha} = \frac{\partial H}{\partial \varphi_i}$$

6. Perturbation problems.

The canonical system is of particularly great advantage if a perturbation problem is involved. Let φ_i be a solution of the canonical equations and let us suppose that we are interested in studying a small perturbation $\varphi_i + \varepsilon \bar{\varphi}_i$, caused by slightly different boundary conditions or the introduction of a weak external field. Now the "second variation" of a complicated Lagrangean L can easily become a hopelessly complicated differential operator. In the canonical form, however, only the second derivatives of the algebraic operator H are demanded. The perturbation equations become

(18) $$\beta^{\alpha}_{i\mu} \ \frac{\partial \bar{\varphi}_\mu}{\partial X_\alpha} = \frac{\partial^2 H}{\partial p_i \ \partial \varphi_\nu} \ \bar{\varphi}_\nu$$

These are a linear first order system of equation with variable coefficients because the quantities $\frac{\partial^2 H}{\partial \varphi_i \partial \varphi_\nu}$, after substituting for φ_i the explicit solution in terms of the X_j, become explicit functions of the x_j.

7. The conservation laws.

In ordinary dynamics the conservation law for the total energy H is a simple consequence of the Hamiltonian system :

$$\dot{H} = \frac{\partial H}{\partial q_\alpha} \dot{q}_\alpha + \frac{\partial H}{\partial p_\alpha} \dot{p}_\alpha$$

$$= - \dot{p}_\alpha \dot{q}_\alpha + \dot{q}_\alpha \dot{p}_\alpha$$

In the uniformized formulation :

$$\dot{H} = \frac{\partial H}{\partial q_\alpha} \dot{q}_\alpha = \beta_{\alpha\mu} \dot{q}_\mu \dot{q}_\alpha$$

$$= \frac{1}{2}(\beta_{\alpha\mu} + \beta_{\mu\alpha}) \dot{q}_\mu \dot{q}_\alpha = 0 .$$

8. Field equations.

The analogous treatment in the realm of partial differential equations becomes

$$\frac{\partial H}{\partial X_i} = \frac{\partial H}{\partial \varphi_\mu} \frac{\partial \varphi_\mu}{\partial X_i} = \beta^\alpha_{\mu\nu} \frac{\partial \varphi_\nu}{\partial X_\alpha} \frac{\partial \varphi_\mu}{\partial X_i} = \beta^\alpha_{\mu\nu} \varphi_{\nu,\alpha} \varphi_{\mu,i}$$

$$(19) \qquad = \frac{1}{2} \beta^\alpha_{\mu\nu}(\varphi_{\nu,\alpha} \varphi_{\mu,i} - \varphi_{\mu,\alpha}\varphi_{\nu,i})$$

$$= \frac{1}{2} \frac{\partial}{\partial X_i}(\beta^\alpha_{\mu\nu} \varphi_{\nu,\alpha}\varphi_\mu) - \frac{1}{2} \beta^\alpha_{\mu\nu}(\varphi_\mu \varphi_{\nu,i})_\alpha$$

Hence

$$(20) \qquad \frac{\partial}{\partial X_i}\underbrace{(H - \frac{1}{2} \varphi_\mu \beta^\alpha_{\mu\nu} \varphi_{\nu,\alpha})}_{-L} + \frac{1}{2} \frac{\partial}{\partial X_\alpha}(\beta^\alpha_{\mu\nu}\varphi_\mu \frac{\partial \varphi_\nu}{\partial X_i}) = 0$$

We put

$$(21) \qquad t^\alpha_i = - L \delta^\alpha_i + \theta^\alpha_i$$

where

$$(22) \qquad \theta^\alpha_i = \frac{1}{2} \varphi_\mu \beta^\alpha_{\mu\nu} \frac{\partial \varphi_\nu}{\partial X_i}$$

The conservation laws of energy and momentum appear in the following form :

$$\frac{\partial t_i^\alpha}{\partial X_\alpha} = 0 \tag{23}$$

The construction of the tensor θ_i^α is exceptionally simple if we utilize the canonical scheme : "The tensor θ_i^α is obtained by replacing in the canonical matrix form the operation $\frac{\partial}{\partial X_\alpha}$ by $\frac{\partial}{\partial X_i}$". Of great importance is the following result : No matter how complicated the Hamiltonian function H may be (and thus, no matter how strongly non-linear our field equations may be), the function H appears solely in the scalar term $H\,\delta_i^k$; all the other terms of the tensor t_i^k are strictly quadratic in the canonical variables. This fact is fundamental for the dynamical evaluation of the conservation laws.

9. The graviational equations of Einstein.

The graviational equations $R_{ik} - \frac{1}{2} R\, g_{ik} = 0$ of Einstein are derivable from an action principle and can thus be brought into the canonical form. Here the Lagrangean is the scalar Riemannian curvature, multiplied by $\sqrt{|g|}$:

$$L = \sqrt{|g|}\, g^{ik} \left[\left(\frac{1}{2} \frac{\partial \Gamma^\alpha_{i\alpha}}{\partial X_k} + \frac{1}{2} \frac{\partial \Gamma^\alpha_{k\alpha}}{\partial X_i} - \frac{\partial \Gamma^\alpha_{ik}}{\partial X_\alpha} \right) + \Gamma^\beta_{i\alpha} \Gamma^\alpha_{k\beta} - \Gamma^\beta_{ik} \Gamma^\alpha_{\beta\alpha} \right] \tag{24}$$

The canonical form is automatically provided by "Palatini's Principle" according to which we can vary the g_{ik} and the Γ^m_{ik} independently in Einstein's variational principle. We thus have 50 dynamical variables. Moreover, the given Lagrangean is already in the canonical form, without further transformations, if we write it in the form

$$L = \left(\Gamma^\alpha_{ik} - \frac{1}{2} \Gamma_i\, \delta^\alpha_k - \frac{1}{2} \Gamma_k\, \delta^\alpha_i \right) \frac{\partial\, \sqrt{|g|}\, g^{ik}}{\partial X_\alpha} + \left(\Gamma^\beta_{i\alpha}\, \Gamma^\alpha_{k\beta} - \Gamma^\alpha_{ik}\, \Gamma_\alpha \right) \sqrt{|g|}\, g^{ik} \tag{25}$$

which is variationally equivalent to the original L . The canonical variables are thus the quantities

$$u^{ik} = \sqrt{|g|}\, g^{ik}$$

and the "momenta"

$$p^\alpha_{ik} = \Gamma^{*\alpha}_{ik} = \Gamma^\alpha_{ik} - \frac{1}{2} \Gamma_i\, \delta^\alpha_k - \frac{1}{2} \Gamma_k\, \delta^\alpha_i \qquad (\Gamma_i = \Gamma^\alpha_{i\alpha})$$

One of the advantages of the canonical system is that the "raising" and "lowering" of subscripts is completely avoided. We do not encounter the g_{ik} directly, but only in the combination $\sqrt{g}\, g^{ik}$. The difficult algebraic transformation which relates the g_{ik} and the g^{ik}, does not come into play at all.

According to the general rule, given above, we obtain the "canonical momentum-energy tensor" (also called the "pseudo-tensor" since it possesses the general covariance properties of an arbitrary tensor only with respect to <u>linear</u> transformations), in the following form :

$$
\begin{aligned}
t_i^k &= -L\delta_i^k + p^k_{\mu\nu} \frac{\partial u^{\mu\nu}}{\partial X_i} \\
&= -L\,\delta_i^k + \left(\Gamma^k_{\mu\nu} - \frac{1}{2}\Gamma_\mu\,\delta^k_\nu - \frac{1}{2}\Gamma_\nu\,\delta^k_\mu\right) \frac{\partial\, \sqrt{g}\, g^{\mu\nu}}{\partial X_i}
\end{aligned} \tag{26}
$$

We make use of the well-known fact that the second term of the Lagrangean (25) is equal to the first term, multiplied by $-\frac{1}{2}$. Hence

$$
L = \frac{1}{2}\, p^\alpha_{\mu\nu} \frac{\partial u^{\mu\nu}}{\partial X_\alpha} = \frac{1}{2}(t^\alpha_\alpha + L\,\delta^\alpha_\alpha)
$$

and thus

$$
-L = \frac{1}{2}\, t^\alpha_\alpha = \frac{1}{2}\, t \tag{27}
$$

The relation (26) can thus be written in the following alternate form :

$$
t_i^k - \frac{1}{2}\, t\,\delta_i^k = \left(\Gamma^k_{\mu\nu} - \frac{1}{2}\Gamma_\mu\,\delta^k_\nu - \frac{1}{2}\Gamma_\nu\,\delta^k_\mu\right) \frac{\partial \sqrt{g}\, g^{\mu\nu}}{\partial X_i} \tag{28}
$$

If the same tensor is constructed <u>without</u> the canonical method, we obtain ten much more complicated expression [2]

$$
\begin{aligned}
t_i^k - \frac{1}{2}\, t\,\delta_i^k = [&-2\,\Gamma^\mu_{i\nu}\,\Gamma^k_{\mu\rho}\, g^{\rho\nu} + \Gamma_i\,\Gamma^k_{\mu\nu}\, g^{\mu\nu} \\
&+ \Gamma_\nu(\Gamma^\nu_{i\mu}\, g^{\mu k} + \Gamma^k_{i\mu}\, g^{\mu\nu} - \Gamma_i\, g^{k\nu})]\sqrt{g}
\end{aligned} \tag{29}
$$

10. <u>The symmetric conservation laws.</u>

The canonical energy-momentum tensor t_i^k has the great disadvantage that it is <u>not symmetric</u> in i and k. We must demand, however, that the energy-momentum

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([2]) The fundamental paper of Einstein gives only the first term, since he normalized the reference system by the condition $g = 1$, that is $\Gamma_i = 0$.

tensor of physical action should be a symmetric tensor. Without this condition even the fundamental relation $E = mc^2$, the equivalence of inertial mass and energy, cannot be proved. Nor is it possible to derive a motion law for a particle submerged in an external field. We will now investigate under what conditions the energy-momentum tensor t_i^k can be symmetrized.

Let us assume that an arbitrary tensor b_{ik} , generally not symmetric, has the property that it can be generated as the divergence of a tensor of third order :

$$b_{ik} = \frac{\partial \Phi_{ik\alpha}}{\partial X_\alpha} = \Phi_{ik\alpha,\alpha} \tag{30}$$

We will now construct the following new tensor :

$$B_{ik} = \tfrac{1}{2}(\Phi_{ik\alpha} + \Phi_{ki\alpha} + \Phi_{i\alpha k} + \Phi_{k\alpha i} - \Phi_{\alpha ki} - \Phi_{\alpha ik})_{,\alpha}$$

Then

$$\begin{aligned} B_{ik,k} &= \tfrac{1}{2}(\Phi_{ik\alpha} + \Phi_{i\alpha k})_{,\alpha k} \\ &+ \tfrac{1}{2}(\underbrace{\Phi_{ki\alpha} - \Phi_{\alpha ik}}_{0})_{,\alpha k} + \tfrac{1}{2}(\underbrace{\Phi_{k\alpha i} - \Phi_{\alpha ki}}_{0})_{,\alpha k} \\ &= \Phi_{ik\alpha,\alpha k} = b_{ik,k} \end{aligned}$$

The tensor B_{ik} is symmetric :

$$B_{ki} = B_{ik}$$

We see that we succeeded in constructing a tensor which is symmetric and which has the same divergence, (with respect to k), as the original tensor.

We can prove directly that if $\Phi_{ik\alpha} = \Phi_{ki\alpha}$, we obtain

$$B_{ik} = B_{ki} = b_{ik}$$

Hence a symmetric tensor remains unaltered by our construction. On the other hand, if b_{ik} is anti-symmetric, and correspondingly

$$\Phi_{ki\alpha} = - \Phi_{ik\alpha}$$

we obtain for the substitute tensor

(31) $$B_{ik} = (\Phi_{i\alpha k} + \Phi_{k\alpha i})_{,\alpha}$$

We come to the conclusion that for the purpose of symmetrizing an arbitrary tensor without changing its divergence it is sufficient if the anti-symmetric part of the tensor is representable in the form (30).

The non-symmetric conservation law (23) was the mere consequence of the existence of a variational principle. But all the Legrangeans of the field equations of theoretical physics have certain definite invariance properties. In classical physics L allows an arbitrary rotation of the coordinate axes, without changing its form. Special relativity has shown that the invariance with respect to three-dimensional rotations had to be extended to invariance with respect to four-dimensional rotations (Lorentz-invariance). This invariance with respect to arbitrary rotations of the coordinates has the consequence that the anti-symmetric part of t_i^k can be written in the form (30) and can thus be symmetrized.

An arbitrary infinitesimal rotation of the coordinates X_i may be written in the form

(32) $$\delta X_i = \varepsilon\, \rho_{i\alpha}\, X_\alpha$$

where

(33) $$\rho_{ik} = -\rho_{ki}$$

Let the corresponding variation of φ_i be $\delta\varphi_i$ and let us form the variation of L, caused by this infinitesimal transformation:

(34) $$\delta L = \frac{1}{2}\, \delta\varphi_\mu\, \beta^\alpha_{\mu\nu}\, \varphi_{\nu,\alpha} + \frac{1}{2}\, \varphi_\mu\, \beta^\alpha_{\mu\nu}\, (\delta\varphi_\nu)_{,\alpha}$$
$$= \frac{1}{2}\, \varphi_\mu\, \beta^\alpha_{\mu\nu}\, \varphi_{\nu,\sigma}\, \varepsilon\rho_{\sigma\alpha} - \frac{\partial H}{\partial\varphi_\mu}\, \delta\varphi_\mu$$

In view of (17) we can put

(35) $$\delta L = -\frac{1}{2}\, \delta\varphi_\mu\, \beta^\alpha_{\mu\nu}\, \varphi_{\nu,\alpha} + \frac{1}{2}\, \varphi_\mu\, \beta^\alpha_{\mu\nu}\, (\delta\varphi_\nu)_{,\alpha}$$
$$+ \frac{1}{2}\, \varphi_\mu\, \beta^\alpha_{\mu\nu}\, \varphi_{\nu,\sigma}\, \varepsilon\rho_{\sigma\alpha}$$
$$= \frac{1}{2}(\varphi_\mu\, \beta^\alpha_{\mu\nu}\, \delta\varphi_\nu)_{,\alpha} + \varepsilon\, \theta^\alpha_\mu\, \rho_{\mu\alpha}$$

Since we have assumed that L is an <u>invariant</u> of an arbitrary orthogonal transformation, we have $\delta L = 0$ and thus we obtain

(36) $$\varepsilon\,\Theta^{\alpha}_{\mu}\,\rho_{\mu\alpha} = +\frac{1}{2}(\varphi_{\mu}\,\beta^{\alpha}_{\mu\nu}\,\delta\varphi_{\nu})_{,\alpha}$$

Now φ_i is a "vector" of an abstract configuration space but not a vector of ordinary space. It may represent an arbitrary combinaition of vectors, tensors, or even "spinors" of any kind. But we can state that under all circumstances the infinitesimal change $\delta\varphi_i$ of φ_i under the impact of an arbitrary infinitesimal rotation will be <u>linear</u> in the φ_μ and also <u>linear</u> in the coefficients $\rho_{\alpha\beta} = -\rho_{\beta\alpha}$. Hence we can put

(37) $$\delta\varphi_{\nu} = \varepsilon\,\varphi_{\mu}\;\rho_{ik}\,C^{ik}_{\mu\nu}$$

where the coefficients $C^{ik}_{\mu\nu}$ (anti-symmetric in i , k), are given constants which will depend on the covariant character of the quantities φ_i. Substitution in (36) gives :

(38) $$\varepsilon\,\Theta^{k}_{i}\;\rho_{ik} = +\frac{\varepsilon}{2}(\varphi_{\mu}\,\beta^{\alpha}_{\mu\nu}\,\varphi_{\sigma})_{,\alpha}\,C^{ik}_{\sigma\nu}\;\rho_{ik}$$

We will now abandon the notation Θ^{k}_{i} and replace it by Θ_{ik}, with the understanding that the second subscript takes the place of the previous "contravariant" subscript. In view of the fact that the coefficients $\rho_{ik} = -\rho_{ki}$ of an infinitesimal rotation can be chosen arbitrarily, we obtain from (38)

(39) $$\frac{1}{2}(\Theta_{ik} - \Theta_{ki}) = +\frac{1}{2}(\varphi_{\mu}\,\beta^{\alpha}_{\mu\nu}\,\varphi_{\sigma}\,C^{ik}_{\sigma\nu})_{,\alpha}$$

We define a tensor of third order (anti-symmetric in i , k) by putting

(40) $$\Phi_{ik\alpha} = +\frac{1}{2}\;\varphi_{\mu}\;\beta^{\alpha}_{\mu\nu}\,\varphi_{\sigma}\,C^{ik}_{\sigma\nu}$$

Then the relation (39) shows that the anti-symmetric part of Θ_{ik} appears in the form (30). Hence we can symmetrize this tensor according to (31) and obtain in the place of the original Θ_{ik} the new symmetric tensor

(41) $$S^{ik} = S^{ki} = \frac{1}{2}(\Theta_{ik} + \Theta_{ki}) + (\Phi_{i\alpha k} + \Phi_{k\alpha i})_{,\alpha}$$

The <u>symmetric conservation law</u> now becomes

(42) $$\frac{\partial(S^{i\alpha} - L\,\delta^{\alpha}_{i})}{\partial X_{\alpha}} = 0$$

The new (symmetric) tensor S^{ik} has once more the property that it is strictly quadratic in the canonical variables, quite irrespective of the structure of the Hamiltonian function H. The tensor $S^{ik} - L\,\delta_i^k$ can be conceived as the true energy-momentum tensor of the given field equations. The tensor character of S^{ik} holds solely for orthogonal transformations.

11. The case of the Einstein equations.

In the case of Einstein's gravitational equations special conditions prevail. Here we see that the Lagrangean (25) is an invariant not only with respect to orthogonal but even with respect to arbitrary linear transformations, if u^{ik} is transformed as a contravariant tensor, while Γ_{ik}^{m} is transformed as a tensor od third order, covariant in i, k, contravariant in m. Hence in this case the coefficients ρ_{ik} of the infinitesimal transformation (32) need not be restricted by the condition (33), but can be considered as arbitrarily prescribed constants. Accordingly we will obtain not only the anti-symmetric part of θ_i^k but the full θ_i^k expressed in the form of a divergence. In order to construct this espression, let us remember that the right side of (36) demands the following procedure : "Take the canonical matrix form. Replace the derivative $\partial\varphi_\nu/\partial X_\alpha$ by $\delta\varphi_\nu$ and apply the operation $\partial/\partial X_\alpha$ to the whole expression". This means in our case, in view of the canonical Lagrangean (25) :

$$+\left(\Gamma^{*\alpha}_{\mu\nu}\,\delta u^{\mu\nu}\right)_{,\alpha}$$

Now $u^{\mu\nu}$ transforms according to the following law :

$$\delta u^{\mu\nu} = \varepsilon\left(u^{\alpha\nu}\rho_{\mu\alpha} + u^{\mu\alpha}\rho_{\nu\alpha}\right) \tag{43}$$

and we obtain for the right side of (36) :

$$+\varepsilon\left[\Gamma^{*\alpha}_{\mu\nu}\left(\rho_{\mu\sigma}u^{\sigma\nu} + \rho_{\nu\sigma}u^{\mu\sigma}\right)\right]_{,\alpha} = +2\varepsilon\left(\Gamma^{*\alpha}_{\mu\nu}\rho_{\mu\sigma}u^{\sigma\nu}\right)_{,\alpha}$$

In view of the arbitrariness of the ρ_{ik} we obtain from (36) :

$$\begin{aligned}\theta_i^k &= +2\left(\Gamma^{*\alpha}_{i\nu}\,u^{k\nu}\right)_{,\alpha}\\ &= +2\left(\Gamma^{*\alpha}_{i\nu}\sqrt{|g|}\,g^{k\nu}\right)_{,\alpha}\end{aligned} \tag{44}$$

Let us introduce the following notation :

$$B_i^{\alpha k} = \Gamma^{*\alpha}_{i\nu}\, u^{k\nu} = \sqrt{|g|}\, \Gamma^{*\alpha}_{i\nu}\, g^{\nu k}$$

(46)

$$= \sqrt{|g|}\left(\Gamma^{\alpha}_{i\nu}\, g^{\nu k} + \frac{1}{2}\, \Gamma_i\, g^{\alpha k} + \frac{1}{2}\, \Gamma^k\, \delta^{\alpha}_i \right)$$

We now define the <u>symmetric energy-momentum tensor</u> S^{ik} of the gravitational field by the following relation :

(47) $$S^{ik} - \frac{1}{2} S\, \delta^{ik} = -\left(B_i^{\alpha k} + B_k^{\ \ i} + B_i^{k\alpha} + B_k^{i\alpha} - B_\alpha^{ik} - B_\alpha^{ki}\right)_{,\alpha}$$

<u>The tensor</u> S^{ik} <u>is symmetric and its divergence vanishes</u> :

(48) $$\frac{\partial S^{i\alpha}}{\partial x_\alpha} = 0$$

The study of the motion of a particle in a gravitational field can be based on this tensor. The covariance of this tensor is still more restricted than that of the Einsteinian t_i^α . Instead of arbitrary <u>linear</u> transformations only orthogonal transformations are at our disposal.

NOTE. - Formula (46) differs in sign from that obtained by Einstein in account of the different definition of $\Gamma^{\lambda}_{\mu\lambda}$ adopted by the present author.