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ANALYSIS OVER INFINITE DIMENSIONAL SPACES AND APPLICATIONS TO QUANTUM FIELD THEORY

par James GLIMM

Elliptic operators, roughly of the form

$$H = \sum_{i=1}^{\infty} \frac{\partial^2}{\partial g_i^2} + P(g) \tag{1}$$

arise in the study of quantum fields. Although there is some mathematical theory for such generators, experience with physics indicates that a very substantial part of this theory is as yet undeveloped.

In this lecture, I want to begin by posing three general questions, and after answering them, I will sketch some parts of the quantum field applications. The questions are

1. Why do we need analysis over infinite dimensional spaces ?
2. What parts of the mathematical theory correspond to physics ?
3. What parts of the physics corresponds to mathematics ?

To answer the first question, we should realize that quantum fields are not the first application of mathematics to use analysis over infinite dimensional spaces. Weiner measure, which defines an integral over the infinite dimensional space $C(R^1)$ of continuous functions on R^1, is used to describe Brownian motion. It also occupies a basic role in the theory of stochastic processes. As a second example, some partial differential equations, for example the Navier-Stokes equation can be written as ordinary differential equations taking values in a Banach space. The Hopf construction of weak solutions for the Navier-Stokes equation via the Galerkin approximations has this character, as does the Ebin-Marsden solution of the Euler equation. As the third example, we explain why quantum fields lead to analysis over infinite dimensional spaces.

In quantum mechanics, the dynamics is governed by the Schrödinger equation,

$$i \frac{\partial}{\partial t} u(g,t) = H u(g,t) , \tag{2}$$

$$H = -\Delta + V(g) . \tag{3}$$

For a system of n particles,

$$g = g_1, \dots, g_n \in R^{3n} , \; g_i \in R^3$$

describe the positions (or configuration) of the particles, and

$$R^{3n} = \text{configuration space}$$

is the set of all possible configurations. Quantum mechanics is essentially synonomous with the study of H , that is with the study of certain elliptic operators defined over configuration space. The states in quantum mechanics lie in a Hilbert space H , and (ignoring the complications of spin and statistics), we may realize H as

$$H = L_2(R^{3n}, dg) = L_2 \text{ (configuration space)} . \tag{4}$$

Fields are better known to mathematicians under another name. A field is just the solution of some given hyperbolic equation. For our discussion, we consider the equation

$$u_{tt} = \Delta u + P'(u) = 0 , \tag{5}$$

which has the conserved energy

$$\int u_t^2 + \nabla u^2 + P(u) \, dx \tag{6}$$

In order for the energy to be positive or semibounded, we require that P be semibounded. The classical theory of such equations has been developed by Jürgens(2), Segal(3), and Morawetz and Strauss(4), where further restrictions are made on P , depending on the number of space dimensions. For such an equation, the solution is uniquely determined by the initial conditions $\{u(x,o), u_t(x,o)\}$. The set of all initial <u>positions</u> $u(x,o)$, is the configuration space. Thus the field configuration space is some space of functions of the variable X . A convenient choice turns out to be

$$S'(R^d) = \text{Field configuration space} , \tag{7}$$

when $X \in R^d$ (d is the number of space dimensions). We conclude that fields have infinite dimensional configuration spaces, and that quantum field theory is equivalent to the study of elleptic operators defined over infinite dimensional spaces, and acting for example on a Hilbert space

$$H = L_2(S'(R^d)) \tag{8}$$

To form H and L_2, we must specify a measure on $S'(R^d)$. This is a nontrivial problem, which we postpone for the moment.

The second general question, what parts of the mathematical theory correspond to physics, has a simple answer. Self adjointness of H corresponds to existence questions, or to the construction of the quantum fields. The choice of the measure on $S'(R^d)$ also corresponds to existence questions. The spectral properties of H - its eigenfunctions and eigenvalues - correspond to particles, bound states and scattering. In terms of the measure on $S'(R^d)$, these questions can be expressed in terms of decay rates of correlations between measurements in widely separated regions.

The third question has many possible answers. We simply say that either in terms of the operators (1) or the (non Gaussian) measurer on $S'(R^d)$ which enter in (8), the physics gives a qualitatively new class of examples, together with some strong hints about the structure of these examples.

Having discussed general principles at some length, we now turn to the mathematical problem of constructing quantum fields and establishing their basic properties. The simplest nonlinear field occurs in two space time dimensions with $P(\xi) = \xi^4 + \frac{1}{2} m_o^2 \xi^2$ in (6). The corresponding quantum field has been studied extensively. It has been solved by distinct constructions, due to Glimm-Joffe[(5)], Glimm-Joffe-Spencer[(6)], Nelson-Guerra-Rosen-Simon[(7),(8)], and Dobruskyn-Minlos[(9)].

The constructions contain explicitly or implicitly the following two features : (a) approximation by n-degree of freedom, $n < \infty$ (b) estimates to control the limit $n \to \infty$. Eckmann-Magnon-Sénéor[(10)] have shown that the field theory can be constructed by Borel summation from its formal power series, for small coupling.

A very appealing form of the construction of quantum fields uses the Feynman-Kac formula. The advantage of this formula[(11)] is that it gives an explicit formula for the measure in (8). To explain the ideas in a simple fashion, we first present the case of a finite number of degrees of freedom.

The equation

$$(2\pi t)^{-d/2} e^{-|x-y|^2/2t} = F(x-y,t)$$

defines the fundamental solution for the heat equation in R^d. It also defines the density for the transition probability for a Wiener path $\omega = \omega(s)$ to pass from

$$\omega(o) = x \quad \text{to} \quad \omega(t) = y$$

in time t. Thus for initial data $u(x,o)$, the solution

$$u(y,t) = \int F(x-y,t)\, u(x,o)dx$$

of the heat equation has by definition a Wiener integral representation,

$$U(y,t) = \int_{W(y,t)} u(\omega(o))d\omega$$

where $d\omega$ is (a conditional) Wiener measure, and

$$W(y,t) = \{\omega : \omega(t) = y\}$$

is a set of Wiener paths. In this notation, the Feynman-Kac formula gives

$$(9) \qquad v(y,t) = \int_{W(y,t)} e^{-\int_o^t P(\omega(s))ds}\, v(\omega(o))d\omega$$

as a solution of the heat equation

$$v_t = -Hv$$

with potential P:

$$H = -\tfrac{1}{2}\Delta + P .$$

Before generalizing this discussion to quantum field theory, we want to transform (9). The idea is that the Lebesgue measure dg, defined on R^d, has no limit as $d \to \infty$. If we write Ω_H as the ground state of H, $\Omega_H \in L_2(R^d)$, then Ω_H also has no limit as $d \to \infty$. However $\Omega_H^2 dg$ does have a limit, and is in fact the required measure in (8). Our transformation of (9) is equivalent to the similarity transformation

$$H \to \Omega_H^{-1} H \Omega_H = \hat{H} .$$

We note that $\hat{H}$ is self adjoint on

$$H = L_2(R^d, \Omega_H^2\, dg) \tag{10}$$

and that the ground state of $\hat{H}$ is 1 .
The transformation of (9) is

$$< G(g), e^{-s\hat{H}} F(g) > = \lim_{t\to\infty} \frac{\int F(\omega(o))\, e^{-\int_{-t}^{t} P(\omega(\sigma))d\sigma}\, G(\omega(s))}{\int e^{-\int_{-t}^{t} P(\omega(\sigma))d\sigma}\, d\omega} \tag{11}$$

$$= \int F(\omega(o))\, G(\omega(s))\, \hat{d\omega}$$

$$\hat{d\omega} = \lim_{t\to\infty} \frac{e^{-\int_{-t}^{t} P(\omega(\sigma))d\sigma}\, d\omega}{\int e^{-\int_{-t}^{t} P(\omega(\sigma))d\sigma}\, d\omega}$$

and in (11) , the inner product $< , >$ is in the Hilbert space (10).

To extend this formula to Field theory, we have only to replace the path $\omega(s)$ taking values

$$\omega(s) \in R^d$$

in the classical configuration space by a path $\omega(s)$ taking values

$$\omega(s) \in S'(R^d)$$

in the field configuration space. Now a path (= function) from one variable, s , to a space of functions of $X \in R^d$, is naturally regarded as a function of

$$(s,x) \in R^{d+1} .$$

Thus the field theory path space is a function space over R^{d+1} , and as in (1), we take

$$S'(R^{d+1}) = \text{Field path space} \tag{12}$$

For a given mass m_o , we define the Gaussian measure $d\Phi$ on (12), with covariance $(-\Delta + m_o^2)^{-1}$. $d\Phi$ corresponds directly to the Wiener measure in the case of a finite number of degrees of freedom. For a given interaction potential $P(\Phi)$, we define

(13)
$$dg = \lim_{\Lambda \to \infty} \frac{e^{-\int_\Lambda P(\Phi(x))dx} \, d\Phi}{\int e^{-\int_\Lambda P(\Phi(x))dx} \, d\Phi} \quad .$$

This limit exists in some sense which is sufficient to reconstruct the underlying quantum field theory. Formally, the measure in (8) is just the time $t = 0$ conditional expectation of the measure (13), as one can see for the case of a finite number of degrees of freedom, in (11), and statements very close to this have been proved(12).

Furthermore a study of the properties of the measure defined by (13) leads to properties of the corresponding quantum field theory :

THEOREM(13). The two dimensional $P(\varphi)$ quantum field model with weak coupling describes particles with a positive mass. For $P = \varphi^4$ and weak coupling, there are no even bound states.

REMARK. For $P = \varphi^6 - \varphi^4$ and weak coupling, there is a strong indication(13) that bound states occur. The above theorem makes the Hang-Ruelle scattering theory applicable to the weakly coupled $P(\varphi)_2$ field theory, and yields the existence of an isometric S matrix.

For the more interesting and more difficult Yukawa interaction, there are infinite ultraviolet renormalizations even in two space time dimensions. Recent progress in this Y_2 model includes the verification of the Hang-Kastler axioms by McBryan and Park, as well as new formulations of the Euclidean formalism for fermions by Osterwalder and Frolich and by Schroder and Uhlenbrock.

FOOTNOTES

(1) See for example Gross, J. Funct. Analysis 1 (1961), p. 123.

(2) Jürgens, Math. Ann. 138 (1959) p. 179 and Math Z. 17 (1961), p. 265.

(3) Segal, Ann. of Math. 18 (1963) p. 339.

(4) Morawetz and Strauss, Comm. Pure Appl. Math. 25 (1972) p. 1.

(5) Glimm and Joffe, Acta Math. 125 (1970) p. 203.

(6) Glimm, Joffe and Spencer, Ann. of Math., to appear and a contribution in : Constructive Quantum Field Theory, Ed. by Velsand Wightman, Springer Verlag, Berlin (1973).

(7) Nelson, in : Constructive Quantum Field Theory, Ed. by Velsand and Wightman, Springer Verlag, Berlin (1973).

(8) Guerra, Rosen and Simon, Ann. of Math., to appear.

(9) Dobrushya and Minlos, J. of Functional Analysis and its applications (Russian).

(10) Eckmann, Magnon and Sénéra. To appear.

(11) This approach has been basic to many papers on constructive quantum field theory, starting with Symanzik, NYU , 1964 (see also earlier papers of Schwinger) as well as Nelson, in : Mathematical theory of elementary particles, Ed. by Goodman and Segal, MIT press, 1966. It was central to the approach of Joffe and the author ; see for example Glimm-Joffe, Ann. of Math. 91 (1970), or Glimm, Comm. Math. Physics 8 p. 12 (1968).

In 1968, a covariant form of the Feynman-Kac formula was known to Symanzin, as part of his formal program for a covariant construction of Euclidean quantum fields. This paper appears in : focal quantum field theory, proceedings of the International School of Physics "Enrier Ferme" Course 45, Ed. by Jost. Academic Press, New York (1969).

Three years later, Symanzik's covariant Feynman-Kac formula was used by Nelson as part of a simplified bound on the vacuum energy per unit volume, in : Proceedings of the Summer Institute of Partial Differential Equations, Berkeley 1971, Amer. Math. Soc. Providence R.I. 1973. Symanzik's Feynman-Kac formula was subsequently used by Guerra, Phys. Rev. Lett. 28 p. 1213 (1972) to obtain new results on the vacuum energy per unit volume. Following the Nelson and Guerra papers, Symanzik's covariant Feynman-Kac formula came into wide usage ;

see footnote references 6, 7, 8 and 12.

(12) The passage from the Euclidean field theory, defined on $L_2(S'(R^{d+1}))$ to the Hilbert space $H = L_2(S'(R^d))$ is given by the theory of Euclidean axioms of Nelson, J. Funct. Anal. 12, p. 91 (1973) and Osterwalder-Schroder, Comm. Math. Phys. 31 p. 83 (1973) and in Constructive Quantum Field Theory, Ed. Velo and Wightman, Springer Verlag, Berlin (1973).

(13) See footnote reference 6, as well as further progress of Spencer, to appear.