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Sur les équations aux dérivées partielles

Andrzej Granas

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SÉMINAIRE

SUR LES ÉQUATIONS AUX DÉRIVÉES PARTIELLES

IV

TOPICS IN INFINITE DIMENSIONAL TOPOLOGY

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INTRODUCTION

These notes are based upon a series of lectures given at the Collège de France in the spring of 1970. Our aim is to survey some of the recent results related, on the one hand, to the generalizations of the Lefschetz Fixed Point Theorem and, on the other, to the branch of infinite dimensional topology known as the theory of compact vector fields. The notes consist of two independent parts and the link between them is provided by the classical Leray-Schauder theory. In Part A we shall be concerned mainly with the fixed point theorems for non-compact spaces, and in Part B with the infinite dimensional cohomology theories.

Let E be an infinite dimensional normed space. A continuous mapping $f : X \to Y$ between two subsets X and Y of E is called a compact vector field provided it is of the form $f(x) = x - F(x)$, where $F : X \to E$ is a compact mapping (i.e., the closure of $F(X)$ is a compact subset of E). Two such fields $f,g : X \to Y$ are compactly homotopic provided there exists a homotopy $h_t : X \to Y$ joining f and g which is of the form $h_t(x) = x - H(x,t)$, where $H : X \times [0,1] \to E$ is compact. Since compact fields compose well, we have the category $\mathfrak{S}$ with subsets of E as objects and compact fields as morphisms. By the Leray-Schauder category $\mathfrak{H}$ we shall understand the subcategory of $\mathfrak{S}$ generated by closed bounded subsets of E. This category arose naturally in connection with the question of solvability of the non-linear equation $x = F(x)$, where F is a compact operator, and was introduced in the early thirties by J. Schauder and J. Leray [B.11], [B.9]. Furthermore, the above authors made the important discovery that many familiar geometrical facts of finite dimensional topology can be carried over to infinitely many dimensions provided attention is restricted to the above category of maps. In particular, for maps of this category, a generalization of Brouwer's degree (or of the equivalent notion of the fixed point index) was established, known presently under the name of the Leray-Schauder theory, and with its aid various applications were obtained.

For some time, it was thought that the Leray-Schauder theory had rather loose ties with topological fixed point theorems. This is not the case, however, and one of our aims is to show that several results in the fixed point theory can be reduced to the suitably modified and supplemented theory of the Leray-Schauder index [A.26], [A.15].

To be more specific, let U be an open subset of a normed (or more generally locally convex) space E and $F : U \to E$ be a compact map with a compact set of fixed points. To every such F, one can assign an integer $\mathrm{Ind}(F)$, the Leray-Schauder index of F, which satisfies a number of naturally expected properties ; in particular, when $F : U \to U$, it is equal to the (generalized) Lefschetz number $\Lambda(F)$ of F and,

hence, $\Lambda(F) \neq 0$ implies that F has a fixed point. Now, more generally, let X be a space which is r-dominated by an open set U in E (for example a metric ANR), $r : U \to X$ and $s : X \to U$ be a pair of maps with $rs = 1_X$ and $G : X \to X$ be a compact map; then, $sGr : U \to U$ is also compact and $\Lambda(G) = \Lambda(sGr) = \mathrm{Ind}(sGr)$. Consequently, if $\Lambda(G) \neq 0$, then G has a fixed point. The last theorem (established by the author in [A.24] contains several known results in topology (eg. the Lefschetz Fixed Point Theorem for compact ANR-s) and non-linear functional analysis (eg. the Schauder Fixed Point Theorem and the Birkhoff-Kellogg Theorem). As another interesting consequence of the Leray-Schauder index, we note the well-known fixed point index theory for compact ANR-s. We recall that some years ago, such a theory was deduced (by a device due to A. Deleanu [1.14]) from the fixed point index theory for convexoid spaces established (by combinatorial means) by J. Leray [A.37] in 1954 (see also [A.10, [A.12]).

Alongside the index function Ind we consider in Part A the related notion of the local fixed point index ind. The local index is defined for arbitrary compact maps of metric ANR-s and (although more restrictive than Ind) is topologically invariant. Its theory, however, is based on ideas other than that of the Leray-Schauder index, which go back essentially to W. Hurewicz.

We emphasize that in the definition of the generalized Lefschetz number $\Lambda(F)$ and in all the development of the fixed point theory for non-compact spaces, an essential use is made of the notion of the generalized trace due to J. Leray [A.39] . In particular, using the Leray trace, we define, following C. Bowszyc, new topological invariants (the Euler-Poincaré characteristic and the Lefschetz power series of a map) which turn out to be convenient tools for the treatment of periodic points, even in the finite dimensional case [A.8].

We now give, in brief, some general idea about the results of K. Gęba and the author [B.5] on infinite dimensional cohomologies which are to be presented in Part B. To this end, we recall the following theorem, proved by J. Leray (with the aid of the degree) in [B.8] : If X and Y are two equivalent objects of the Leray-Schauder category $\mathfrak{S}$, then the complements $E - X$ and $E - Y$ have the same number of components. In connection with this theorem, the following problems arise :

Problem 1 : If X and Y are two equivalent (or more generally homotopically equivalent) objects of $\mathfrak{S}$, are the homology groups $H_n(E - X)$ and $H_n(E - Y)$ isomorphic for each n ?

Problem 2 : If X and Y are equivalent in $\mathfrak{S}$ will the fundamental groups $\pi_1(E - X)$ and $\pi_1(E - Y)$ be isomorphic ?

The answer to the latter is "no" and the corresponding example shows that, from the geometrical point of view, the Leray-Schauder category is as "rich" as the category of compacta in R^n . One of our aims will be to give an affirmative answer to the

first (and more involved) problem and thus to establish the Alexander-Pontrjagin Invariance in E. We note that the Leray-Schauder degree is not adequate for this purpose and therefore a theory of an essentially algebraic character is needed. Thus, we are lead to infinite dimensional cohomology theories. The construction of such theories and their applicability will constitute our primary concern in the second part of this notes.

In conclusion, I wish to thank the Collège de France for the opportunity to deliver these lectures and to Professor Leray for his kind invitation. Looking back over my own researches, I cannot but express my deep gratitude to K. Borsuk, J. Leray and L. Lusternik. I have received from them, in different periods, much advice and encouragement, and it is in their work that I have found especial source of ideas. My thanks go also to K. Gęba for his help and collaboration, and to the participants of a topological seminar in Gdańsk for many useful discussions.

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Part A

SOME NEW RESULTS IN THE THEORY OF FIXED POINTS

PART A

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In the topics of the fixed point theory which we propose to discuss in the first part of these lectures, an essential use will be made of the generalized theory of the trace due to J. Leray [39]*). We shall begin therefore by recalling the basic definitions and facts of this theory.

I. THE LERAY TRACE AND THE GENERALIZED LEFSCHETZ NUMBER

In what follows all the vector spaces are taken over a fixed field K.

1. The ordinary trace.

For an endomorphism $\varphi : E \to E$ of a finite dimensional vector space E, we let $\operatorname{tr} \varphi$ denote the ordinary trace of φ.

We recall the following two basic properties of the trace function tr:

(1.1) Assume that in the category of finite dimensional vector spaces, the following diagram commutes

$$\begin{array}{ccc} E' & \xrightarrow{f} & E'' \\ {\scriptstyle\varphi}\uparrow & \nwarrow^{g} & \uparrow{\scriptstyle\psi} \\ E' & \xrightarrow{f} & E'' \end{array} .$$

Then $\operatorname{tr} \varphi = \operatorname{tr} \psi$; in other words $\operatorname{tr}(gf) = \operatorname{tr}(fg)$.

(1.2) Given a commutative diagram of finite dimensional vector spaces with exact rows

$$\begin{array}{ccccccccc} 0 & \to & E' & \to & E & \to & E'' & \to & 0 \\ & & \downarrow{\scriptstyle\varphi'} & & \downarrow{\scriptstyle\varphi} & & \downarrow{\scriptstyle\varphi''} & & \\ 0 & \to & E' & \to & E & \to & E'' & \to & 0 \end{array}$$

*) For some other advances in the fixed point theory (not related to the theory of the generalized trace) see the expository article of E. Fadell (Bull. Am. Math. Soc. Nr. 1, 1970).

we have $tr(\varphi) = tr(\varphi') + tr(\varphi'')$.

Definition. Let $E = \{E_q\}$ be a graded vector space. We say that E is of a finite type provided : (i) $\dim E_q < \infty$ for all q ; (ii) $E_q = 0$ for almost all q . If $\varphi = \{\varphi_q\}$ is an endomorphism of such a space (i.e., $\varphi_q : E_q \to E_q$) then the (ordinary) Lefschetz number $\lambda(\varphi)$ of φ is defined by

$$\lambda(\varphi) = \sum_q (-1)^q \, tr(\varphi_q)$$

and the Euler-Poincaré characteristic $\chi(E)$ of a space $E = \{E_q\}$ of finite type is given by

$$\chi(E) = \sum_q (-1)^q \dim E_q \, .$$

Clearly, $\chi(E) = \lambda(1_E)$.

2. The Leray trace Tr :

Let $\varphi : E \to E$ be an endomorphism of an arbitrary vector space E . Denote by $\varphi^{(n)} : E \to E$ the n-th iterate of φ and observe that the kernels

$$\ker \varphi \subset \ker \varphi^{(2)} \subset \ldots \subset \operatorname{Ker} \varphi^{(n)} \subset \ldots$$

form an increasing sequence of subspaces of E .

Let us put now

$$N(\varphi) = \bigcup_{n \geq 1} \operatorname{Ker} \varphi^{(n)} \quad \text{and} \quad \tilde{E} = E/N(\varphi) \, .$$

By definition

$$x \in N(\varphi) \Leftrightarrow \varphi^{(n)}(x) = 0 \quad \text{for some } n \, .$$

Clearly, φ maps $N(\varphi)$ into itself and, therefore, induces the endomorphism

$$\tilde{\varphi} : \tilde{E} \to \tilde{E}$$

on the factor space $\tilde{E} = E/N(\varphi)$.

(2.1) We have $\varphi^{-1}(N\varphi) = N\varphi$; consequently, the kernel of the induced map $\tilde{\varphi} : \tilde{E} \to \tilde{E}$ is trivial, i.e., $\tilde{\varphi}$ is a monomorphism.

Proof. If $x \in \varphi^{-1}(N\varphi)$, then $\varphi(x) \in N\varphi$. This implies that for some n we have $\varphi^n(\varphi(x)) = 0 = \varphi^{n+1}(x)$ and hence $x \in N(\varphi)$. Conversely, if $x \in N(\varphi)$, then $\varphi^n(x) = 0$ for some n ; then $\varphi^n \varphi(x) = 0$ and, hence, $\varphi(x) \in N(\varphi)$, i.e. $x \in \varphi^{-1}(N\varphi)$.

Definition. Let $\varphi : E \to E$ be an endomorphism of a vector space E . We say that φ is admissible provided the factor space $\tilde{E} = E/N(\varphi)$ is finite

dimensional. For such φ, we define the (generalized) trace $\mathrm{Tr}\,\varphi$ of φ by putting $\mathrm{Tr}(\varphi) \underset{df}{=} \mathrm{tr}(\tilde{\varphi})$.

(2.2) Let $\varphi : E \to E$ be an endomorphism. If $\dim E < \infty$, then $\mathrm{Tr}(\varphi) = \mathrm{tr}(\varphi)$.

The following properties of the generalized trace can be deduced from the corresponding properties of the ordinary trace (cf. [39]).

(2.3) Assume that in the category of arbitrary vector spaces the following diagram commutes :

$$\begin{array}{ccc} E' & \xrightarrow{f} & E'' \\ {\scriptstyle\varphi}\uparrow & \overset{c}{\nwarrow} & \uparrow{\scriptstyle\psi} \\ E' & \xleftarrow{f} & E'' \end{array} .$$

Then if any of the maps φ or ψ is admissible, then so is the other and in that case $\mathrm{Tr}\,\varphi = \mathrm{Tr}\,\psi$.

(2.4) Given a commutative diagram of vector spaces with exact rows

$$\begin{array}{ccccccccc} 0 & \to & E' & \to & E & \to & E'' & \to & 0 \\ & & \downarrow{\scriptstyle\varphi'} & & \downarrow{\scriptstyle\varphi} & & \downarrow{\scriptstyle\varphi''} & & \\ 0 & \to & E' & \to & E & \to & E'' & \to & 0 \end{array}$$

the endomorphism φ is admissible if and only if both φ' and φ'' are admissible and in that case

$$\mathrm{Tr}(\varphi) = \mathrm{Tr}(\varphi') + \mathrm{Tr}(\varphi'') .$$

3. The Leray endomorphisms.

Let $\varphi = \{\varphi_q\}$ be an endomorphism of a graded vector space $E = \{E_q\}$ into itself. By $\tilde{\varphi} = \{\tilde{\varphi}_q\}$ we denote the induced endomorphism on the graded vector space $\tilde{E} = \{\tilde{E}_q\}$.

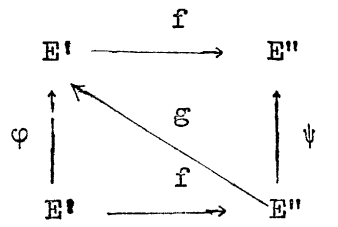
Definition (cf. [39] and [8]). We say that φ is a Leray endomorphism provided the graded vector space $\tilde{E} = \{\tilde{E}_q\}$ is of a finite type. For such φ we define the (generalized) Lefschetz number $\Lambda(\varphi)$ of φ by putting

$$\Lambda(\varphi) \underset{df}{=} \sum_q (-)^q \mathrm{Tr}(\varphi_q)$$

and the Euler-Poincaré characteristic $\chi(\varphi)$ of φ by

$$\chi(\varphi) \underset{df}{=} \sum_q (-1)^q \dim \tilde{E}_q = \chi(\tilde{E}) .$$

(3.1) Assume that in the category of graded vector spaces the following diagram commutes :



Then if any φ or ψ is a Leray endomorphism then so is the other and in that case $\Lambda(\varphi) = \Lambda(\psi)$.

Proof. This clearly follows from (2.3).

(3.2) Let

$$\begin{array}{ccccccccc} \cdots \to & E'_q & \to & E_q & \to & E''_q & \to & E'_{q-1} & \to \cdots \\ & \downarrow{\scriptstyle \varphi'_q} & & \downarrow{\scriptstyle \varphi_q} & & \downarrow{\scriptstyle \varphi''_q} & & \downarrow{\scriptstyle \varphi'_{q-1}} & \\ \cdots \to & E'_q & \to & E_q & \to & E''_q & \to & E'_{q-1} & \to \cdots \end{array}$$

be a commutative diagram of vector spaces in which the rows are exact. If both $\varphi = \{\varphi_q\}$ and $\varphi' = \{\varphi'_q\}$ are the Leray endomorphisms on $E = \{E_q\}$ and $E' = \{E'_q\}$ respectively, then so is $\varphi'' = \{\varphi''_q\}$ on $E'' = \{E''_q\}$. Moreover, in that case, we have

$$\Lambda(\varphi'') = \Lambda(\varphi) - \Lambda(\varphi') .$$

Proof. This follows from (2.4).

Among the properties of the Leray endomorphisms we note also the following generalization of a theorem of H. Hopf (cf. [28] and [9]).

Given a chain complex $C = \{C_q, \partial_q\}$ denote by $H = \{H_q\}$ the graded homology of C .

(3.3) (The Hopf Lemma). Let $c = \{c_q\}$ be a chain map of a complex $C = \{C_q, \partial_q\}$ into itself and $c_* = h = \{h_q\}$ be the induced endomorphism on $H = \{H_q\}$. If c is a Leray endomorphism, then so is c_* and in that case we have

$$\Lambda(c) = \Lambda(c_*) .$$

Proof. Denote by Z_q and B_q the spaces of cycles and boundaries. We have clearly the following commutative diagrams with exact rows

$$\begin{array}{ccccccccc} 0 & \to & B_q & \to & Z_q & \to & H_q & \to & 0 \\ & & \downarrow{\scriptstyle b_q} & & \downarrow{\scriptstyle z_q} & & \downarrow{\scriptstyle h_q} & & \\ 0 & \to & B_q & \to & Z_q & \to & H_q & \to & 0 \end{array}$$

$$\begin{array}{ccccccccc} 0 & \to & Z_q & \to & C_q & \xrightarrow{\partial_q} & B_{q-1} & \to & 0 \\ & & \downarrow{\scriptstyle z_q} & & \downarrow{\scriptstyle c_q} & & \downarrow{\scriptstyle b_{q-1}} & & \\ 0 & \to & Z_q & \to & C_q & \xrightarrow{\partial_q} & B_{q-1} & \to & 0 \end{array}$$

Now, applying (2.4) to the above diagrams, we have for each q

$$\mathrm{Tr}(z_q) = \mathrm{Tr}(b_q) + \mathrm{Tr}(h_q) \quad \text{and} \quad \mathrm{Tr}(c_q) = \mathrm{Tr}(z_q) + \mathrm{Tr}(b_{q-1})$$

and, by simple calculation, theorem (3.3) follows .

4. Lefschetz maps.

Now we may pass to the topological situation. To this end, consider a category $\mathfrak{S}$ of pairs of topological spaces and continuous mappings and fix a homology functor H from the category $\mathfrak{S}$ to the category of graded vector spaces over K . (Remark : In all that follows H is either the singular homology or the Čech homology functor).

Thus, for a topological pair (X,A) in $\mathfrak{S}$ $H(X,A) = \{H_q(X,A)\}$ is the graded vector space, $H_q(X,A)$ being the q-dimensional relative homology group with coefficients in K . For a continuous map $f : (X,A) \to (Y,B)$, $H(f)$ is the induced linear map $f_* = \{f_q\}$, where $f_q : H_q(X,A) \to H_q(Y,B)$.

Definition. A continuous mapping $f : (X,A) \to (X,A)$ is called a Lefschetz map (with respect to H) provided $f_* : H(X,A) \to H(X,A)$ is a Leray endomorphism. For such f we define the Lefschetz number $\Lambda(f)$ of f by putting

$$\Lambda(f) = \Lambda(f_*)$$

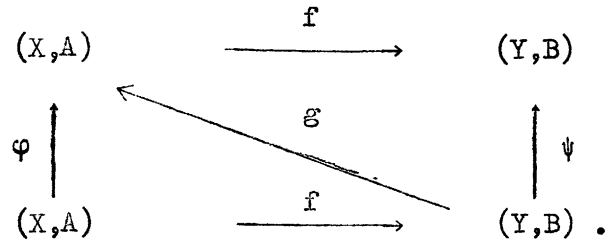
and the Euler-Poincaré characteristic $\chi(f)$ of f by

$$\chi(f) = \chi(f_*) .$$

The following simple and evident property is of importance :

(4.1) If the maps f and g are homotopic, then their Lefschetz numbers (if defined) coincide, i.e., $\Lambda(f) = \Lambda(g)$. Similarly $f \sim g$ implies that $\chi(f) = \chi(g)$.

(4.2) Assume that in the category $\mathfrak{S}$ of pairs of spaces and continuous maps, the following diagram commutes :

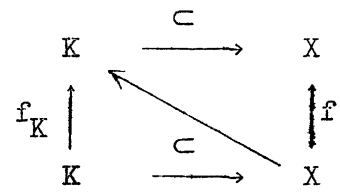


Then (i) if any of the maps φ or ψ is a Lefschetz map, then so is the other and in that case $\Lambda(\varphi) = \Lambda(\psi)$; (ii) φ has a fixed point if and only if ψ does.

Proof. The first assertion follows clearly (by applying the homology functor H to the above diagram) from the corresponding property of the Leray endomorphisms (3.1). The second assertion is evident.

The following are typical instances in which the above proposition is used :

Example 1. Let $f : X \to X$ be a map such that $f(X) \subset K \subset X$. Then we have the commutative diagram

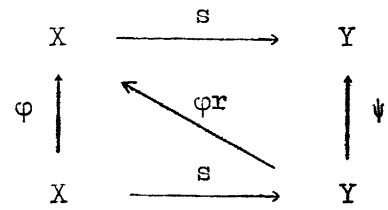


with the obvious contractions*).

Example 2. Let $r : Y \to X$, $s : X \to Y$ be a pair of continuous mappings such that $rs = 1_X$. In this case X is said to be r-dominated by Y and r is said to be an r-map (cf. [6]). In this situation, given a map $\varphi : X \to X$, we have the commutative diagram

---

*) Let $f : X \to Y$ be a map such that $f(A) \subset B$, where $A \subset X$ and $B \subset Y$. By the contraction of f to the pair (A,B) , we understand a map $f' : A \to B$ with the same values as f . A contraction of f to the pair (A,Y) is simply the restriction $f|A$ of f to A . The same terminology will be used for maps of pairs of spaces.



with $\psi = s \varphi r$.

Given a continuous mapping $f : (X,A) \to (X,A)$ we denote by $f_X : X \to X$ and $f_A : X \to X$ and $f_A : A \to A$ the evident contractions of f .

(4.3) Let $f : (X,A) \to (X,A)$ be a mapping. If f_X and f_A are Lefschetz maps, then so is the map f and in that case

$$\Lambda(f) = \Lambda(f_X) - \Lambda(f_A) \quad .$$

Proof. For the proof, write the endomorphism of the exact homology sequence of the pair (X,A) induced by f and then apply proposition (3.2).

Remark : The above definitions and properties are clearly valid also in the contravariant case, when H is the cohomology-type functor.

In what follows we shall use the following terminology.

A continuous mapping $f : X \to X$ is called homologically trivial with respect to the functor H provided the induced homomorphisms

$f_{*q} : H_q(X) \to H_q(X)$ are trivial for $q \geq 1$ and $f_{*o} : H_o(X) \simeq H_o(X)$.

A space X is said to be acyclic (with respect to the H) provided (i) X is non-empty ; (ii) $H_o(X) \approx K$; (iii) $H_q(X) = 0$ for all $q \geq 1$.

(4.4) Let $f : X \to X$ be continuous and assume that any of the following conditions is satisfied :

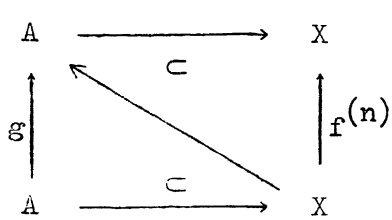
(i) $f^{(n)}(X)$ is contained in an acyclic subset A of X ;

(ii) $H_o(X) \simeq K$ and $f^{(n)} : X \to X$ is homologically trivial ;

(iii) $H_o(X) \simeq K$ and $f^{(n)} : X \to X$ is homotopic to a constant map.

Then, f is a Lefschetz map and $\Lambda(f) = 1$.

Proof. To prove (i) we write the commutative diagram :



with obvious contractions. Since A is acyclic, we infer by (4.2) that $f^{(n)}$ is a Lefschetz map. Moreover,

$$\mathrm{Tr}\,(f^{(n)})_{*q} \begin{cases} 0 & \text{for } q \geq 1 \\ 1 & \text{for } q = 0 \end{cases}$$

By functoriality we have for all $q \geq 0$, $[f_{*q}]^{(m)} = [f^{(m)}]_{*q}$. This, in view of (2.2) implies easily that

$$\mathrm{Tr}(f_{*q}) = \begin{cases} 0 & \text{for all } q \geq 1 \\ 1 & \text{for } q = 0 . \end{cases}$$

Consequently $\Lambda(f) = 1$ and the proof of (i) is completed. The proof of (ii) is similar and (ii) implies (iii).

II. COMPACT MAPS OF THE ANR-SPACES

We shall propose now the first application of the theory of generalized trace by establishing a general fixed point theorem which on the one hand contains the classical Lefschetz Theorem for compact ANR-s and on the other hand contains various fixed point theorems of the non-linear functional analysis.

1. The Hopf-Lefschetz theorem.

Let H be the singular homology functor with coefficients in K from the category of topological pairs to the category of graded vector spaces. A pair (X,A) is said to be of a finite type provided the graded vector space $H(X,A) = \{H_q(X,A)\}$ is of a finite type. Clearly, every continuous self-map $f : (X,A) \to (X,A)$ of such a pair is a Lefschetz map and $\Lambda(f)$ coincides with the ordinary (relative) Lefschetz number

$$\lambda(f) = \sum_q (-1)^q \mathrm{tr}(f_{*q})$$

of the map f.

In what follows we shall make use of the following (relative) version of the Hopf-Lefschetz fixed point theorem remarked independently by W. Holsztyński and C. Bowszyc (cf. [28] and [7], [9]).

(1.1) (The Hopf-Lefschetz Theorem for polyhedral pairs) : Let (X,A) be a pair of finite polyhedra and $f : (X,A) \to (X,A)$ be continuous. Then $\lambda(f) \neq 0$ implies that f has a fixed point in $\overline{X-A}$.

The proof of (1.1) is similar to that in the absolute case (cf. [45]).

2. ANR-spaces.

We denote by ANR (respectively AR) the class of metrizable absolute neighbourhood retracts (resp. absolute retracts). We recall (cf. [6]) that $Y \in$ ANR (resp. $Y \in$ AR) provided for any metrizable pair (X,A), with A closed in X and any continuous $f_o : A \to Y$, there exists an extension $f : U \to Y$ of f_o over a neighbourhood U of A in X (resp. an extension $f : X \to Y$ of f_o over X.

In what follows we shall make use of the following two facts from general topology :

(2.1) (Kuratowski [6]) : Every metrizable space is embeddable into a Banach space ; in particular, any topologically complete metrizable space can be embedded as a closed subset of a Banach space.

(2.2) (Arens-Eells [1]) : Every metrizable space can be embedded as a closed subset of a linear normed space.

The above embedding theorems permit to give the following simple characterization of the ANR-s (resp. topologically complete ANR-s).

(2.3) In order that $Y \in ANR$ (resp. $Y \in AR$) it is necessary and sufficient that Y be r-dominated by an open set of a normed space (resp. by a normed space).

Proof. Let $Y \in ANR$. By theorem (2.2) there exists an embedding $\varphi : Y \to E$ of Y into a normed space E such that $\varphi(Y)$ is closed in E. Take a retraction $r : U \to \varphi(Y)$ of an open set $U \supset \varphi(Y)$. Then $\varphi^{-1} r : U \to Y$ is clearly an r-map. The converse follows from the general properties of the ANR-s [5]. The proof of the second part is similar.

By applying (2.1), instead of (2.2), we obtain analogously :

(2.4) A metrizable space Y is a topologically complete ANR (resp. a topologically complete AR) if and only if it is r-dominated by an open set in a Banach space (resp. by a Banach space).

3. Compact maps.

A continuous map $f : X \to Y$ between topological spaces is called compact provided it maps X into a compact subset of Y. Let $h_t : X \to Y$ be a homotopy and $h : X \times I \to Y$ be defined by $h(x,t) = h_t(x)$ for $(x,t) \in X \times I$; then h_t is said to be a compact homotopy provided the map h is compact.

We shall make use of the following :

(3.1) (Approximation Theorem [41], [26]) : Let U be an open subset of a normed space E and let $f : X \to U$ be a compact mapping. Then for every $\varepsilon > 0$ there exists a finite polyhedron $K_\varepsilon \subset U$ and a mapping $f_\varepsilon : X \to U$, called an ε-approximation of f, such that

(i) $\|f(x) - f_\varepsilon(x)\| < \varepsilon$ for all $x \in X$;

(ii) $f_\varepsilon(X) \subset K_\varepsilon$;

(iii) f_ε is homotopic to f.

Proof. Given $\varepsilon > 0$ (which we may assume to be sufficiently small), $f(X)$ is contained in the union of a finite number of open balls $V(y_i, \varepsilon) \subset U$ $(i = 1,2,\ldots,k)$. Putting for $x \in X$,

$$f_\varepsilon(x) = \frac{\sum_{i=1}^{k} \lambda_i(x) y_i}{\sum_{j=1}^{k} \lambda_j(x)} ,$$

where

$$\lambda_i(x) = \max \{0, \varepsilon - \|f(x) - y_i\|\} ,$$

we obtain the map f_ε satisfying (i) and (ii). Clearly, the values of f_ε are in a finite polyhedron $K_\varepsilon \subset U$ with vertices $y_1, y_2, \ldots, y_k$.

Next, we make the following simple but important observation :

(3.2) Let $f : X \to X$ be a compact map of a metric spaces X and assume that f is the uniform limit of a sequence $\{f_n\}$ of maps $f_n : X \to X$. If, for almost every n there is $x_n \in X$ such that $f_n(x_n) = x_n$, then f has a fixed point.

Proof. Since f is compact and $f_n \to f$ uniformly, there is a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that

(1) $$\rho(f(x_{n_k}), x_{n_k}) \to 0 ,$$

(2) $$f(x_{n_k}) \to x \quad \text{for some} \quad x \in X .$$

From (2) and (1) we get $x_{n_k} \to x$ and, hence, by continuity of f,

(3) $$f(x_{n_k}) \to f(x) .$$

Comparing (2) and (3) we have $f(x) = x$.

4. The Lefschetz Fixed Point Theorem for arbitrary ANR-s.

We consider first the following special case of our main theorem.

THEOREM 1. Let $f : U \to U$ be a compact map of an open set U in a normed space E. Then

(a) f is a Lefschetz map and

(b) $\Lambda(f) \neq 0$ implies that f has a fixed point.

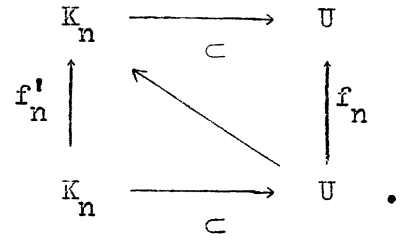
Proof. By applying to f the Approximation Theorem (3.1) we get a sequence $\{K_n\}$ of finite polyhedra $K_n \subset U$ and a sequence $\{f_n\}$ of maps $f_n : U \to U$ such that

(i) $f_n \to f$ uniformly on U ;

(ii) $f_n(U) \subset K_n$ for every n ;

(iii) f_n is homotopic to f for every n.

Now, for every n, we have the commutative diagram



Since every K_n is of a finite type, f'_n is a Lefschetz map. Consequently, by the property (I.4.2) of Lefschetz maps (see Example 1), f_n is also a Lefschets map and $\Lambda(f'_n) = \Lambda(f_n)$ for every n. Now (iii) implies that f is a Lefschetz map ; and, moreover, we have

(iv) $\Lambda(f) = \Lambda(f_n) = \Lambda(f'_n)$ for every n.

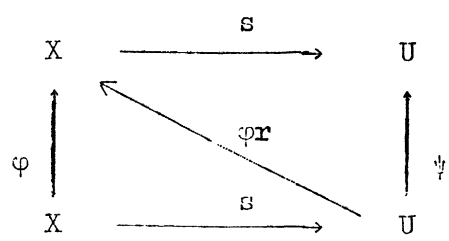
To prove (b) assume that $\Lambda(f) \neq 0$. Then, in view of (iv) we have $\Lambda(f'_n) \neq 0$, for every n. Now we apply the Hopf-Lefschetz Theorem to $f'_n : K_n \to K_n$ for each n and obtain a sequence $\{x_n\}$ of points $x_n \in U$ such that $x_n = f'_n(x_n) = f_n(x_n)$. Now, because of (i), we may apply (3.2). By (3.2) there exists a fixed point for f and the proof is completed.

Now we state the main result in full generality (cf. [24]).

THEOREM 2. Let X be an ANR and $\varphi : X \to X$ be a compact mapping. Then (a) φ is a Lefschetz map and

(ii) $\Lambda(\varphi) \neq 0$ implies that φ has a fixed point.

Proof. By (2.3) X is r-dominated by an open set U in a normed space E. Let $s : X \to U$ and $r : U \to X$ be a pair of maps with $rs = 1_X$. Then we have the commutative diagram



with $\psi = s\varphi r$. Clearly, the compactness of φ implies that of ψ. Consequently, by Theorem 1, ψ is a Lefschetz map. From the commutativity of the above diagram it follows, in view of (I.4.2), that φ is also a Lefschetz map and $\Lambda(\varphi) = \Lambda(\psi)$. Now, if $\Lambda(\varphi) \neq 0$, then $\Lambda(\psi) \neq 0$ and, by Theorem 1, ψ has a fixed point. From this, by applying again (I.4.2), we infer that φ has a fixed point and the proof is completed.

5. Corollaries.

We may draw now some consequences of Theorem 2.

COROLLARY 1 (Lefschetz Fixed Point Theorem [36]) : Let X be a compact ANR and $f : X \to X$ be continuous. Then $\lambda(f) \neq 0$ implies that f has a fixed point.

COROLLARY 2. Let X be an acyclic ANR or, in particular, an AR . Then any compact map $f : X \to X$ has a fixed point.

COROLLARY 3 (Schauder Fixed Point Theorem [44]). Let X be a convex (not necessarily closed) subset of a normed (or locally convex metrizable) linear space. Then any compact $f : X \to X$ has a fixed point.

Proof. By a theorem of Dgundji [16], X is an AR and hence the assertion follows from Corollary 2.

COROLLARY 4 (Birkhoff-Kellog Theorem [3]) : Let $S = \{x \in E \,;\, \|x\| = 1\}$ be the unit sphere in a normed space E and $f : S \to E$ be a compact operator satisfying

$$(*) \qquad \|f(x)\| \geq \alpha > 0 \quad \text{for all} \quad x \in S \ .$$

Then there exists an invariant direction for f , i.e., for some $x_0 \in S$ and $\mu > 0$, we have

$$f(x_0) = \mu x_0 \ .$$

Proof. Let us put for each $x \in S$

$$\varphi(x) = \frac{f(x)}{\|f(x)\|} \ .$$

Then (*) implies that the map $\varphi : S \to S$ is compact. Since S is clearly an acyclic ANR , φ has a fixed point, i.e.,

$$\varphi(x_0) = \frac{f(x_0)}{\|f(x_0)\|} = x_0$$

for some x_0 and the proof of our assertion is completed.

COROLLARY 5. Let X be an ANR and $f : X \to X$ be a compact map. Assume further that one of the following conditions holds :

(i) $f^{(n)}$ maps X into an acyclic subset of X ;

(ii) $f^{(n)} : X \to X$ is homotopic to a constant map.

Then $\Lambda(f) = 1$ and f has a fixed point.

Proof. This, in view of Theorem 2, follows from (I.4.4)

COROLLARY 6 (Browder-Eells [13]). Let X be a Banach (or more generally a Frechet) manifold and $f : X \to X$ a compact map. Then $\Lambda(f)$ is defined and $\Lambda(f) \neq 0$ implies that f has a fixed point*).

Proof. It is known (cf. R. Palais, Lectures on Infinite-dimensional Manifolds, 1965) that a space which is locally an ANR is an ANR. Since X is locally an ANR, the assertion follows.

6. An alternate proof of Theorem 2.

Another method of proving Theorem 2**) is based on a result of J. Dugundji which characterizes metric ANR-s in terms of homotopic domination by polyhedra. We shall state a part of the Dugundji theorem which will be sufficient for our purposes.

In all that follows, by polyhedron, we shall understand a simplicial complex with the weak topology [45].

Definition. Let $\varepsilon > 0$ and $h_t : X \to Y$ be a homotopy into a metric space (Y,ρ) ; then h_t is said to be an ε-homotopy provided $\rho(h_t(x),h_{t'}(x)) < \varepsilon$ for all $x \in X$ and $t,t' \in [0,1]$. If two given maps $f,g : X \to Y$ can be joined by an ε-homotopy, then we write $f \underset{\varepsilon}{\sim} g$ and say that f is ε-homotopic to g ; clearly, $f \underset{\varepsilon}{\sim} g$ implies, in particular, that $\rho(f(x),g(x)) < \varepsilon$ for all $x \in X$.

Definition. Let X be a metric space and $\varepsilon > 0$. We say that

$$T_{X,\varepsilon} = (P,s,r,d_t,\varepsilon)$$

is an ε-dominating system for X provided (i) P is a polyhedron ;
(ii) $s : X \to P$ and $r : P \to X$ is a pair of maps satisfying $rs \underset{\varepsilon}{\sim} 1_X$;
(iii) $d_t : X \to X$ is an ε-deformation joining rs and 1_X . If such a system exists, X is said to be ε-dominated by a polyhedron P .

(6.1) (J. Dugundji). Let X be a metric ANR. Then for every $\varepsilon > 0$ there exists a polyhedron which ε-dominates X ***).

Now we turn to a proof of Theorem 2. To this end, let $f : X \to X$ be a compact map of a metric ANR into itself.

For $\varepsilon > 0$, take an arbitrary ε-dominating system

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*) We remark that the method employed in [13] will be used later on in proving fixed point theorems for multi-valued compact maps of topologically complete ANR-s.

**) Cf. J.W. Jaworowski and M.J. Powers, Λ-spaces and fixed point theorems, Fund. Math. 1969.

***) J. Dugundji, Absolute neighbourhood retracts and local connectedness in arbitrary metric spaces, Comp. Math. 13 (1958).

$$T_{X,\varepsilon} = (P,s,r,d_t,\varepsilon)$$

for X . We have

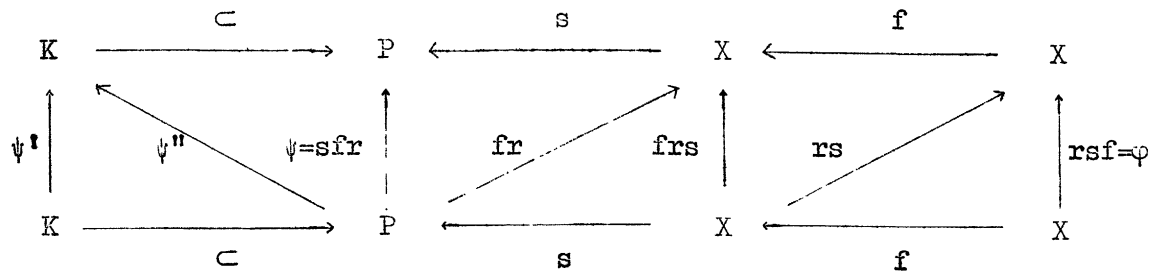
(1) $$rsf \underset{\varepsilon}{\sim} f$$

and hence

(2) $$\rho(rsf(x),f(x)) < \varepsilon \quad \text{for all} \quad x \in X .$$

Next, consider a map $\psi : P \to P$ given by $\psi = sfr$; since ψ is compact, there exists a finite polyhedron K such that $\psi(P) \subset K$.

Now write the following commutative diagram :



in which ψ' and ψ'' stand for the obvious contractions of ψ . By Lemma I.4.2 all the vertical arrows represent Lefschetz maps and

(3) $$\Lambda(\psi') = \Lambda(\psi) = \Lambda(frs) = \Lambda(rsf) .$$

Since f is homotopic to rsf , f is a Lefschetz map and

(4) $$\Lambda(f) = \Lambda(rsf) .$$

Assume that $\Lambda(f) \neq 0$; then, in view of (3) and (4), $\Lambda(\psi') \neq 0$ and, since K is a finite polyhedron, ψ' has a fixed point, by the Hopf-Lefschetz theorem. Consequently, by I.4.2 and the commutativity of the diagram above $\varphi = rsf$ has a fixed point, i.e.,

(5) $$rsf(x_o) = x_o \quad \text{for some} \quad x_o \in X$$

and hence, in view of (2) ,

(6) $$\rho(x_o,f(x_o)) < \varepsilon \quad \text{for some} \quad x_o \in X .$$

Since $\varepsilon > 0$ was arbitrary, the compactness of f implies that f has a fixed point and the proof is completed.

III. THE FIXED POINT INDEX

We shall encounter two approaches to the fixed point index theory for compact maps corresponding to the two different ways of proving Theorem II.2 . First, we consider the fixed point index in a somewhat restricted sense (the local index : ind) and, using an approach based on the Theorem of Dugundji, we extend to the case of compact maps of the ANR-s the classical local index theory for finite polyhedra. Then we turn to more general axioms and, with the aid of the Approximation Theorem, we extend to infinite-dimensional case the fixed point theory due to A. Dold [15] . This leads to the Leray-Schauder index : Ind and to a version of the Leray-Schauder theory [41], [42] which is suitable for establishing a relation between the Lefschetz number and the fixed point index of a compact map.

1. The local fixed point index.

Notation : Let U be open in a space X and $f : X \to X$ be a map ; call (X,f,U) a triple provided f is fixed-point free on the boundary $\dot{U}$ of U . Given a class of self-maps $\mathcal{F}$, we shall denote by $\mathcal{F}^*$ the corresponding class of all possible triples.

Definition. Let $\mathcal{C}$ be a category of topological spaces and H a homology (or cohomology) functor from $\mathcal{C}$ to the category of graded vector spaces. A class $\mathcal{F} = \mathcal{F}(\mathcal{C})$ of distinguished self-maps and self-homotopies in $\mathcal{C}$ is said to be admissible for a local fixed point index on $\mathcal{C}$ provided :

(i) if h is invertible in $\mathcal{C}$ and $f \in \mathcal{F}$, then $hfh^{-1} \in \mathcal{F}$;

(ii) every $f \in \mathcal{F}$ is a Lefschetz map with respect to H ;

(iii) the fixed point set $\nu_f = \{x \in X ; f(x) = x\}$ is compact for every $f : X \to X$ in $\mathcal{F}$;

(iv) for every homotopy, $h_t : X \to X$ in $\mathcal{F}$, the set $\nu(\{h_t\}) = \bigcup_{0 \le t \le 1} \nu(h_t)$ is compact.

Example : Let $\mathcal{C}_1$ be the category of all polyhedra and $\mathcal{C}_2$ the category of all metric ANR-s . Assume that $\mathcal{F}_i = \mathcal{F}(\mathcal{C}_i)$, $i = 1,2$, consists of those self-maps and self-homotopies in $\mathcal{C}_i$ which are compact. Then $\mathcal{F}_i$ is an admissible class for a local fixed point index on $\mathcal{C}_i (i=1,2)$.

Definition. (comp. [39], [12]) Let $\mathcal{F} = \mathcal{F}(\mathcal{C})$ be an admissible class and $\mathcal{F}^*$ the corresponding class of triples. Then, a local fixed-point index on $\mathcal{F}$ is a function $\mathrm{ind} : \mathcal{F}^* \to K$ satisfying the following axioms :

I (Excision). If (X,f,U) and (X,g,U) are in $\mathcal{F}^*$ and $f = g$ on $\bar{U}$, then $\mathrm{ind}\,(X,f,U) = \mathrm{ind}\,(X,g,U)$.

II (Additivity). If $(X,f,U)\in\mathcal{F}^*$ and U contains mutually disjoint open sets $U_j (j = 1,2,\ldots,k)$ such that f is fixed-point free on $U - \bigcup_{j=1}^{k} U_j$, then

$$\operatorname{ind}(X,f,U) = \sum_{j=1}^{k} \operatorname{ind}(X,f,U_j) .$$

III (Fixed Points). If $\operatorname{ind}(X,f,U) \neq 0$, then f has a fixed point in U.

IV. (Homotopy). If $h_t : X \to X$ is a homotopy in $\mathcal{F}$ and $(X,h_t,U) \in \mathcal{F}^*$ for all t, then

$$\operatorname{ind}(X,h_0,U) = \operatorname{ind}(X,h_1,U) .$$

V (Commutativity). If for two maps $f : X \to Y$, $g : Y \to X$ in $\mathcal{C}$ the triples (X,gf,U), $(Y,fg,g^{-1}(U))$ are in $\mathcal{F}^*$, then

$$\operatorname{ind}(X,gf,U) = \operatorname{ind}(Y,fg,g^{-1}(U)) .$$

VI (Normalization). For every $f : X \to X$ in $\mathcal{F}$

$$\operatorname{ind}(X,f,X) = \Lambda(f) .$$

Remark : It should be noted that the commutativity implies the following property of the local index :

VII (Topological invariance). Let $h : X \to Y$ be an invertible map in $\mathcal{C}$. Then for any $(X,f,U) \in \mathcal{F}^*$

$$\operatorname{ind}(X,f,U) = \operatorname{ind}(Y,hfh^{-1},h(U)) .$$

2. The local index for compact maps of polyhedra.

We begin by stating the following classical result which goes back essentially to H. Hopf :

(2.1) (The local index for finite polyhedra).*) Let $\mathcal{C}$ be the category of all finite polyhedra, H be the singular homology (or cohomology) functor with rational coefficients, and let $\mathcal{F} = \mathcal{F}(\mathcal{C})$ consist of all continuous self-maps in $\mathcal{C}$. Then there exists a unique integer valued function $\operatorname{ind} : \mathcal{F}^* \to Z$ satisfying the properties I - VI .

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* cf. H. Hopf, Über die algebraische Anzahl von Fixpunkten, Math. Z. 29 (1929) ; for the uniqueness and complete self-contained proof of (1.1) (in terms of cohomology) we refer to B. O'Neill, Essential sets and fixed points, Amer. J. Math 75 (1953).

We shall use the following property of the local index which is a consequence of the Commutativity Axiom :

VIII (Contraction). Let $(K,f,U) \in \mathcal{F}^*$ and $K' \subset K$ be such that $f(K) \subset K'$. Denote by $f' : K' \to K'$ the contraction of f and put $U' = K' \cap U$. Then

$$\text{ind } (K,f,U) = \text{ind } (K',f',U') \ .$$

Definition. Let P be an arbitrary polyhedron, U open in P and $f : P \to P$ a compact map which is fixed-point free on $\dot{U}$. The compactness of f implies that there exists a finite polyhedron $K \subset P$ such that $f(P) \subset K$. Denote by $f_K : K \to K$ the corresponding contraction and put $U_K = U \cap K$.
Clearly, (K,f_K,U_K) is a triple. We let

$$(*) \qquad \text{ind } (P,f,U) \underset{\text{df}}{=} \text{ind } (K,f_K,U_K) \ .$$

The Contraction property of the local index for finite polyhedra implies that ind (P,f,U) is well defined.

Theorem 1. Let $\mathcal{C}$ be the category of polyhedra and $\mathcal{F} = \mathcal{F}(\mathcal{C})$ the class consisting of all self-maps and self-homotopies in $\mathcal{C}$ that are compact. Then, the index function ind : $\mathcal{F}^* \to Z$, defined by the formula $(*)$, satisfies all the properties I - VI. Moreover, the above function is unique.

Using properties of the weak topology and the Contraction property, Theorem 1 follows in a straightforward manner from Theorem (2.1).

3. The local index for compact maps of the ANR-s.

Notation : Let (X, ρ) be a metric ANR, U open in X and $f : X \to X$ a compact map which is fixed-point free on the boundary $\dot{U}$ of U. For $\varepsilon > 0$, we let

$$V_\varepsilon = \{x \in X \ ; \ \text{dist } (x,\dot{U}) < \varepsilon\}$$

be an ε-neighbourhood of $\dot{U}$ in X and for a number η such that

$$(1) \qquad 0 < \eta < \text{dist } (\kappa_f,\dot{U})$$

we let $V_1 = V_{(\eta)}$ and $V_{\frac{1}{2}} = V_{(\frac{\eta}{2})}$.

The compactness of f implies that for some $\vartheta > 0$ we have

$$(2) \qquad \rho \ (z,f(z)) \geq \vartheta, \quad \text{for all} \quad z \in \bar{V}_1$$

We let

$$(3) \qquad \delta = \min \ (\vartheta, \ \eta) \ .$$

Let

$T_{X,\varepsilon} = (P,s,r,d_t,\varepsilon)$ and $T_{X,\varepsilon'} = (P',s',r',d'_t, \varepsilon')$

be two dominating systems for X with constants ε and ε' respectively .
We have

(4) $$rs \underset{\varepsilon}{\sim} 1_X \quad \text{and} \quad r's' \underset{\varepsilon'}{\sim} 1_X$$

(5) $$rsf \underset{\varepsilon}{\sim} f \quad \text{and} \quad r's'f \underset{\varepsilon'}{\sim} f$$

(6) $$\rho(rsf,f) < \varepsilon \text{ and } \rho(r',s',f,f) < \varepsilon' \ .$$

We define

$h_t : P \to P$ and $h'_t = P' \to P'$ $(0 \leqq t \leqq 1)$

by putting

$h'_t = s'fd_tr'$ and $h_t = sd'_tfr$ $(0 \leqq t \leqq 1)$.

(3.1) <u>Lemma</u>. <u>Assume that both</u> ε <u>and</u> ε' <u>are smaller than</u> $\frac{\delta}{2}$. <u>Then</u>

(i) h'_t <u>is a compact homotopy joining</u> $s'fr'$ <u>and</u> $(s'fr)(sr')$;

(ii) h'_t <u>is a fixed-point free on</u> $r'^{-1}(V_{\frac{1}{2}})$;

(iii) h_t <u>is a compact homotopy joining</u> sfr <u>and</u> $(sr')(s'fr)$;

(iv) h_t <u>is fixed-point free on</u> $r^{-1}(V_{\frac{1}{2}})$.

<u>Proof</u>. (i) and (iii) are evident. To prove (ii) , suppose to the contrary that for some $t\in[0,1]$

$h'_t(y) = y = s'fd_tr'(y)$ for a point $y \in r'^{-1}(V_{\frac{1}{2}})$.

Then, in view of (6) and (3) , we have

(7) $$r'(y) = r's'fd_tr'(y)$$

(8) $$r'(y) \in V_{\frac{1}{2}} \quad \text{and} \quad z = d_tr'(y) \in V_1 \ .$$

From (7) and (6) we get

(9) $$\rho(fd_tr'(y) \ , \ r'(y)) < \varepsilon < \frac{\delta}{2}$$

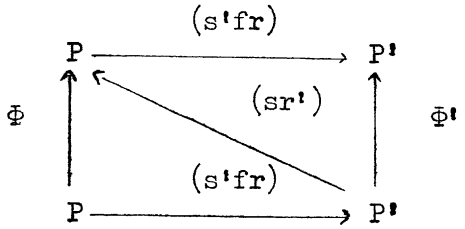
(10) $$\rho(d_tr'(y) \ , \ r'(y)) < \varepsilon' < \frac{\delta}{2}$$

and hence, in view of (3) ,

$$\rho(z,f(z)) < \delta < \vartheta$$

which is a contradiction because of (8) and (2) , This completes the proof of (ii) .
The proof of (iv) is similar and is omitted.

(3.2) Lemma. Let $T_{X,\varepsilon}$ and $T_{X,\varepsilon'}$ be two dominating systems for X with constants ε and ε' smaller than $\frac{\delta}{2}$. Consider the following commutative diagram



Then

(i) $(sr')\,(s'fr) \sim sfr \quad (\text{rel } r^{-1}(U))^*)$;

(ii) $(s'fr)\,(sr') \sim s'fr' \quad (\text{rel } r'^{-1}(U))$;

(iii) $\text{ind}\,(P,\Phi,r^{-1}(U)) = \text{ind}\,(P',\Phi',r'^{-1}(U))$;

(iv) $\text{ind}\,(P,\Psi,r^{-1}(U)) = \text{ind}\,(P,\Psi',r'^{-1}(U))$.

Proof. (i) and (ii) are consequences of Lemma (4.1) ; and the formula $Fr(f^{-1}(A)) \subset f^{-1}Fr(A)$ (iv) follows clearly by the Homotopy from (i), (ii) and (iii) .

To prove (iii), we note that since both Φ and Φ' are compact, the commutativity of the index in $\mathfrak{S}_1$ implies

$$\text{ind}\,(P,\Phi,r^{-1}(U)) = \text{ind}\,(P',\Phi',(sr')^{-1}r^{-1}(U)) = \text{ind}\,(P',\Phi',(rsr')^{-1}(U)) .$$

Next, we observe that

$$(rsr')^{-1}(U) \subset r'^{-1}(U) \cup r'^{-1}(V_1)$$

and

$$\kappa_{\Phi'} \cap r'^{-1}(U) \subset (rsr')^{-1}(U) .$$

Consequently, by the Excision, we get

$$\text{ind}\,(P',\Phi',(rsr')^{-1}(U)) = \text{ind}\,(P',\Phi',r'^{-1}(U) \cup r'^{-1}(V_1)) =$$

$$= \text{ind}\,(P',\Phi',r'^{-1}(U))$$

and the proof of (iii) is completed.

Definition. Let (X,f,U) be a triple such that X is a metric ANR and $f : X \to X$ is a compact map. Take an arbitrary ε-dominating system $T_{X,\varepsilon} = (P,s,r,d_t,\varepsilon)$ for X with constant ε less than $\frac{\delta}{2}$. Using Theorem 1, we let

$$(*) \qquad \text{ind}\,(X,f,U) \underset{\text{df}}{=} \text{ind}\,(P,sfr,r^{-1}(U)) .$$

It follows from Lemma (3.2) that ind (X,f,U) is well defined.

---

*) If (X,f,U) and (X,g,U) are triples, we write $f \sim g$ rel U to mean that f and g can be joined by a homotopy which is fixed point free on $\dot{U}$.

Theorem 2. Let $\mathfrak{S}$ be the category of metric ANR-s and $\mathcal{F} = \mathcal{F}(\mathfrak{S})$ be the class consisting of all compact self-maps and compact self-homotopies in $\mathfrak{S}$. Then, the function ind : $\mathcal{F}^* \to Z$ defined by the formula (*) satisfies all the properties I - VI. In V it is assumed that one of the maps f or g is compact.

4. Proof of Theorem III.2.

The Normalization property was already established (see the proof of Theorem II.2 given in section II.6). All the remaining properties (except the commutativity) follow easily from the corresponding properties of the local index in $\mathfrak{S}_1$. Let us prove for instance Property IV.

Property IV : Let $h_t : X \to X$ be a compact homotopy such that $(X,h_t,U) \in \mathcal{F}^*$ for all $t \in \{0,1\}$. Take a number η satisfying

$$0 < \eta < \operatorname{dist}(\varkappa(\{h_t\}), \dot{U}) .$$

The compactness of the homotopy $\{h_t\}$ implies that for some $\vartheta > 0$ we have

$$\rho(z, h_t(z)) > \vartheta \quad \text{for all} \quad z \in \overline{V}_\eta \quad \text{and} \quad t \in [0,1]$$

Put $\delta = \min(\vartheta, \eta)$ and take an arbitrary $T_{X,\varepsilon} = (P,s,r,d_t,\varepsilon)$ with $\varepsilon < \frac{\delta}{2}$. Then $sh_t r : P \to P$ is a compact homotopy such that $(P, sh_t r, r^{-1}(U)) \in \mathcal{F}^*$ for all $t \in [0,1]$.

Consequently,
$\operatorname{ind}(X,h_0,U) = \operatorname{ind}(P, sh_0 r, r^{-1}(U)) = \operatorname{ind}(P, sh_1 r, r^{-1}(U)) = \operatorname{ind}(X,h_1,U)$ and the assertion follows.

Property V : For the proof of the commutativity (which is somewhat more involved) we shall need two lemmas.

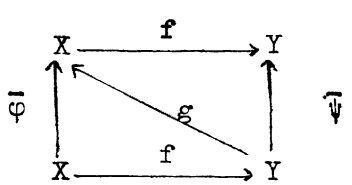
(4.1) Lemma. Let $\psi_\bullet : P \to P$ be a compact map of a polyhedron and $\mu : P \to Y$ a map into a metric space Y ; assume that U is open in Y and $(P,\psi,\mu^{-1}(U)) \in \mathcal{F}^*(\mathfrak{S}_1)$. There is a $\delta > 0$ such that for any ε-deformation $d'_t : Y \to Y$ with $\varepsilon < \delta$

(i) $\quad (P,\psi,\mu^{-1}d'^{-1}_t(U)) \in \mathcal{F}^*(\mathfrak{S}_1)$ for all $t \in [0,1]$;

(ii) $\quad \operatorname{ind}(P,\psi,\mu^{-1}(U)) = \operatorname{ind}(P,\psi,\mu^{-1}d'^{-1}_t(U))$.

Proof. This reduces to the case of finite polyhedra.

Consider now the following commutative diagram



in which X and Y are metric ANR-s and f is compact. Assume that $\bar{\varphi}$ is fixed-point free on the boundary $\dot{U}$ of $U \subset X$; this implies that $\bar{\psi}$ is fixed-point free on the boundary $\dot{U}_y$ of $U_Y = g^{-1}(U) \subset Y$. We let $U_X = f^{-1}(U_Y)$.

(4.2) Lemma. For every $\delta > 0$ there exist $T_{X,\varepsilon} = (P,s,r,d_t,\varepsilon)$ and $T_{Y,\varepsilon'} = (P',s',r',d_t',\varepsilon')$ with ε and ε' smaller than δ such that the following two conditions are satisfied :

(a) the homotopy $h_t' : P' \to P'$ given by

$$h_t' = s'fd_tgr' \qquad (0 \leqq t \leqq 1) \tag{1}$$

is fixed-point free on $r'^{-1}(\dot{U}_Y)$;

(b) the homotopy $h_t : P \to P$ given by

$$h_t = sgd_t'fr \qquad (0 \leqq t \leqq 1) \tag{2}$$

is fixed-point free on $(fr)^{-1}(r's')^{-1}(\dot{U}_Y)$.

The proof of Lemma 4.2 is given separately in section 5 .

Now, after the above preliminaries, we turn to the proof of Property V .

By Lemma 4.2 there exist dominating systems

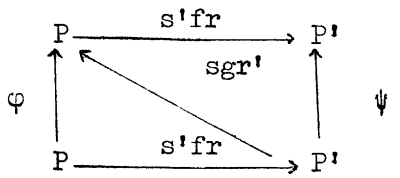
$$T_{X,\varepsilon} = (P,s,r,d_t,\varepsilon) \quad \text{and} \quad T_{Y,\varepsilon} = (P',s',r',d_t',\varepsilon')$$

such that the conditions (a) and (b) of Lemma 4.2 are satisfied and

$$\text{ind}\,(X,\bar{\varphi},U_X) = \text{ind}\,P,s\,\bar{\varphi}\,r,r^{-1}(U_X) \tag{*}$$

$$\text{ind}\,(Y,\bar{\psi},U_Y) = \text{ind}\,(P',s'\,\psi\,r'^{-1}(U_Y)) . \tag{**}$$

Consider the following commutative diagram



We assert that

$$\text{ind}\,(Y,fg,U_Y) \overset{(i)}{=} \text{ind}\,(P',s'fgr',r'^{-1}-U_Y)) \quad \overset{(ii)}{=} \text{ind}\,(P',\psi,r'^{-1}(U_Y)) =$$
$$\overset{(iii)}{=} \text{ind}\,(P,\varphi,(s'fr)^{-1}r'^{-1}(U_Y)) = \text{ind}\,(P,\varphi,(fr)^{-1}(r's')^{-1}(U_Y)) =$$
$$\overset{(iv)}{=} \text{ind}\,(P,sgfr,(fr)^{-1}(r's')^{-1}(U_Y)) \overset{(v)}{=} \text{ind}\,(P,sgfr,r^{-1}(U_X)) =$$
$$\overset{(vi)}{=} \text{ind}\,(X,gf,U_X) \overset{(vii)}{=} \text{ind}\,(X,gf,U) .$$

Indeed,

(i) and (vii) are evident in view of (*) and (**) ; (ii) holds by the

Homotopy because, in view of Lemma 4.2 ,

$$\psi \sim s'fgr' \qquad (\text{rel } r'^{-1}(U_Y)) \ ;$$

(iii) is a consequence of the Commutativity of the index in $\mathfrak{S}_1$; (iv) follows, by the Homotopy, because (in view of Lemma 4.2)

$$\varphi \sim sgfr \quad \text{rel } (fr)^{-1}(r's')^{-1}(U_Y) \ ;$$

(vi) is a consequence of Lemma 4.1 aplied to the map $\mu = fr : P \to Y$; (vii) holds by the Excision of the index in $\mathfrak{S}_2$.

Thus, we proved that

(***) $$\text{ind } (X,gf,U) = \text{ind } (Y,fg,g^{-1}(U)) \ .$$

It follows then, by the Excision, that

$$\text{ind } (Y,fg,V) = \text{ind } (X,gf,f^{-1}(V))$$

and, consequently, (***) holds assuming that one of the maps f or g is compact. The proof of the commutativity is completed.

5. Proof of Lemma 4.2 .

Let $\delta > 0$; we fix

$$T_{X,\varepsilon} = (P,s,r,d_t,\varepsilon)$$

such that $\varepsilon < \delta$ and

(3) $$fd_tg(y) \neq y \quad \text{for all} \quad y \in \dot{U}_Y \quad \text{and} \quad t \in [0,1] \ .$$

We assert now that

(4) There exists $\eta > 0$, satisfying $\eta < \delta$, such that for any $T_{Y,\varepsilon'} = (P',s',r',d'_t,\varepsilon')$ with $\varepsilon' < \eta$ we have
$$h'_t(x) = s'fd_tgr'(x) \neq x$$
for all $x \in r'^{-1}(\dot{U}_Y)$ and $t \in [0,1]$.

For suppose not. Then there is a sequence

$$T_{Y,\alpha_n} = (P'_n,s'_n,r'_n,(d'_t)^n,\alpha_n)$$

such that every homotopy

$$h'^{(n)}_t = s'_nfd_tgr'_n : P'_n \to P'_n$$

has a fixed point on $r'^{-1}_n(\dot{U}_Y)$ and $\{\alpha_n\}$ satisfies

(5) $$\lim_{n=\infty} \alpha_n = 0 \ , \quad \alpha_n < \alpha_{n-1} \ , \quad \alpha_n > 0 \ .$$

Denote by $x_n \in r'^{-1}_n(\dot{U}_Y)$ a fixed point for $h'^{(n)}_t$.

Thus, for some $t_n \in [0,1]$ we have

$$h'^{(n)}_{t_n}(x_n) = (s'_n f d_{t_n} g r'_n)(x_n) = x_n$$

and hence

$$r'_n s'_n f d_{t_n} g(y_n) = y_n \quad \text{with} \quad y_n = r'_n(x_n) \in \dot{U}_Y .$$

From this we get

$$\rho(f d_{t_n} g(y_n), y_n) < \alpha_n \qquad n = 1,2,\dots$$

This in turn implies, in view of the compactness of the homotopy fd_tg and the interval $[0,1]$, that

$$f d_t g(y) = y \quad \text{for some} \quad t \in [0,1] \quad \text{and} \quad y \in \dot{U}_Y$$

which is in contradiction with (3).

Next we claim that

(5) For some $T_{Y,\varepsilon'} = (P',s',r',d'_t,\varepsilon')$ with $\varepsilon' < \delta$ we have for all $z \in (fr)^{-1}(r's')^{-1}(\dot{U}_Y)$ and $t \in [0,1]$

$$h_t(z) = sgd_t fr(z) \neq z .$$

For suppose to the contrary that (5) is false. Then for each $n = 1,2,\dots$ and

$$T_{Y,\frac{1}{n}} = (P'_n, s'_n, r'_n, d'^{n}_{t}, \tfrac{1}{n})$$

there is $t_n \in [0,1]$ and $z_n \in (fr)^{-1}(r'_n s'_n)^{-1}(\dot{U}_Y)$ so that

$$sgd'^{n}_{t_n} fr(z_n) = z_n$$

and hence

$$\text{(6)} \qquad frsg\,[d'^{n}_{t_n} fr(z_n)] = fr(z_n) .$$

Consider the sequences $\{t_n\}$, $\{z_n\}$ and put for each n

$$\text{(7)} \begin{cases} x_n = fr(z_n) \\ x'_n = r'_n s'_n fr(z_n) = r'_n s'_n(x_n) \in \dot{U}_Y \\ x''_n = d'^{n}_{t_n} fr(z_n) = d'^{n}_{t_n}(x_n) . \end{cases}$$

From (6) and (7) we get

$$\text{(8)} \qquad frsg(x''_n) = x_n$$

$$\text{(9)} \qquad \rho(x_n, x'_n) < \frac{1}{n} \qquad \text{and} \qquad \rho(x_n, x''_n) < \frac{1}{n}$$

Now, using the compactness of f and (8), (9) we obtain

$$\mathrm{frsg}(x) = x \quad \text{for some} \quad x \in \dot{U}_Y$$

which is in contradiction with (3) . Thus (5) is proved. Now, by comparing the statements (3) and (5) the assertion of the lemma follows.

6. More general axioms.

Now we shall consider a more general notion of the fixed point index.

Definition. Let $f : U \to X$ be a continuous map between topological spaces. Call f admissible provided U is an open subset of X and the fixed point set of f

$$\varkappa_f = \{x \in U,\ f(x) = x\} \subset U$$

is compact. A homotopy $h_t : U \to X$ will be called admissible provided the set $\varkappa(\{h_t\}) = \bigcup_{0 \leq t \leq 1} \varkappa(h_t)$ is compact.

Definition (comp. [15]). Let $\mathfrak{S}$ be a category of topological spaces in which a class $\mathfrak{F} = \mathfrak{F}(\mathfrak{S})$ of admissible maps and homotopies is distinguished. By a fixed-point index on $\mathfrak{F}$ we shall understand a function $\mathrm{Ind} : \mathfrak{F} \to K$ which satisfies the following conditions :

I (Excision). If $U' \subset U$ and $\varkappa_f \subset U'$, then the restriction $f' = f|U' : U' \to X$ is in $\mathfrak{F}$ and

$$\mathrm{Ind}(f) = \mathrm{Ind}(f') .$$

II (Additivity). Assume that $U = \bigcup_{i=1}^{k} U_i$, $f_i = f|U_i$ and the fixed point sets $\varkappa_i = \varkappa_f \cap U_i$ are mutually disjoint, $\varkappa_i \cap \varkappa_j = \emptyset$ for $i \neq j$. Then

$$\mathrm{Ind}\ f = \sum_{i=1}^{k} \mathrm{Ind}\ f_i .$$

III (Fixed points) . If $\mathrm{Ind}\ f \neq 0$, then $\varkappa_f \neq 0$, i.e., the map f has a fixed point.

IV (Homotopy). Let $h_t : U \to X$, $0 \leq t \leq 1$, be an admissible homotopy in $\mathfrak{F}$. Then $\mathrm{Ind}\ (h_0) = \mathrm{Ind}\ (h_1)$.

V (Multiplicativity). If $f_1 : U_1 \to X_1$ and $f_2 : U_2 \to X_2$ are in $\mathfrak{F}$ then so is the product map $f_1 \times f_2 : U_1 \times U_2 \to X_1 \times X_2$ and

$$\mathrm{Ind}\ (f_1 \times f_2) = \mathrm{Ind}\ (f_1)\ \mathrm{Ind}\ (f_2) .$$

VI (Commutativity). Let $U \subset X$, $U' \subset X'$ be open and assume $f : U \to X'$, $g : U' \to X$ are maps in $\mathfrak{S}$. If one of the composites

$$gf : V = f^{-1}(U') \to X \quad \text{or} \quad fg : V' = g^{-1}(U) \to X'$$

is in $\mathcal{F}$, then so is the other and, in that case,

$$\mathrm{Ind}\,(gf) = \mathrm{Ind}\,(fg)\ .$$

VII (Normalization). If $U = X$ and $f : X \to X$ is in $\mathcal{F}$, then $\mathrm{Ind}(f) = \Lambda(f)$.

7 . The Fixed Point Index in R^n .

In the following definition H is the singular homology $\mathbb{H}$ over the integers Z . Let us fix for each n an orientation $1 \in H_n(S^n)$ of the n-th sphere $S^n = \{x \in R^{n+1};\ ((x)) = 1\}$ and accordingly identify $H_n(S^n) \approx Z$ with the integers Z .

Definition (cf [15]). Let $f : U \to R^n$ be an admissible map. Denote by $K = \kappa_f$ the fixed point set for f and by

$$(i-f) : (U, U-K) \to (R^n, R^n - \{0\})$$

the map given by $(i-g)(x) = x - f(x)$. The fixed point index $\mathrm{Ind}\, f$ of the map f is defined to be the image of 1 under the composite map

$$Z = H_n(S^n) \to H_n(S^n, S^n - K) \xrightarrow{\sim} H_n(U, U-K) \xrightarrow{(i-f)_*} H_n(R^n, R^n - 0) = Z\ .$$

The following theorem was established by A. Dold [15]

(7 .1) (The Fixed Point Index in R^n) : Let $\mathcal{E}$ be the category of open subsets of Euclidean spaces and $\mathcal{F}(\mathcal{E})$ the class of all continuous admissible maps in $\mathcal{E}$. Then the function $f \to \mathrm{Ind}\, f$ defined above satisfies the properties I-VII. In VII it is assumed that f is compact.

We note that the excision and the commutativity implies the following property of the index :

VIII (Contraction). Let U be open in R^{n+1} and $f : U \to R^{n+1}$ be an admissible map such that $f(U) \subset R^n$. Denote by $f' : U' \to R^n$ the contraction of f , where $U' = U \cap R^n$. Then $\mathrm{Ind}(f) = \mathrm{Ind}(f')$.

8. The Leray-Schauder Index.

Let U be an open subset of a normed space E and let $f : U \to E$ be an admissible compact map. Take an open set $V \subset U$ such that $\kappa_f \subset V$. Then the number $\epsilon = \inf \{\|x - f(x)\| \text{ for } x \in \dot{V}\}$ is positive.

Let $g = f|V : V \to E$. From the definition of ε , it follows that :

(i) every ε -approximation $g_\varepsilon : V \to E$ of g is admissible ;

(ii) given two ε -approximations $g'_\varepsilon, g''_\varepsilon : V \to E$ of g , there exists an admissible finite dimensional compact homotopy $h_t : V \to E$, $0 \leq t \leq 1$, such that $h_0 = g'_\varepsilon$, $h_1 = g''_\varepsilon$

Definition. Let $f : U \to E$ be an admissible compact map and $g_\varepsilon : V \to E$ be an ε -approximation of $g = f|V$ as above. Denote by E^n a finite dimensional subspace

of E such that $g_\varepsilon(V) \subset E^n$ and let $g'_\varepsilon : V_n \to E^n$, where $V_n = V \cap E^n$, be the contraction of g_ε. Using (7,1), we define the Leray-Schauder index of f by putting $\mathrm{Ind}(f) = \mathrm{Ind}(g'_\varepsilon)$.

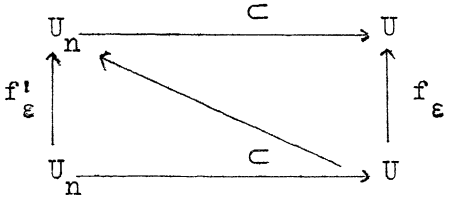
It follows from (i), (ii), and the properties I, IV and VIII of the index in R^n, that $\mathrm{Ind}(f)$ is well defined.

Theorem 3. Let $\mathcal{C}$ be the category of open subsets in linear normed spaces and let $\mathcal{F}$ be the class of all admissible compact maps in $\mathcal{C}$. Assume that all admissible homotopies are compact. Then, defined on $\mathcal{F}$, the Leray-Schauder index $f \to \mathrm{Ind}(f)$ satisfies the properties I-VII. In VI it is assumed that both f and g are compact.

Proof. Using the Approximation Theorem I.3.1, properties I-V follow from the corresponding properties of the index in R^n. These once proved, property VI follows (similarly as in [15]) from I, IV and V.

Proof of property VIII : Given a compact map $f : U \to U$ let $f_\varepsilon : U \to U$ be an ε-approximation of f such that its values are in some finite dimensional subspace E^n of E and let $U_n = U \cap E^n$.

Consider the following commutative diagram in which all the arrows represent either the obvious inclusions or the contractions of the map f :



By the definition $\mathrm{Ind}(f) = \mathrm{Ind}(f'_\varepsilon)$. By Lemma (I.4.2), we have $\Lambda(f'_\varepsilon) = \Lambda(f_\varepsilon)$ and, consequently, in view of (7.1), (property VII), $\mathrm{Ind}(f) = \Lambda(f_\varepsilon)$. Since f is homotopic to f_ε, this implies that $\mathrm{Ind}(f) = \Lambda(f)$.

9. Remarks on the non-metrizable case. First we remark that the Approximation Theorem (I.3.1) extends (with appropriate modifications), to the case when U is open in locally convex topological space E.

This fact permits to extend the Leray-Schauder index to the case of locally convex spaces and to state Theorem 3 in the following more general form :

Theorem 3'. Let $\mathcal{C}$ be the category of open subsets of locally convex topological spaces. Let $\mathcal{F} = \mathcal{F}(\mathcal{C})$ be the class of all admissible compact maps and assume that all admissible homotopies are compact. Then, there exists a function $\mathrm{Ind} : \mathcal{F} \to Z$ (the Leray-Schauder index) which satisfies the properties I-VII. In VII it is assumed in addition that both f and g are compact.

We remark that Theorem 3 is not completely satisfactory (because the question of topological invariance of the Ind f remains open). Nevertheless, some useful applications of the Leray-Schauder index can be given.

Theorem 4. Let X be a space which is r-dominated by a set U open in a locally convex topological space E. Let $r : U \to X$ and $s : X \to U$ be a corresponding pair of maps with $rs = 1_X$. Assume that $f : X \to X$ is a compact map. Then (i) f is a Lefschetz map; (ii) $\Lambda(f)$ is equal to the Leray-Schauder index of the map sfr, $\Lambda(f) = \mathrm{Ind}\,(sfr)$; (iii) if $\Lambda(f) \neq 0$, then the map f has a fixed point.

Proof. (i) and (ii) follow from Lemma (I.4.2) (Example 2) and Theorem 3 (Property VII); (iii) is a consequence of (ii), Theorem 3 (Property III) and again of Lemma I.42.

Remark. Since every metrizable ANR is r-dominated by an open set in a normed space, Theorem 4 includes as a special case Theorem II.2; moreover, it gives more precise information about the Lefschetz number $\Lambda(f)$ of a compact map f by relating it to the Leray-Schauder index of the map sfr.

10. The fixed point index for compact (non metrizable) ANR-s.

We shall give now an application of the Leray-Schauder index to the fixed point index theory for the compact ANR-s for normal spaces. Such a theory was established previously by combinatorial means (and in a different form) by several authors (cf. J. Leray [37], A. Deleanu [14], D. Bourgin [10], F. Browder [12]).

Let X be a compact ANR for normal spaces and $h : X \to E'$ be an embedding of X into a locally convex space E'. It can be shown that the linear span E of the compact set h(X) in E' is normal. It follows that X is r-dominated by a set open in a locally convex space.

Definition (comp. [19]). Let X be a compact ANR for normal spaces and $f : U \to X$ be an admissible map. To define Ind(f) take an open set V in a locally convex space E which r-dominates X; let $s : X \to V$, $r : V \to X$ be a pair of maps with $rs = 1_X$. Since the composite map

$$r^{-1}(U) \xrightarrow{r} U \xrightarrow{f} X \xrightarrow{s} V$$

is compact, its index is defined by Theorem 3' and we let

$$\mathrm{Ind}\, f = \mathrm{Ind}\,(sfr). \tag{*}$$

The Excision and the Commutativity of the Leray-Schauder index imply that this definition is independent of the choices involved.

Theorem 5. Let $\mathcal{C}$ be the category of compact ANR-s for normal spaces and $\mathcal{F}$ be the class of all continuous admissible maps in $\mathcal{C}$. Then the function $f \to \mathrm{Ind}\, f$ defined by (*) satisfies all the properties I-VII.

IV. OTHER GENERALIZATIONS OF THE LEFSCHETZ FIXED POINT THEOREM

In the previous lectures it was shown how the Lefschetz Fixed Point Theorem for compact ANR-s can be extended to the case of non-compact spaces. Now we turn to some other generalizations of the above theorem.

1. Fixed Point Theorems for Approximative ANR-s .

First, using the general properties of the Lefschetz maps and Theorem II.2, we extend the Lefschetz Theorem to a class of compacta*) introduced by H. Noguchi in [43] and called here the class of approximative ANR-s. Until further notice H stands for the Čech-Vietoris homology functor with compact carriers.

Definition. Let (X,A) be a pair of metric spaces and ε be a positive number. A continuous map $r_\varepsilon : X \to A$ is called an ε-retraction provided $\rho(r_\varepsilon(a),a) < \varepsilon$ for all $a \in A$. A subspace $A \subset X$ is said to be an approximative retract of X provided for each $\varepsilon > 0$ there exists an ε-retraction $r_\varepsilon : X \to A$.

(1.1) Assume that a compactum A is an approximative retract of a space X. Then the map $i_* : H(A) \to H(X)$ induced by the inclusion $i : A \to X$ is a monomorphism.

Definition. A compactum X is said to be an approximative ANR (resp. approximative AR) provided for each embedding $h : X \to Y$ into a metric space Y, the set $h(X)$ is an approximative retract of some open set U in Y (resp. an approximative retract of Y).

Although not necessarily locally connected, the approximative ANR-s enjoy many familiar properties of the ANR spaces (cf. [43] and [20]). In particular :

(1.2) Every compact approximative ANR is of a finite type.

The following property of the approximative ANR-s is of importance :

(1.3) Let $f : X \to Y$ be a map into a compact approximative ANR. There exists an $\varepsilon > 0$ such that for each $g : X \to Y$ the condition

$$\rho(f(x),g(x)) < \varepsilon \quad \text{for all} \quad x \in X$$

implies $f_* = g_*$.

Proof. In view of (II.2.1), we may assume without loss of generality, that Y is contained in a Banach space E, and hence there is an open set U in E such that $Y \subset U$ is an approximative retract of U. Let $\varepsilon > 0$ be a number smaller than the distance $\operatorname{dist}(Y,\dot{U})$ of the compact set Y to the boundary $\dot{U}$ of U

*) We recall (cf. [5] and [32]) that there exist in R^3 locally connected acyclic continua without the fixed point property. This shows that the Lefschetz fixed point theorem cannot be extended to arbitrary compacta.

in E. Let $g : X \to Y$ be a map such that

(*) $$\| g(x) - f(x) \| < \varepsilon \quad \text{for all} \quad x \in X .$$

Denote by $j : Y \to U$ the inclusion and put $f' = jf$, $g' = jg$. From (*) and the definition of ε we infer that for each $x \in X$ the internal $tf'(x) + (1-t)g'(x)$ where $0 \leqq t \leqq 1$ is entirely contained in U. This implies that f' and g' are homotopic. Consequently, $f'_* = g'_*$, i.e., $j_* f_* = j_* g_*$. Since j_* is by (1.1) a monomorphism we get $f_* = g_*$.

THEOREM 1 (cf. [25]). Let X be an approximative compact ANR and $f : X \to X$ be continuous. Then $\lambda(f) \neq 0$ implies that f has a fixed point.

Proof. We may assume, without loss of generality, that X is an approximative retract of an open set U contained in a Banach space E. For each $n = 1,2,\ldots$, let $r_n : U \to X$ be a $\frac{1}{n}$ - retraction from U to X. We have

(1) $$\|x - r_n(x)\| < \frac{1}{n} \quad \text{for all} \quad x \in X .$$

Let $j : X \to U$ be the inclusion and define for each $n = 1,2,\ldots$ a map $g_n : U \to U$ by putting

(2) $$g_n = jfr_n .$$

Consider now for each n the diagram

$$\begin{array}{ccc} X & \xrightarrow{\ j\ } & U \\ {\scriptstyle D_n = f}\uparrow & \swarrow {\scriptstyle fr_n} & \uparrow{\scriptstyle g_n} \\ X & \xrightarrow{\ j\ } & U \end{array}$$

and its image under the functor H in the category of graded vector spaces

$$\begin{array}{ccc} H(X) & \xrightarrow{\ j_*\ } & H(U) \\ {\scriptstyle H(D_n) = f_*}\uparrow & \swarrow {\scriptstyle (fr_n)_*} & \uparrow{\scriptstyle (g_n)_*} \\ H(X) & \xrightarrow{\ j_*\ } & H(U) . \end{array}$$

In view of (2) we have

(3) $$(g_n)_* = (j_*)(fr_n)_* \quad \text{for all} \quad n .$$

In view of (1) the identity map $1 : X \to X$ is the uniform limit of the sequence $\{r_n j\}$ of maps $r_n j : X \to X$. Applying to the map $1 : X \to X$ proposition (1.3), we conclude that there exists an integer n_o such that $1_* = (r_n j)_*$ for

all $n \geq n_o$. This implies $f_* = (fr_n)_* \circ j_*$ for $n \geq n_o$, and hence in view of (3), the diagram $H(D_n)$ commutes for $n \geq n_o$.

Applying now Proposition (I.3.1) to the diagram $H(D_n)$ we have

$$\lambda(f) = \Lambda(f) = \Lambda(g_n) \quad \text{for} \quad n > n_o . \tag{4}$$

Now let us assume that $\lambda(f) \neq 0$. We shall prove that f has a fixed point. In view of (2) and (4) each $g_n : U \to U$ is a compact map with $\Lambda(g_n) \neq 0$ for $n \geq n_o$.

Applying Theorem II.2, we find a sequence $\{x_n\}$ of points in X such that

$$g_n(x_n) = x_n \quad \text{for} \quad n \geq n_o . \tag{5}$$

Let $\{x_{k_n}\}$ be a subsequence of $\{x_n\}$ such that

$$\lim_{n=\infty} x_{k_n} = x . \tag{6}$$

In view of (1) we have

$$\rho(x_{k_n}, r_{k_n}(x_{k_n})) \leq \frac{1}{k_n} \tag{7}$$

and hence, in view of (6),

$$\lim_{n=\infty} r_{k_n}(x_{k_n}) = x . \tag{8}$$

By continuity of f, we have from (8)

$$\lim_{n=\infty} f(r_{k_n}(x_{k_n})) = f(x) . \tag{9}$$

In view of (5) and (2), $x_{k_n} = g_{k_n}(x_{k_n}) = f(r_{k_n}(x_{k_n}))$ and, therefore, in view of (6), we have

$$\lim_{n=\infty} f(r_{k_n}(x_{k_n})) = x . \tag{10}$$

Comparing (9) and (10), we conclude that $x = f(x)$ and the proof is completed.

COROLLARY. Acyclic compact approximative ANR-s, and in particular approximative AR-s, have the fixed point property.

2. Fixed Point Theorems for Pairs of Spaces.

Next, some generalizations of some of the proved theorems to the case of pairs of spaces. The corresponding results due to C. Bowszyc (cf. [7], [9]) assure not only the existence of, but also provide certain information about the localization of fixed points. By H we denote the singular homology functor with coefficients in K.

(2.1) Let (X,A) be a pair of ANR-s and $f : (X,A) \to (X,A)$ be a compact map*). Then f is a Lefschetz map and

$$\Lambda(f) = \Lambda(f_X) - \Lambda(f_A) .$$

Proof. This clearly follows from Theorem II.2 and (I.4.3).

(2.2) Let (U,V) be a pair of open subsets of a normed space E and $f : (U,V) \to (U,V)$ be a compact map. Then $\Lambda(f) \neq 0$ implies that f has a fixed point in $U-V$.

Proof. First we apply the Approximation Theorem for $\varepsilon = \frac{1}{n}$ $(n = 1,2,\ldots)$ and find a sequence of finite polyhedral pairs (K_n,L_n) and a sequence $\{f_n\}$ of maps $f_n : (U,V) \to (U,V)$ such that $(K_n,L_n) \subset (U,V)$ for all $n > N$ and

(i) $\|f(x) - f_n(x)\| < \frac{1}{n}$ for all $x \in U$ and $n > N$

(ii) $f_n(U,V) \subset (K_n,L_n)$ for all $n > N$

(iii) $f \sim f_n$ for all $n > N$.

Then, using the Hopf-Lefschetz Theorem for the pairs of polyhedra and the argument analogous to that in the absolute case, we get a sequence of points $\{x_n\}$ such that for almost all n

$$f_n(x_n) = x_n \quad \text{and} \quad x_n \in K_n - L_n .$$

This implies, in view of (i), that

$$\|f(x_n) - x_n\| \to 0 \quad \text{and} \quad \operatorname{dist}(x_n,U-V) \to 0$$

and hence, in view of the compactness of the map f , implies the existence of a fixed point for f in $U-V$.

(2.3) Let (X,A) be a pair of ANR-s with A open in X and let $f : (X,A) \to (X,A)$ be a compact map. Then $\Lambda(f) \neq 0$ implies that f has a fixed point in $X-A$.

Proof. By the same argument as in the proof of Theorem II.2, this follows from (2.2) and (I.4.2).

(2.4) Let (X,A) be a pair of ANR-s with A closed in X and let $f : (X,A) \to (X,A)$ be a compact map. Then $\Lambda(f) \neq 0$ implies that f has a

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*) A map $f : (X,A) \to (Y,B)$ between the pairs of spaces if compact provided it maps (X,A) into a pair of compact spaces contained in (Y,B) .

fixed point in $\overline{X-A}$.

Proof. In view of the compactness of the map f , it is sufficient to show that for each $\varepsilon > 0$ there exists a point $x_o \in X-A$ such that $\rho(f(x_o),x_o) < \varepsilon$.

Let $\varepsilon > 0$ be given. We may assume without loss of generality that X is a closed subset of a normed space E . Take a retraction $r : U \to X$ of an open set $U \subset E$ onto X . We may assume (by making U smaller if necessary) that

$$\|r(x) - x\| < \frac{\varepsilon}{2} \quad \text{for all} \quad x \in U . \tag{1}$$

Let $B = r^{-1}(A)$; clearly for the pair of maps

$$i : (X,A) \to (U,B) \qquad \text{and} \qquad r : (U,B) \to (X,A) ;$$

we have $r \circ i = 1_{(X,A)}$. We define a compact map $g : (U,B) \to (U,B)$ by putting

$$g = i \circ f \circ r \tag{2}$$

and we let

$$\delta = \tfrac{1}{2} \min(\varepsilon, \delta_1) \quad \text{where} \quad \delta_1 = \operatorname{dist}(\overline{g(U)}, E-U) > 0 . \tag{3}$$

We claim now that there exists an open set V in E and a continuous map $g_1 : (U,B) \to (U,B)$ such that the following four properties are satisfied :

(i) g_1 is compact ;

(ii) $g_1(V) \subset B \subset V$;

(iii) $\|g_1(x) - g(x)\| < \delta$ for all $x \in U$;

(iv) g_1 is homotopic to g .

To this end we shall define four open sets U_o , U_1 , U_2 and V satisfying

$$B \subset V \subset \bar{V} \subset U_2 \subset \bar{U}_2 \subset U_1 \subset U_o \subset U \tag{4}$$

as follows : Since B is closed in U and A is an ANR , there exists an extension $\tilde{r}_B : U_o \to A$ of $r_B : B \to A$ over an open set $U_o \subset U$; thus U_o is defined.

Next, before defining U_1 , we let $\tilde{g} : U_o \to A$ be given by

$$\tilde{g} = f_A \circ \tilde{r}_B ; \tag{5}$$

clearly $\tilde{g}$ is a compact map and

$$\tilde{g}(x) = g(x) \in A \quad \text{for every} \quad x \in B . \tag{6}$$

Now we let

$$U_1 = \{x \in U_o ; \|\tilde{g}(x) - g(x)\| < \delta\} . \tag{7}$$

Finally, we define U_2 and V arbitrarily.

Now let $\lambda : U \to [0,1]$ be a real-valued function such that

(8) $\lambda(x) = 0$ for $x \in U - U_2$ and $\lambda(x) = 1$ for $x \in \bar{V}$.

We define the function g_1 on U as follows :

(9) $$g_1(x) = \begin{cases} g(x) & \text{for } x \in U - U_2 \\ g(x) + \lambda(x)(\tilde{g}(x) - g(x)) & \text{for } x \in U_1 \end{cases} .$$

From (4) and (8) it follows that g_1 is well defined and continuous. Furthermore, it is not difficult to check that $g_1 : (U,B) \to (U,B)$ and that the conditions (i) - (iv) are satisfied.

The various inter-relations between the relevant maps may be displayed now in the following two diagrams :

$$\begin{array}{ccc} (U,B) & \xrightarrow{\subset} & (U,V) \\ {\scriptstyle g_1}\uparrow & \nwarrow & \uparrow{\scriptstyle \tilde{g}_1} \\ (U,B) & \xrightarrow[\subset]{} & (U,V) \end{array} \qquad \begin{array}{ccc} (X,A) & \xrightarrow{i} & (U,B) \\ {\scriptstyle f}\uparrow & \overset{fr}{\nwarrow} & \uparrow{\scriptstyle g = ifr} \\ (X,A) & \xrightarrow[i]{} & (U,B) \end{array} .$$

Since g_1 is a Lefschetz map (by (2.1)) and $g_1 \sim g$ we conclude by (I.4.2) that

$$\Lambda(f) = \Lambda(g) = \Lambda(g_1) = \Lambda(\tilde{g}_1) .$$

If $\Lambda(f) \neq 0$ then by (2.2) there exists a fixed point for $\tilde{g}_1$. More precisely, we have

$$\tilde{g}_1(y_0) = g_1(y_0) = y_0 \quad \text{for some } y_0 \in U - V .$$

Let us put $x_0 = r(y_0)$; clearly, x_0 does not belong to A (because $r^{-1}(A) = B$). Further, in view of (1), (2), (iii) and (3), we have :

$$\|x_0 - y_0\| = \|r(y_0) - y_0\| < \varepsilon/2$$

$$\|y_0 - f(x_0)\| = \|g_1(y_0) - fr(y_0)\| = \|g_1(y_0) - g(y_0)\| < \varepsilon/2$$

and consequently

$$\|f(x_0) - x_0\| < \varepsilon \quad \text{with} \quad x_0 \in X - A .$$

The proof is completed.

The preceeding discussion is summarized in the following :

THEOREM 2 (cf. [7]). Let (X,A) be a pair of ANR-s such that A is either closed or open in X and assume that $f : (X,A) \to (X,A)$ is a compact mapping. Then :

(i) f is a Lefschetz map ;

(ii) $\Lambda(f) = \Lambda(f_X) - \Lambda(f_A)$;

(iii) $\Lambda(f) \neq 0$ implies that f has a fixed point in $\overline{X-A}$.

3. Some applications of the Mayer-Vietoris functor.

Notation : Let $\{A_i\}$ be a finite family of sets $(i = 1,2,\dots,n)$ and

$$f : \bigcup_{i=1}^{n} A_i \to \bigcup_{i=1}^{n} A_i$$

be a map satisfying $f(A_i) \subset A_i$ for every $i = 1,2,\dots,n$. Given a multi-index

$$j = (j_1, j_2, \dots, j_k) , \quad j_1 < j_2 < , \dots , < j_k \quad \text{with} \quad |j| = k \leq n$$

we let

$$A_j = A_{j_1} \cap A_{j_2} \cap \dots \cap A_{j_k}$$

and denote by

$$f_j : A_j \to A_j$$

the corresponding contraction. By $\mathfrak{M}\{A_i\}$ we denote the smallest lattice of sets containing all the members of the family $\{A_i\}$ and by $N(\{A_i\})$ the nerve of the family $\{A_i\}$.

Definition. An ordered pair of spaces $\{X_1, X_2\}$ is called an excisive couple (cf. [45]) provided the excision map $(X_1, X_1 \cap X_2) \to (X_1 \cup X_2, X_2)$ induces an isomorphism of the corresponding homology groups. More generally, a family of spaces $\{A_i\}$ $(i = 1,2,\dots,n)$ is said to be excisive provided for any members X_1 and X_2 of the lattice $\mathfrak{M}\{A_i\}$ the couple $\{X_1, X_2\}$ is excisive.

Clearly we have the category of excisive couples and on this category there is defined the Mayer-Vietoris functor which assigns to an excisive $\{X_1, X_2\}$ the exact Mayer-Vietoris sequence

$$\dots H_q(X_1 \cap X_2) \to H_q(X_1) \oplus H_q(X_2) \to H_q(X_1 \cup X_2) \to H_{q-1}(X_1 \cap X_2) \dots$$

and to a morphism $f : \{X_1, X_2\} \to \{Y_1, Y_2\}$ it assign the map of the corresponding exact sequences.

(3.1) Let A_i be closed in a metric space X for $i = 1,2,\dots,n$, and assume that for every multi-index j the set A_j is an ANR. Then (i) every member X of the lattice $\mathfrak{M}\{A_i\}$ is an ANR and (ii) $\{A_i\}$ is the excisive family of spaces.

(3.2) Assume that

$$A = \bigcup_{i=1}^{n} A_i$$

and $\{A_i\}$ is the excisive family. Let $f : A \to A$ be a continuous map such that $f(A_i) \subset A_i$ for all $i = 1,2,\dots,n$. If, for every multi-index j, $f_j : A_j \to A_j$ is a Lefschetz map, then so is the map f and we have :

$$\Lambda(f) = \sum_j (-1)^{|j|+1} \Lambda(f_j) .$$

Proof. We indicate the induction by considering the case $n = 2$. We have the map of the Mayer-Vietoris exact sequence into itself induced by f, i.e. the following commutative diagram with exact rows :

$$\begin{array}{ccccccccc}
\to & H_q(A_1 \cap A_2) & \to & H_q(A_1) \oplus H_q(A_2) & \to & H_q(A_1 \cup A_2) & \to & H_{q-1}(A_1 \cap A_2) & \to \\
 & \downarrow{\scriptstyle f_{(1,2)q}} & & \downarrow{\scriptstyle f_{1q} \oplus f_{2q}} & & \downarrow{\scriptstyle f_q} & & \downarrow{\scriptstyle f_{(1,2)q-1}} & \\
\to & H_q(A_1 \cap A_2) & \to & H_q(A_1) + H_q(A_2) & \to & H_q(A_1 \cup A_2) & \to & H_{q-1}(A_1 \cap A_2) & \to
\end{array}$$

By (I.3.2) $\{f_q\}$ is a Leray endomorphism and

$$\begin{aligned}
\Lambda(f) = \Lambda(\{f_q\}) &= \Lambda\{(f_{1q} \oplus f_{2q})\} - \Lambda(\{f_{(1,2)q}\}) \\
&= \Lambda(\{f_{1q}\}) + \Lambda(\{f_{2q}\}) - \Lambda(\{f_{(1,2)q}\}) \\
&= \Lambda(f_1) + \Lambda(f_2) - \Lambda(f_{(1,2)}) .
\end{aligned}$$

THEOREM 3 ([9]). Let X be an ANR, $\{A_i\}_{i=1}^n$ a family of ANR-s such that all $A_i \subset X$ are either open or closed in X and every A_j is an ANR. Assume further that

$$A = \bigcup_{i=1}^{n} \quad \text{and} \quad f : (X,A) \to (X,A)$$

is a compact map satisfying $f(A_i) \subset A_i$ $(i = 1,2,\dots,n)$. Then

(i) $\Lambda(f) = \Lambda(f_X) - \sum_j (-1)^{|j|+1} \Lambda(f_j)$,

(ii) in particular, if all A_j are either empty or acyclic

$$\Lambda(f) = \Lambda(f_X) - \chi(N(\{A_i\})) ,$$

(iii) $\Lambda(f) \neq 0$ *implies that* f *has a fixed point in* $X-A$.

Proof. (i) follows from (3.1), (3.2) and remark that any finite family of open sets is excisive ; (ii) follows from the Euler-Poincaré formula, since $\Lambda(f_j) = 1$ for every j ; (iii) is just the restatement of Theorem 2.

COROLLARY 1. *Let* X *be an* AR *and*

$$A = \bigcup_{i=1}^{n} A_i \quad (n \geq 2)$$

be the disjoint union of AR-s *which are either all open or all closed and such that every* A_j *is an* AR . *Assume further that* $f : (X,A) \to (X,A)$ *is a continuous map such that* (a) $f(A_i) \subset A_i$ *and* (b) $f_X : X \to X$ *and all the maps* $f_i : A_i \to A_i$ *are compact. Then* f *has a fixed point in* $X-A$.

Proof. Clearly f is compact , $\Lambda(f_X) = 1$ and $\chi(N\{A_i\}) = n$. Hence by Theorem 3, $\Lambda(f) = 1-n \neq 0$ and our assertion follows.

COROLLARY 2 ([11]). *Let* X *be a compact* AR *and* U *be the union of* n*-open sets* $U_1, U_2, \ldots, U_n$ *such that* (i) $\bar{U}_i \cap \bar{U}_j \neq 0$, (ii) *every* $\bar{U}_i$ *is an* AR , (iii) $n \geq 2$. *Assume further that* $f : X - U \to X$ *is a map satisfying* $f(\dot{U}_i) \subset \bar{U}_i$ *for every* $i = 1,2,\ldots,n$. *Then* f *has a fixed point.*

Proof. Let

$$A = \bigcup_{i=1}^{n} \bar{U}_i \, .$$

It follows from the assumptions that there is a map $g : (X,A) \to (X,A)$ such that $g(x) = f(x)$ for all $x \in X-U$. We have clearly $\Lambda(g) = 1-n \neq 0$ and hence $g(x) = x$ for some $x \in \overline{X-A} = X-U$. Consequently, $f(x) = g(x) = x$.

4. Common fixed points.

Let X be a space and $\mathcal{F}_X = \{f\}$ be a family of maps $f : X \to X$; call $\mathcal{F}_X$ *divisible* [9] provided for any $f_1, f_2 \in \mathcal{F}_X$ there exists $h \in \mathcal{F}_X$ such that for some n_1 , n_2

$$f_1 = h^{n_1} \quad \text{and} \quad f_2 = h^{n_2} \, .$$

(4.1) *Assume that* $\mathcal{F}_X = \{f\}$ *is a divisible family such that for any* $f \in \mathcal{F}_X$ *the fixed point set* $\nu(f) = \{x \in X \, , \, f(x) = x\}$ *is non-empty and compact. Then*

$$\bigcap_{f \in \mathcal{F}_X} \nu(f)$$

is also non-empty.

Proof. Let $f_1, f_2, \ldots, f_k \in \mathcal{F}_X$; there exists an $h \in \mathcal{F}_X$ such that, for some $n_1, \ldots, n_k$

$$f_1 = h^{n_1}, \quad f_2 = h^{n_2}, \ldots, f_k = h^{n_k}.$$

Consequently, $\varkappa(h) \subset \varkappa(f_i) = \varkappa(h^{n_i})$ for each $i = 1,2,\ldots,k$ and hence the intersection

$$\varkappa(f_1) \cap \varkappa(f_2) \cap \ldots \cap \varkappa(f_k)$$

is non-empty. This shows that the family $\{\varkappa(f)\}_{f \in \mathcal{F}_X}$ is a centered family of compact spaces and our assertion follows.

Definition. Let (X,A) be a pair of spaces and $\{f_t\}$ $(t \geq 0)$ a continuous family of maps $f_t : (X,A) \to (X,A)$, depending on the parameter $t \in R^+$; $\{f_t\}$ is said to be a semi-flow on (X,A) provided

(a) $f_{t_1+t_2} = f_{t_1} \circ f_{t_2}$ for any $t_1, t_2 \in R^+$,

(b) $f_0 = 1_{(X,A)}$.

If for a point $x \in X$, $f_t(x) = x$ for all $t \geq 0$, then x is said to be a fixed point for a semi-flow $\{f_t\}$.

We note the following evident proposition :

(4.2) Let $f_t : (X,A) \to (X,A)$, $t \in R^+$ be a semi-flow. Then (i) f_{t_1} is homotopic to f_{t_2} for any $t_1, t_2 \in R^+$; (ii) the family $\{f_t\}$, indexed by the positive rationals, is divisible.

THEOREM 4. Let (X,A) be a pair of ANR-s such that A is either open or closed in X and assume that the relative Euler-Poincaré characteristic $\chi(X,A)$ of (X,A) is finite and different from 0 . Let $f_t : (X,A) \to (X,A)$ be a semi-flow on (X,A) such that f_t is compact for each $t > 0$. Then $\{f_t\}$ has a fixed point $x \in \overline{X-A}$.

Proof. We have clearly $\Lambda(f_t) = \chi(X,A) \neq 0$ for every $t \geq 0$ and hence our assertion follows, in view of (4.1), (4.2), and the continuity of the family $\{f_t\}$, from Theorem 2.

5. Vector fields on the manifolds with boundary.

Let M be a C^k-manifold with boundary ∂M $(k \geq 2)$; for a point $p \in M$ we let $T_p(M)$ be the tangent space to M at p . If $p \in \partial M$ then $T_p(M)$ decomposes into two closed half-spaces $T_p^+(M)$ and $T_p^-(M)$ such that $T_p(\partial M) = T_p^+(M) \cap T_p^-(M)$. A tangent vector ξ_p is said to be outwardly (resp. inwardly) directed at a point $p \in \partial M$ provided $\xi_p \in T_p^+(M)$ (resp. $\xi_p \in T_p^-(M)$) .

Let $\xi = \{\xi_p\}_{p\in M}$ be a C^{k-1}-vector field on the manifold M and assume that on any component of the boundary ∂M either (i) all ξ_p we outwardly directed or (ii) all ξ_p are inwardly directed . Denote by

$$A = A_1 \cup \dots \cup A_k$$

the union of all the components A_i of the first kind and by

$$B = B_1 \cup \dots \cup B_l$$

the union of all components of the second kind.

We may state now the following generalization of the well-known result of H. Hopf (cf. [9]) :

THEOREM 5. *If the relative Euler-Poincaré characteristic $\chi(M,A)$ is different from zero, then the field $\xi = \{\xi_p\}$ vanishes at some point $p\in M$; the same assertion holds when $\chi(M,B) \neq 0$.*

Proof. First, by taking a tabular neighbourhood $N_A \overset{\sim}{\text{diff}} A \times I$ of A , we enlarge M to a manifold $\bar{M}$. We have $\partial N_A = A \cup A'$ and $\partial \bar{M} = B \cup A'$.

Since the pair (M,A) is clearly a deformation retract of (M,N_A) we have

$$\chi(M,A) = \chi(\bar{M},N_A) \quad (*)$$

Next, we define the vector-field $\bar{\xi} = \{\bar{\xi}_p\}$ on $\bar{M}$ as follows :

Let $\xi' = \{\xi'_p\}$ be defined on $M \cup A'$ by

$$\xi'_p = \xi_p \text{ for } p\in M \quad \text{and} \quad \xi'_p = 0 \text{ for } p\in A'$$

we let $\bar{\xi}$ be an arbitrary C^{k-1}-extension of ξ' over M . Since on N_A the field $\bar{\xi}$ is inwardly directed, it determines differentiable semi-flow

$$f_t : (M,N_A) \to (M,N_A) .$$

in view of (*) and Theorem 4 , there is a point $p\in M = \overline{\bar{M} - N_A}$ such that $f_t(p) = p$ for all $t \geq 0$. For this point p , we have $\bar{\xi}_p = \xi_p = 0$ and our first assertion follows. The second assertion follows clearly from the first by considering the vector field $\eta = -\xi$.

V. THE LEFSCHETZ POWER SERIES AND THE EXISTENCE OF PERIODIC POINTS

Next, some applications of the Leray trace to the theory of periodic points. The corresponding results, due to C. Bowszyc [8] are expressed in terms of the Euler-Poincaré characteristic $\chi(f)$ and the Lefschetz power series $L(f)$ of a map f.

1. Algebraic preliminaries.

For a field K we denote by $K\{x\}$ the integral domain consisting of all formal power series

$$s = a_0 + a_1 x + a_2 x^2 + \dots = \sum_{n=0}^{\infty} a_n x^n$$

with coefficients $a_n \in K$; $K\{x\}$ contains the polynominal ring $K[x]$, the field K and $1 \in K$.

(1.1) *A power series* $s = \sum_{n=0}^{\infty} a_n x^n \in K\{x\}$ *is invertible if and only if* $a_0 \neq 0$. *For an element* $s = 1-\lambda x$ *we have*

$$(1-\lambda x)^{-1} = \sum_{n=0}^{\infty} \lambda^n x^n .$$

(1.2) *Assume that*

$$s = \sum_{n=0}^{\infty} a_n x^n \in K\{x\}$$

is of the form $s = uv^{-1}$, *where* $u, v \in K[x]$, $u \neq 0$ *and* $\deg u < \deg v = K$. *Then for any natural* m *at least one of the coefficients*

$$a_{m+1} , a_{m+2} , \dots , a_{m+k}$$

must be different from zero.

Definition. Denote by $d : K\{x\} \to K\{x\}$ the ordinary derivation in $K\{x\}$; for an invertible element $s \in K\{x\}$ we define the *logarithmic derivative* $D(s)$ of s by

$$D(s) = s^{-1} d(s) .$$

(1.3) *If the elements* $s_1 , s_2 , \dots , s_k , s \in K\{x\}$ *are invertible, then*

$$D\Big(\prod_{i=1}^{k} s_i\Big) = \sum_{i=1}^{k} D(s_i) \quad \text{and} \quad D(s^{-1}) = -D(s) .$$

Definition. Assume that a power series s is of the form $s = uv^{-1}$ where

$$u = \sum_{n=0}^{k-1} a_n x^n , \quad v = \sum_{n=0}^{k} b_n x^n \quad \text{and} \quad b_0 , b_k \neq 0 .$$

We define the conjugate s^* of s by putting

$$s^* = \Big(\sum_{n=0}^{k-1} a_n x^{k-1-n}\Big)\Big(\sum_{n=0}^{k} b_n x^{k-n}\Big)^{-1} .$$

(1.4) If the elements $s_1, s_2, \dots, s_m$ have the conjugates, then so does their sum and

$$\Big(\sum_{i=1}^{m} s_i\Big)^* = \sum_{i=1}^{m} s_i^* .$$

2. The Lefschetz power series of an endomorphism.

We recall that for an endomorphism $\varphi : E \to E$ of a vector space E we let $\tilde{\varphi} : \tilde{E} \to \tilde{E}$ be induced by φ on

$$\tilde{E} = E/N \quad \text{where} \quad N\varphi = \bigcup_{n=0}^{\infty} \ker \varphi^n ,$$

if φ is admissible (i.e., $\dim \tilde{E} < \infty$) then $\tilde{\varphi}$ is an automorphism and we denote by $w(\varphi)$ the characteristic polynomial of $\tilde{\varphi}$.

Since $N\varphi = N\,\varphi^n$ we have clearly

(2.1) φ is admissible if and only if φ^n is admissible.

(2.2) Assume that K is algebraically closed and $\varphi : E \to E$ is an admissible endomorphism. Then all the roots $\lambda_1, \dots, \lambda_m$ $(m = \dim \tilde{E})$ of the characteristic polynomial $w(\varphi)$ are different from zero and for any natural n we have

$$\operatorname{Tr}(\varphi^n) = \sum_{j=1}^{m} \lambda_j^n .$$

Proof. By the Jordan Theorem, there exists a basis in $\tilde{E}$ such that the corresponding matrix representation for $\tilde{\varphi}$ has a triangular form. Consequently

$$\operatorname{Tr}\varphi = \sum_{j=1}^{m} \lambda_j \quad , \quad \operatorname{Tr}\varphi^2 = \sum_{j=1}^{m} \lambda_j^2, \dots$$

and the assertion follows.

Assume now that $\varphi = \{\varphi_q\}$ is a Leray endomorphism of a graded vector space $E = \{E_q\}$; we recall that in this case

$$\tilde{E} = \{\tilde{E}_q\} \quad (\text{where } \tilde{E}_q = E_q/N\varphi_q)$$

is of a finite type, the Lefschetz number of φ is given by

$$\Lambda(\varphi) = \sum_q (-1)^q \operatorname{Tr}(\varphi_q) = \sum_q (-1)^q \operatorname{tr}(\tilde{\varphi}_q)$$

and the Euler-Poincaré characteristic $\chi(\varphi)$ of φ by

$$\chi(\varphi) = \chi(\tilde{E}) = \sum_q (-1)^q \dim(\tilde{E}_q) .$$

We note that (2.1) implies

(2.3) φ is a Leray endomorphism if and only if φ^n is a Leray endomorphism ; in that case $\chi(\varphi) = \chi(\varphi^n)$.

Definition (cf. [8] and [2]). The Lefschetz power series $L(\varphi)$ of a Leray endomorphism $\varphi = \{\varphi_q\}$ is an element of $K\{x\}$ defined by

$$L(\varphi) = \chi(\varphi) + \sum_{n=1}^{\infty} \Lambda(\varphi^n)x^n = \sum_{n=0}^{\infty} \lambda(\tilde{\varphi}^n)x^n$$

and the characteristic polynomial $w(\varphi)$ of φ is given by

$$w(\varphi) = \prod_q w_q^{(-1)^q}$$

where w_q is the characteristic polynomial of $\tilde{\varphi}_q$.

The following fact is of basic importance [8] :

(2.4) Let $\varphi = \{\varphi_q\}$ be a Leray endomorphism of a graded vector space $E = \{E_q\}$ into itself. Then

(1) $$L(\varphi) = (D(w))^*$$

where w is the characteristic polynomial of the endomorphism φ . Consequently, the Lefschetz power series $L(\varphi)$ admits a representation of the form

(2) $$L(\varphi) = uv^{-1}$$

where u and v are relatively prime polynomials with $\deg u < \deg v$ $(u \neq 0)$.

Proof. We shall indicate the proof for the case when K is algebraically closed ; to this end, denote by λ_{qj} $(j = 1,2,\ldots, \dim \tilde{E}_q)$ all the roots of the characteristic polynomial w_q of $\tilde{\varphi}_q$.

We have

(3) $$w_q = \prod_j (x - \lambda_{qj}) .$$

Taking into account (1.3) and (1.4) and the definition of w we get

$$Dw_q = \sum_j D(x - \lambda_{qj}) = \sum_j (x - \lambda_{qj})^{-1}$$

$$Dw = \sum_{qj} (-1)^q (x - \lambda_{qj})^{-1}$$

$$((x - \lambda_{qj})^{-1})^* = (1 - \lambda_{qj}x)^{-1}$$

and consequently

$$(4) \qquad (Dw)^* = \Big(\sum_{qj} (-1)^q (x - \lambda_{qj})^{-1}\Big)^* = \sum_{qj} (-1)^q (1 - \lambda_{qj} x)^{-1} .$$

On the other hand, taking into account (1.1) and (2.1), we have

$$L(\varphi) = \sum_{n=0}^{\infty} \lambda(\tilde{\varphi}^n) x^n = \sum_{n=0}^{\infty} \sum_{q=0} (-1)^q \operatorname{tr}(\tilde{\varphi}_q^n) x^n =$$

$$= \sum_{n=0}^{\infty} \sum_{qj} (-1)^q \lambda_{qj}^n x^n = \sum_{qj} (-1)^q (1 - \lambda_{qj} x)^{-1}$$

and hence, in view of the formula (4), our assertion follows.

Definition. Let $L(\varphi) = uv^{-1}$ be a rational representation of $L(\varphi)$ with relatively prime polynomials u and v as in (2.4). We let $P(\varphi)$ be the degree of the polynomial v.

(2.5) Let $\varphi = \{\varphi_q\}$ be a Leray endomorphism. Then

(i) $\chi(\varphi) \neq 0$ implies $P(\varphi) \neq 0$;

(ii) $P(\varphi) \neq 0$ if and only if $\Lambda(\varphi^n)$ for some natural n ;

(iii) if $P(\varphi) = k \neq 0$ then for any natural m, one of the coefficients $\Lambda(\varphi^{m+1})$, $\Lambda(\varphi^{m+2}), \ldots, \Lambda(\varphi^{m+k})$ is different from 0.

We remark further that

(2.6) If the characteristic polynomial w of the Leray endomorphism $\varphi = \{\varphi_q\}$ is represented on the form $w = yz^{-1}$, with relatively prime polynomials y and z, then

$$\chi(\varphi) = \deg y - \deg z \qquad \text{and} \qquad P(\varphi) = a+b$$

where a and b are the numbers of different roots of the polynomials y and z respectively.

3. The Lefschetz power series of a continuous map.

Consider a category $\mathfrak{F}$ of topological spaces (or pairs of topological spaces) and let H be a homology or cohomology functor from $\mathfrak{F}$ to the category of graded vector spaces over K. We recall that a continuous $f : X \to X$ is called a Lefschetz map (with respect to H) provided $H(f)$ is a Leray endomorphism.

It is easily seen (by taking into account (2.3)), that :

(3.1) A map $f : X \to X$ is a Lefschetz map if and only if so is any iterate f^n of f ; in this case

$$\chi(f) = \chi(f^n) .$$

Now the essential part of the proceeding discussion is summarized in the following:

THEOREM 1. The Lefschetz power series $L(f)$ of a map $f : X \to X$ admits a "rational" representation

$$L(f) = uv^{-1}$$

where u and v are relatively prime polynomials with $\deg u < \deg v$. Let us put $P(f) = \deg v$. We have

(a) $\chi(f) \neq 0$ implies $P(f) \neq 0$;

(b) $P(f) = 0$ if and only if $\Lambda(f^n) = 0$ for some n ;

(c) $P(f) = K \neq 0$ implies that for any m at least one of the coefficients

$$\Lambda(f^{m+1}) , \Lambda(f^{m+2}),\dots,\Lambda(f^{m+k})$$

of the series $L(f)$ must be different from zero.

In the next section we shall turn to the applications of Theorem 1 to the theory of periodic points.

4. Lefschetz spaces. The existence of periodic points.

Let $f : Y \to Y$ be a continuous map ; a point $y \in Y$ is said to be a periodic point for f with period n provided $f^n(y) = y$.

In order to increase the generality of our considerations it will be convenient to introduce the following :

Definition (comp. [13]). Let Y be an object of the category $\mathfrak{S}$. Call Y a Lefschetz space (or a Lefschetz pair) with respect to the functor H provided every compact map $f : Y \to Y$ is a Lefschetz map and $\Lambda(f) \neq 0$ implies the existence of a fixed point y for f ; in the case of a pair $Y = (X,A)$ we require additionally that $y \in \overline{X-A}$.

Examples : The following types of topological spaces (resp. pairs of spaces) are all Lefschetz spaces (resp. Lefschetz pairs) :

1° metric ANR-s with respect to the singular homology ;

2° compact metric approximative ANR-s with respect to the Čech-Vietoris homology ;

3° open sets in locally convex spaces ;

4° retracts of Lefschetz spaces with respect to the same homology or cohomology functor ;

5° compact ANR-s for normal spaces with respect to the singular homology ;

6° convexoid spaces in the sense of Leray [37] with respect to the Čech cohomology ;

7° pairs $Y = (X,A)$ of metric ANR-s such that A is either closed or open in X.

THEOREM 2 (cf. [8]). Let $f : Y \to Y$ be a map of a Lefschetz space (or a pair of spaces) such that the n-th iterate f^n of f is compact for some n. If $\chi(f) \neq 0$ or $P(f) \neq 0$ then f has a periodic point y with a period $k < n + P(f)$ (in the case of a pair $Y = (X,A)$ we assert that $y \in \overline{X-A}$).

Next we draw some consequences of Theorem 2.

COROLLARY 1. Let Y be a Lefschetz space of a finite type and $f : Y \to Y$ be an eventually compact map*) such that $f_* : H(Y) \to H(Y)$ is an isomorphism. Then $\chi(Y) \neq 0$ implies that f has a periodic point.

Proof. The fact that f_* is invertible implies that $\chi(f) = \chi(Y)$ and thus our assertion follows from Theorem 2.

Corollary 1 implies the following result due to F.B. Fuller [19] :

COROLLARY 2. Let Y be a compact metric ANR and $f : Y \to Y$ an invertible or, more generally, homotopically invertible map. If the Euler-Poincaré characteristic $\chi(Y) \neq 0$ then f has a periodic point.

COROLLARY 3. Let Y be a Lefschetz space (or a Lefschetz pair) such that $H_{2n}(Y) = 0$ for all $n \geq 0$. Then any eventually compact map $f : Y \to Y$ has a periodic point.

Proof. From the assumptions and the definition of the Euler-Poincaré characteristic of a map, it follows that $\chi(f) \neq 0$ and therefore our assertion is a consequence of Theorem 2.

Corollary 3 contains as a particular case the following result due to O. Hájek**) :

COROLLARY 4. Let Y be a compact ANR such that $H_{2n}(Y) = 0$ for all $n \geq 0$. Then any eventually compact map $f : Y \to Y$ has a periodic point.

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*) A map $f : Y \to Y$ is called eventually compact provided certain iterate f^n of f is compact.

**) O. Hájek, Homological fixed point theorems. I, II, III, Comment. Math. Univ. Carolinae 5 (1964), 13-31 ; 85-92 ; 6(1965), 157-164.

VI. FIXED POINTS FOR MULTI-VALUED COMPACT MAPS

In 1946, S. Eilenberg and D. Montgomery [17] made the important observation that, with the aid of an old theorem of L. Vietoris [46] several results of the fixed-point theory for single-valued mappings could be carried over to the case of multi-valued acyclic maps, i.e., maps for which the image of every point is an acyclic compact set. Thus, the Lefschetz Fixed Point Theorem for compact ANR-s was extended by the above-named authors to arbitrary acyclic maps.*)

We shall propose now an extension of the above Eilenberg-Montgomery theorem to the case of compact multi-valued maps of non-compact ANR-s.

1. Vietoris mappings.

In what follows only metrizable spaces will be considered. The category of such spaces and continuous mappings will be denoted by $\mathfrak{S}$. By H we denote the Čech homology functor with compact carriers and rational coefficients from the category $\mathfrak{S}$ to the category $\mathfrak{AL}$ of graded vector spaces and linear maps of degree zero.

Definition. A continuous mapping $f : X \to Y$ is said to be a Vietoris map provided the following two conditions are satisfied :

(i) f is proper, i.e., for any compact C, the counter image $f^{-1}(C)$ is also compact,

(ii) the set $f^{-1}(y)$ is acyclic for every $y \in Y$.

In our considerations an essential use will be made of the following :

(1.1) (VIETORIS MAPPING THEOREM). If $f : X \to Y$ is a Vietoris map, then the induced map $f_* : H(X) \to H(Y)$ is invertible.

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*) For similar generalizations of some other topological facts see [30], [31] and [27]. We remark that to the special class of acyclic maps consisting of those which are convex-valued various fixed point theorems for compact operators were extended (cf. [4], [18], [23]) as well as the basic facts of the Leray-Schauder theory in Banach spaces (cf. [22], [29]). As in the single-valued case (cf. [38]) fixed point theorems for multi-valued maps prove themselves useful in many branches of mathematics , they found, for instance, applications in the theory of games (cf. [4], [18]) and more recently in the ordinary differential equations (cf. [33]) and optimal control theory (cf. [34]).

Theorem (1.1) clearly follows from the original statement of the Vietoris Mapping Theorem for compacta (cf. [46]).

2. Multi-valued mappings.

Let X and Y be two spaces and assume that for every point $x \in X$ a non-empty subset $\varphi(x)$ of Y is given ; in this case, we say that φ is a multi-valued mapping from X to Y and we write $\varphi : X \to Y$. In what follows, the symbols φ, ψ, χ will be reserved for multi-valued mappings ; the single-valued maps will be denoted by $f, g, h,$ etc.

Let $\varphi : X \to Y$ be a multi-valued map. We associate with φ the following diagram of continuous mappings

$$X \xleftarrow{\ p\ } \Gamma_\varphi \xrightarrow{\ q\ } Y$$

in which

$$\Gamma_\varphi = \{(x,y) \in X \times Y \ , \ y \in \varphi(x)\}$$

is the graph of φ and the natural projections p and q are given by $p(x,y) = x$ and $q(x,y) = y$.

The point-to-set mapping φ extends to a set-to-set mapping by putting

$$\varphi(A) = \bigcup_{a \in A} \varphi(a) \subset Y$$

for $A \subset X$; $\varphi(A)$ is said to be the image of A under φ . If $\varphi(A) \subset B \subset Y$, then the contraction of φ to the pair (A,B) is the multi-valued map $\varphi' : A \to B$ defined by $\varphi'(a) = \varphi(a)$ for each $a \in A$. A contraction of φ to the pair (A,Y) is simply the restriction $\varphi|A$ of φ to A .

Definition. A multi-valued mapping $\varphi : X \to Y$ is said to be continuous provided the graph Γ_φ of φ is closed in the product $X \times Y$; in other words, the conditions $x_n \to x$, $y_n \to y$, $y_n \in \varphi(x_n)$ imply $y \in \varphi(x)$.

We note that if $\varphi = f$ (i.e., φ is single-valued), then the above definition gives the ordinary continuity of f . In what follows only continuous multi-valued mappings will be considered.

Definition. A multi-valued mapping φ , $\varphi : X \to Y$ is called compact provided the image $\varphi(X)$ of X under φ is contained in a compact subset of Y .

The following evident remark is of importance :

(2.1) If $\varphi : X \to Y$ is compact, then the projection $p : \Gamma_\varphi \to X$ is proper as a single-valued mapping.

Definition. Let $\varphi : X \to X$ be a multi-valued mapping. A point x is called a fixed point for φ provided $x \in \varphi(x)$.

3. Acyclic maps.

We shall recall now the statement of the Eilenberg-Montgomery Theorem.

Definition. Let X and Y be two spaces. A multi-valued mapping $\varphi : X \to Y$ is said to be acyclic provided the set $\varphi(x)$ is acyclic for every point $x \in X$.

Assume now that X and Y are compacta and $\varphi : X \to Y$ is an acyclic multi-valued mapping. We observe that, since for every $x \in X$, $p^{-1}(x)$ is homeomorphic to $\varphi(x)$, the projection $p : \Gamma_\varphi \to X$ is a Vietoris map.

Using the Vietoris Mapping Theorem we define the linear map $\varphi_* : H(X) \to H(Y)$ by putting $\varphi_* = q_* \circ p_*^{-1}$; φ_* is said to be induced by the multi-valued mapping φ . It is easily seen that if $\varphi = f$ (i.e., φ is single-valued), then $\varphi_* = f_*$.

Let X be of a compact space of a finite type and $\varphi : X \to X$ be an acyclic multi-valued mapping of X into itself. We define the Lefschetz number $\lambda(\varphi)$ of φ by putting

$$\lambda(\varphi) = \lambda(\varphi_*) .$$

(3.1) (The Eilenberg-Montgomery Theorem). Let X be a compact ANR and $\varphi : X \to X$ an acyclic multi-valued mapping. Then $\lambda(\varphi) \neq 0$ implies that φ has a fixed point.

4. The quasi-category. $\mathfrak{S}$.

In all that follows, the symbol $f : X \Rightarrow Y$ will mean that either (i) f is a Vietoris map or (ii) f is a homeomorphism ; we remark that in either case, the induced map f_* is invertible.

Definition. A multi-valued mapping $\varphi : X \to Y$ is said to be admissible provided either (i) φ is single-valued or (ii) φ is acyclic and compact. The class of all admissible maps will be denoted by $\mathfrak{S}$.

(4.1) If a multi-valued mapping $\varphi : X \to Y$ is admissible, then the diagram of natural projections for φ has the form

$$X \xleftarrow{\ p\ } \Gamma_\varphi \xrightarrow{\ q\ } Y .$$

Proof. If $\varphi = f$, the assertion is evident ; if φ is acyclic and compact, our assertion is a consequence of (2.1), the fact that $p^{-1}(x)$ is homeomorphic to $\varphi(x)$ for every $x \in X$ and the Vietoris Mapping Theorem.

Definition. Two admissible mappings $\varphi : X \to Y$ and $\psi : Y \to Z$ are called composable provided either (i) φ is single-valued or (ii) ψ is the inclusion; in either case, the composite $\psi\varphi : X \to Z$ given by the assignment $x \to \psi(\varphi(x))$ is an admissible mapping from X to Z.

Thus, $\tilde{\mathfrak{S}}$ is equipped with a partially defined operation of composition of maps*). Next we show that the cohomology functor $\tilde{H} : \mathfrak{S} \to \mathfrak{AL}$ can be extended over $\tilde{\mathfrak{S}}$ to a function $\tilde{H} : \tilde{\mathfrak{S}} \to \mathfrak{AL}$ satisfying certain quasi-functorial properties; these turn out to be sufficient for the proofs of our main results.

Definition. Let $\varphi : X \to Y$ be an admissible map. Using (4.1) we define the linear map

$$\tilde{H}(\varphi) = \varphi_* : H(X) \to H(Y)$$

as the composite

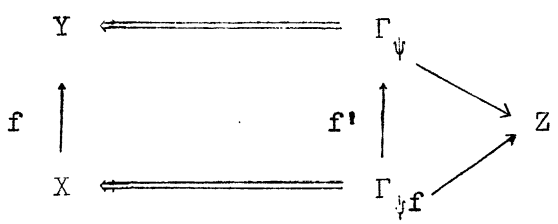
$$H(X) \xrightarrow{(p_*)^{-1}} H(\Gamma_\varphi) \xrightarrow{q_*} H(Y)$$

φ_* is said to be induced by φ; clearly, if $\varphi = f$, then $\varphi_* = f_*$.

(4.2) Let $\varphi : X \to Y$ and $\psi : Y \to Z$ be two composable maps in $\tilde{\mathfrak{S}}$. Then we have $(\psi\varphi)_* = \psi_*\varphi_*$; in other words, $\tilde{H}$ sends commutative triangles in $\tilde{\mathfrak{S}}$ into commutative triangles in $\mathfrak{AL}$.

Proof. Assume first that φ is single-valued and let $\varphi = f$. Then the product mapping $f \times \mathrm{id} : X \times Z \to Y \times Z$ maps $\Gamma_{\psi f} \subset X \times Z$ into $\Gamma_\psi \subset Y \times Z$ and therefore determines the map $f' : \Gamma_{\psi f} \to \Gamma_\psi$.

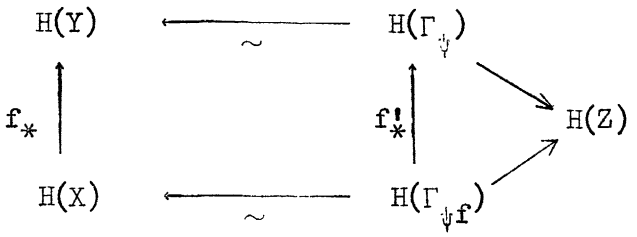
Consider the following diagram



in which all unlabelled arrows represent the natural projections. From the ...

*) We note that, if one of the composites $(\varphi_3\varphi_2)\varphi_1$ or $\varphi_3(\varphi_2\varphi_1)$ is defined, then so is the other and, in that case $\varphi_3(\varphi_2\varphi_1) = (\varphi_3\varphi_2)\varphi_1$. It is not, however, true that the existence of both $\varphi_3\varphi_2$ and $\varphi_2\varphi_1$ implies that of $\varphi_3\varphi_2\varphi_1$.

definition of f' it is clear that this diagram commutes ; consequently, the diagram



also commutes and this shows that $(\psi f)_* = \psi_* f_*$.

The proof of our assertion in the case when ψ is the inclusion is similar.

5. Homotopy and selectors.

Next we introduce for maps in $\tilde{\mathfrak{S}}$ an appropriate notion of homotopy.

Definition. Two admissible mappings $\varphi, \psi : X \to Y$ are called homotopic (written $\varphi \sim \psi$) provided there exists an admissible mapping $\chi : X \times I \to Y$, where $I = [0,1]$, such that

$$\chi(x,0) = \varphi(x) \quad \text{and} \quad \chi(x,1) = \psi(x) \quad \text{for each} \quad x \in X .$$

(5.1) *Let $\varphi, \psi : X \to Y$ be two admissible mappings. Then $\varphi \sim \psi$ implies $\varphi_* = \psi_*$.*

Proof. Let $i_0, i_1 : X \to X \times I$ be two embeddings given by $x \to (x,0)$ and $x \to (x,1)$ respectively, and $\chi : X \times I \to Y$ be an admissible homotopy joining φ and ψ . Then

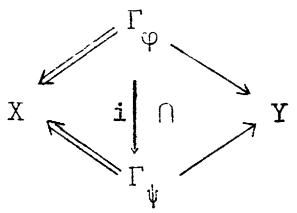
$$\varphi = \chi \circ i_0 \quad \text{and} \quad \psi = \chi \circ i_1 .$$

From this, taking into account that $(i_0)_* = (i_1)_*$, we infer by (4.2) that $\varphi_* = \psi_*$.

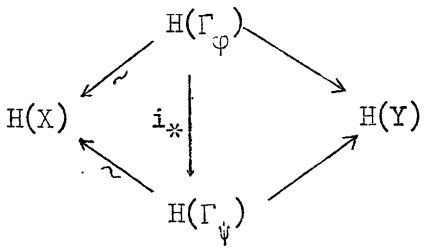
Definition. Let $\varphi, \psi : X \to Y$ be two multi-valued mappings such that $\Gamma_\varphi \subset \Gamma_\psi$, i.e., $\varphi(x) \subset \psi(x)$ for each $x \in X$; in this case, we say that φ is a selector of ψ and indicate this by writing $\varphi \subset \psi$.

(5.2) *Let $\varphi, \psi : X \to Y$ be two admissible mappings. Then $\varphi \subset \psi$ implies $\varphi_* = \psi_*$.*

Proof. Assume that $\varphi \subset \psi$ and note that the diagram



with natural projections is commutative ; consequently, its image under H



also commutes and this shows that $\varphi_* = \psi_*$.

6. Multi-valued Lefschetz maps.

An admissible mapping $\varphi : X \to X$ is said to be a Lefschetz map provided $\varphi_* : H(X) \to H(X)$ is a Leray endomorphism. For such φ we define the Lefschetz number $\Lambda(\varphi)$ of φ by putting $\Lambda(\varphi) = \Lambda(\varphi_*)$.

Note that if X is a compactum of a finite type, then any admissible $\varphi : X \to X$ is a Lefschetz map and $\Lambda(\varphi)$ coincides with the ordinary Lefschetz number $\lambda(\varphi)$ of φ .

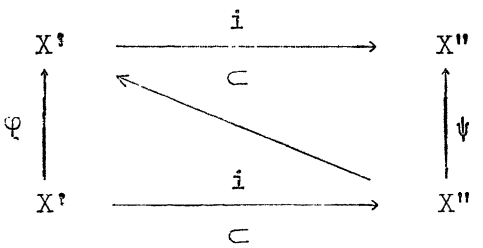
The following two theorems are immediate consequences of (5.1) and (5.2).

(6.1) Let $\varphi, \psi : X \to X$ be two homotopic admissible maps. If φ is a Lefschetz map, then so is ψ and in this case $\Lambda(\varphi) = \Lambda(\psi)$.

(6.2) Let $\varphi, \psi : X \to X$ be two admissible maps such that $\varphi \subset \psi$. If one of them is a Lefschetz map, then so is the other and, in that case, $\Lambda(\varphi) = \Lambda(\psi)$.

We turn now to the property of the Lefschetz maps which will be of importance in the proof of the main theorem.

(6.3) LEMMA. Assume that we are given the following commutative diagram of spaces and admissible multi-valued maps



in which $i : X' \to X''$ stands for the inclusion. Then

(i) if one of the maps φ or ψ is a Lefschetz map, then so is the other and, in that case, $\Lambda(\varphi) = \Lambda(\psi)$;

(ii) φ has a fixed point if and only if ψ does.

Proof. The first assertion clearly follows by applying (2.1), from (I,3,1). The second assertion is evident.

7. The Main Theorem.

The proof of our main result relies essentially on the following simple geometrical fact (comp. [13]) :

(7.1) LEMMA. If U is open in a Banach space E and $X \subset U$ is compact, then there exists a compact absolute neighbourhood retract K such that $X \subset K \subset U$.

Proof. Cover X by a finite number of closed balls $W_1, W_2, \ldots, W_\ell \subset U$, and denote by K_i the convex closure of the compact set $X \cap W_i$. By the Mazur Lemma, every K_i is compact. From the inclusions $K_i \subset W_i \subset U$ we conclude that X is contained in the compact set $K = \cup K_i \subset U$. Now, taking into account the general properties of the ANR spaces [6], we infer that K as the union of a finite number of compact convex sets is an absolute neighbourhood retract and thus our assertion follows.

Before stating our main result we shall prove first the following

THEOREM 1. Let U be open in a Banach space E and $\varphi : U \to U$ be an acyclic compact map. Then (i) φ is a Lefschetz map, and (ii) $\Lambda(\varphi) \neq 0$ implies that φ has a fixed point.

Proof. By assumption, the closure $\overline{\varphi(U)} = X$ is compact and contained in U. By applying to X the proceeding lemma, we find a compact absolute neighbourhood retract K such that $\varphi(U) \subset K \subset U$. Consequently, we have the commutative diagram as in (6.3),

$$\begin{array}{ccc} K & \xrightarrow{i} & U \\ {\scriptstyle \varphi_K}\uparrow & \overset{\varphi'}{\swarrow} & \uparrow{\scriptstyle \varphi} \\ K & \xrightarrow[i]{} & U \end{array}$$

in which i is the inclusion, and φ_K, φ' stand for the obvious contractions of the map φ. Since K is a compact ANR, $\Lambda(\varphi_K)$ is defined ; consequently, by (6.3), φ is a Lefschetz map and $\Lambda(\varphi_K) = \Lambda(\varphi)$.

To prove (ii) assume that $\Lambda(\varphi) \neq 0$. Then we have also $\Lambda(\varphi_K) \neq 0$ and, hence, by the Eilenberg-Montgomery theorem, there exists a point $x \in K$ such that $x \in \varphi_K(x) = \varphi(x)$.

Now we are able to state our principal result in full generality (cf. [21]):

THEOREM 2. Let X be a topologically complete ANR and $\varphi : X \to X$ be a compact acyclic multi-valued map. Then (i) φ is a Lefschetz map and (ii) $\Lambda(\varphi) = 0$ implies that φ has a fixed point.

Proof. Since X is topologically complete we may assume, without loss of generality, that X is a closed subset of a Banach space E. By assumption, there is a retraction $r : U \to X$ of an open set $U \subset E$ onto X. Denoting by $i : X \to U$ the inclusion we have the commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{\ i\ } & U \\ {\scriptstyle\varphi}\uparrow & \nwarrow^{\varphi r} & \uparrow{\scriptstyle\psi = i\varphi r} \\ X & \xrightarrow{\ i\ } & U \end{array}$$

as in (6.3). By assumption, the multi-valued map φ is compact; consequently, so is the map $\psi = i\varphi r$. Theorem 1 implies now that ψ is a Lefschetz map. Applying (6.3), we conclude that φ is also a Lefschetz map.

To prove (ii), assume that $\Lambda(\varphi) \neq 0$. Applying (6.3) again, we have $\Lambda(\varphi) = \Lambda(\psi) \neq 0$. This, in view of Theorem 1, implies that ψ has a fixed point. Applying now (6.3) for the last time we conclude that φ has a fixed point and thus the proof is completed.

Remark. We know that for single-valued maps Theorem 2 is valid without assuming X to be topologically complete; the question whether the same can be proved for multi-valued maps remains open.

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Part B

INFINITE DIMENSIONAL COHOMOLOGY THEORIES

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INFINITE DIMENSIONAL COHOMOLOGY THEORIES

We give first a brief outline of our main results. Let E be an infinite dimensional normed space and $\mathcal{S}$ the corresponding Leray-Schauder category. An "infinite dimensional" or simple a cohomology theory $H^{\infty-*}$ on $\mathcal{S}$ is a sequence of contravariant functors $H^{\infty-n}(X,A)$ from the pairs in $\mathcal{S}$ to the category of abelian groups together with a sequence of natural transformations $\delta^{\infty-n} : H^{\infty-n}(A) \to H^{\infty-n+1}(X,A)$ satisfying the Homotopy, Exactness and Strong Excision axioms ; the graded group $H^{\infty-n-1}(S)$, where S in the unit sphere in E , is the group of coefficients of the theory.

To any (generalized) cohomology theory on the category of finite polyhedra corresponds a cohomology theory on $\mathcal{S}$ with the same group of coefficients ; moreover, the assignment $H^* \to H^{\infty-*}$ is natural with respect to maps of the theories. Thus, in particular, we have the "ordinary" cohomology $H^{\infty-*}(\ ;G)$ over G , the stable cohomotopy $\Sigma^{\infty-*}$ and the Hopf map from $\Sigma^{\infty-*}$ to $H^{\infty-*}(\ ;Z)$.

The "ordinary" cohomology $H^{\infty-n}(X;G)$ is isomorphic to the $(n-1)$-th singular homology group $H_{n-1}(E-X;G)$. A more general result holds in fact, and the second main theorem may be viewed as an extension to the infinite dimensional case of the duality theory due to G. Whitehead [14]. Next a number of consequences follow. Some of them (as the Mayer-Victoris sequence) follow evidently from the axioms alone, while others (as the Alexander-Pontrjagin Invariance or elementary properties of the Leray-Schauder degree) do from both the above results together. The duality combined with the Hurewicz Theorem in S-theory yields to the important Hopf Theorem, relating the ordinary cohomology over Z and the stable cohomotopy on $\mathcal{S}$.

Finally, we discuss briefly the concept of codimension. First, we have for the objects in $\mathcal{S}$ the "basic" codimension Codim defined in terms of the extension problem for compact fields with special ranges $E - E^n$, where $\dim E^n = n$. Our definition coincides in the finite dimensional case with a theorem of P. Alexandroff [1] which characterizes the dimension of compacta by maps into S^n . Further, we define various cohomological codimensions ; we have, in particular, Codim_Z defined in terms of the ordinary cohomology on $\mathcal{S}$ over Z . If the space E is complete, then $\mathrm{Codim} = \mathrm{Codim}_Z$. The proof of this result uses (among others) the representability of the stable cohomotopy on $\mathcal{S}$ and the Homotopy Extension Lemma ; the latter is known to be true in needed generality only under the assumption of completeness[1].

1) Using the Smale-Sard Theorem, K. Geba has extended recently (cf. Fund. Math. 1969) to Banach spaces the framed cobordism theory of L. Pontrjagin. The corresponding bordism groups can be described equivalently as the (suitably defined) homotopy classes of certain σ-proper Fredholm maps. On the other hand, they turn out to be isomorphic to the infinite dimensional stable cohomotopy groups. Thus, the above results of K. Geba provide the important link between the theory of compact vector fields and that of the Fredholm maps. Due to lack of time, however, we shall not be able to give any details.

I. THE LERAY-SCHAUDER CATEGORY

We begin by introducing two categories of primary interest (the category of compact vector fields and the Leray-Schauder category) and two geometrical constructions of further importance (the generalized suspension and the cone functors). With the aid of the generalized suspension, some examples of geometrical interest will be given.

1. Remarks on the notation.

We denote by $E^\infty = (E^\infty, \| \|)$ or simply by E an arbitrary but fixed infinite dimensional linear normed space over the field R. We fix a sequence $\{E^{\infty-n} \oplus E_n\}$ of direct sum decompositions of E such that

(i) $E_0 \subset E_1 \subset \ldots \subset E_n$,

(ii) $E^\infty \supset E^{\infty-1} \supset E^{\infty-2} \supset \ldots \supset E^{\infty-n}$,

(iii) $\operatorname{codim} E^{\infty-n} = \dim E_n = n$.

We let

$$S^{\infty-n} = \{x \in E^{\infty-n+1}; \|x\| = 1\}$$

denote the unit sphere in $E^{\infty-(n-1)}$, $n \geq 1$; and we reserve the symbol $U^{\infty-n}$ for the open set $E^\infty - E^n$.

Next, we let R^∞ be the normed space consisting of all sequences $x = (x_1, x_2, \ldots)$ of real numbers such that $x_i = 0$ for almost all i with the norm

$$\|x\| = \sqrt{\sum_{i=1}^{\infty} x_i^2} \ .$$

The following symbols stand for subsets of R^∞ :

$$R^k = \{x \in R^\infty \ ; \ x_i = 0 \ \text{ for } \ i \geq k+1\} \ ,$$

$$R_+^k = \{x \in R^k \ ; \ x_k \geq 0\} \ ,$$

$$R_-^k = \{x \in R^k \ ; \ x_k \leq 0\}$$

$$S^k = \{x \in R^{k+1} \ ; \ \|x\| = 1\} \ ,$$

$$S_+^k = S^k \cap R_+^{k+1} \ , \quad S_-^k = S^k \cap R_-^{k+1} \ .$$

There are inclusions $R^k \subset R^{k+1}$, $S^k \subset S^{k+1}$ and we have clearly $S^k = S_+^k \cup S_-^k$ and $S^{k-1} = S_+^k \cap S_-^k$.

Finally, we use the following fixed notation :

Ens = the category of sets ;
Ens* = the category of based sets ;
Ab = the category of abelian groups ;
A = either Ab or Ens* ;
$\mathcal{S}(A,B)$ = the set of maps (- morphisms) $f : A \to B$ in a category $\mathcal{S}$.

All standardly used categories are denoted by script letters ; the category of compact vector fields and its subcategories will be denoted by German letters.

2. h-Categories.

An h-category $(\mathcal{S},\sim)$ is a category $\mathcal{S}$ such that for each pair of objects (A and B) in $\mathcal{S}$ there is defined in the set $\mathcal{S}(A,B)$ an equivalence relation $\sim$ (called homotopy) satisfying the following (compositive) property :

$$f_1 \sim f_2, g_1 \sim g_2 \Rightarrow g_1 f_1 \sim g_2 f_2 .$$

If $f \in \mathcal{S}(A,B)$, then by $[f]$ we denote the equivalence (homotopy) class containing f and we let $\pi(A,B)$ be the set of such homotopy classes.

A subcategory $\mathcal{S}_0$ of $\mathcal{S}$ will be called dense provided it has the same objects as $\mathcal{S}$. We say that $(\mathcal{S}_0,\approx)$ is an h-subcategory of an h-category $(\mathcal{S},\sim)$ provided $\mathcal{S}_0 \subset \mathcal{S}$ and the relation $f \approx g$ implies $f \sim g$ for any f and g in $\mathcal{S}_0$.

Remark : In what follows Ens and cA will be considered as h-categories with the relation of homotopy $\sim$ defined by : $f \sim g \Leftrightarrow f = g$.

A map $f : A \to B$ in $(\mathcal{S},\sim)$ is invertible (respectively h-invertible) provided there is a map $f' : B \to A$ such that $f' \circ f = 1_A$ and $f \circ f' = 1_B$ (respectively $f' \circ f \sim 1_A$ and $f \circ f' \sim 1_B$). In the first case, we write $A \sim B$ and call the objects A and B equivalent. In the second case, A and B are said to be homotopically equivalent and we write $A \underset{h}{\sim} B$.

Examples :

1°) The category T (respectively K) of all topological (respectively compact Hausdorff) spaces and all continuous maps with the ordinary relation of homotopy.

2°) Let E be a linear normed space and denote by K_E the full subcategory of K whose objects are compact subsets of E contained in finite dimensional subspaces of E. We say that a polyhedron $K \subset E$ is a geometric subpolyhedron of E if K has a triangulation which is a finite union of geometric simplexes. We denote by P_E the full subcategory of K_E whose objects are geometric subpolyhedra of E and consider K_E ans P_E as h-categories with the ordinary relation of homotopy.

3°) For any concrete h-category D that will appear later on, we shall denote by D^2 the corresponding category of pairs. For example, the objects of T^2 are pairs (X,A) of topological spaces ; the morphisms are continuous maps $f : (X,A) \to (Y,B)$.

A functor $\lambda : \mathfrak{S}_1 \to \mathfrak{S}_2$ between two h-categories is called an h-functor provided it sends homotopy commutative diagrams in $\mathfrak{S}_1$ into such in $\mathfrak{S}_2$.

Clearly if $\lambda : \mathfrak{S}_1 \to \mathfrak{S}_2$ is such a h-functor then

$$X \sim Y \text{ in } \mathfrak{S}_1 \Rightarrow \lambda(X) \sim \lambda(Y) \text{ in } \mathfrak{S}_2 ,$$

and

$$X \underset{h}{\sim} Y \text{ in } \mathfrak{S}_1 \Rightarrow \lambda(X) \underset{h}{\sim} \lambda(Y) \text{ in } \mathfrak{S}_2 .$$

3. The directed set $\mathfrak{L}(E)$.

Let E be an infinite dimensional normed space. By $\mathfrak{L} = \mathfrak{L}(E) = \{L_\alpha, L_\beta, L_\gamma, \dots\}$ we shall denote the family of all finite fimensional linear subspaces of E.

For notational convenience we establish one-to-one correspondence $\alpha \Leftrightarrow L_\alpha$ between the symbols $\alpha, \beta, \gamma, \dots$ and $L_\alpha, L_\beta, L_\gamma, \dots$ and in the formulas to occur we replace occasionally one sort of symbols by another.

We shall write $\alpha \leqq \beta$ if and only if $L_\alpha \subset L_\beta$; evidently, the relation $\leqq$ converts the family $\mathfrak{L}$ into a directed set $(\mathfrak{L}, \leqq)$.

Given an element α of $\mathfrak{L}$ we let $d(\alpha)$ denote the dimension of the linear space L_α . A relation $\alpha \leqq \beta$ in $\mathfrak{L}$ will be called elementary provided $d(\beta) = d(\alpha) + 1$. Given an arbitrary relation $\alpha \leqq \beta$ in $\mathfrak{L}$ by a chain joining α and β we shall understand a finite sequence $\alpha = \alpha_0 \leqq \alpha_1 \leqq \dots \leqq \alpha_k = \beta$ of elements in $\mathfrak{L}$ such that $\alpha_i \leqq \alpha_{i+1}$ is elementary for each $i = 0,1,\dots,k-1$.

If X is a subset of E and $\alpha \in \mathfrak{L}$, we let $X_\alpha = X \cap L_\alpha$. Evidently, the subset $\mathfrak{L}_X$ of $\mathfrak{L}$ defined by

$$\mathfrak{L}_X = \{\alpha \in \mathfrak{L} \, ; \, X_\alpha \text{ is non-empty}\}$$

is cofinal in $\mathfrak{L}$.

If X and Y are two subsets of E and $f : X \to Y$ is a mapping such that $f(X_\alpha) \subset Y_\alpha$ then by $f_\alpha : X_\alpha \to Y_\alpha$ we denote the contraction of f to the pair (X_α, Y_α).

4. Compact and finite dimensional mappings.

In what follows compact mappings will be denoted by the capital letters F, G, H.

(4.1) Definition. Let α be an element of the directed set $\mathfrak{L}$ and $F : X \to E$ be a compact mapping into a normed space E; we say that F is an α-mapping provided F is compact and $F(X) \subset L_\alpha$. If $F : X \to E$ is an α-mapping for some α it is called a finite dimensional mapping.

We state for the reference the following two well-known facts :

(4.2) Lemma (Approximation Lemma). Let U be open in E and $F : X \to U$ be a compact mapping. Then for each $\varepsilon > 0$ there exists a finite polyhedron $P_\varepsilon \subset U$ and a finite dimensional mapping $F_\varepsilon : X \to U$ such that

(i) $F_\varepsilon(X) \subset P_\varepsilon$,

(ii) $\|F(x) - F_\varepsilon(x)\| < \varepsilon$ for each $x \in X$,

(iii) F and F_ε are homotopic.

(4.3) Lemma (On Extension of Compact Mappings). Let A be closed in a metric space X and $F : A \to E$ be a compact mapping. If either (i) is complete or (ii) F is an -mapping, then there is a compact mapping $F : X \to E$ being an extension of F over X and such that $F(X)$ conv$(F(A))$.

Proof. Since the convex hull of a relatively compact set in a complete E is also relatively compact, our assertion follows at once from the Dugundji Extension Theorem.

Remark. It is not known whether a compact mapping $F : A \to E$ admits a compact extension over X, without assuming E to be complete.

5. Compact vector fields.

Notation. Given two subsets X and Y of E and a continuous mapping $f : X \to Y$ we denote by the same but capital letter the mapping $F : X \to E$ defined by

$$F(x) = x - f(x), \qquad x \in X .$$

(5.1) Definition. Let X and Y be arbitrary subsets of E. A mapping $f : X \to Y$ is said to be a compact vector field (or simply a compact field) provided the map $F : X \to E$ is compact.

The set of all compact vector fields with domain X and range Y will be denoted by $\mathfrak{F}(X,Y)$ and its elements will be denoted by the small letters f, g, h, etc.

Some simple but important properties of compact fields are summarized in the following proposition

(5.2) Let $f : X \to E$ be a compact vector field. Then (i) if X is closed (respectively bounded) in E, then so is the set $f(X)$; (ii) if $C \subset E$ is relatively compact, then so is $f^{-1}(C)$.

(5.3) If $f : X \to Y$ is a one-to-one compact vector field of a closed set X onto Y, then f is bicontinuous and $f^{-1} : Y \to X$ is a compact field.

(5.4) The class of compact vector fields has the following properties :

(i) if f and g are compact fields (respectively α-fields) then so is their composition gf ;

(ii) if f is a compact field (respectively an α-field) then so is every contraction, and in particular every restriction, of f ;

(iii) the inclusions $i : A \to X$ and in particular the identities $1_X : X \to X$ are α-fields for every $\alpha \in \mathfrak{A}$;

(iv) if $f : X \to Y$ is a continuous mapping between two subsets of E such that X is compact, then f is a compact field ; if, in addition, X and Y are contained in L_α then f is an α-field.

It follows from (5.4) that subsets of E as objects and compact vector fields as maps for m a category. This category will be denoted by $\mathfrak{F}(E)$ and called the category of compact vector fields in E. For each α we have a dense subcategory $\mathfrak{F}_\alpha(E)$ of $\mathfrak{F}(E)$ whose maps are α-fields between the subsets of E.

Clearly, if $\alpha < \beta$ is a relation in $\mathfrak{A}$ then

$$\mathfrak{F}_\alpha(E) \subset \mathfrak{F}_\beta(E) .$$

Now we define a category

$$\mathfrak{F}_0(E) = \bigcup \mathfrak{F}_\alpha(E)$$

as the union of all categories $\mathfrak{F}_\alpha(E)$ for $\alpha \in \mathfrak{A}$. Evidently, $\mathfrak{F}_0(E)$ is a dense subcategory of $\mathfrak{F}(E)$. In what follows the maps of $\mathfrak{F}_0(E)$ will be called finite dimensional fields.

6. Homotopy of compact vector fields.

Notation. Given two subsets X and Y of E and a homotopy $h_t : X \to Y$ $(0 \leq t \leq 1)$ we shall denote by $h : X \times I \to Y$ the mapping defined for $(x,t) \in X \times I$ by $h(x,t) = h_t(x)$. By the capital H we shall denote the mapping $H : X \times I \to E$ defined for $(x,t) \in X \times I$ by

$$H(x,t) = x - h_t(x) .$$

(6.1) Definition. Let X and Y be two subsets of E. A family of compact vector fields $h_t : X \to Y$ depending on the parameter t $(0 \leq t \leq 1)$ is called a compact homotopy provided the mapping $H : X \times I \to E$ is compact. Two compact vector fields $f,g : X \to Y$ are said to be compactly homotopic, provided there exists a compact homotopy $h_t : X \to Y$ such that $h_0 = f$, $h_1 = g$.

We write $f \sim g$ to mean that the fields f and g are compactly homotopic.

The relation " $\sim$ " is an equivalence relation in each of the sets $\mathfrak{S}(X,Y)$ and it clearly satisfies the compositive property in the definition of an h-category. Consequently, it converts the category of compact vector fields into an h-category $(\mathfrak{S},\sim)$. When there is no risk of misunderstanding this category will be denoted simply by $\mathfrak{S}$.

(6.2) Let $h_t : X \to E$ be a compact homotopy. Then (i) if X is closed (resp. bounded) in E then so is the set $h(X \times I)$; (ii) if $C \subset E$ is relatively compact, then so is the set $h^{-1}(C)$.

(6.3) Let $X \subset E$, U be an open set in E and let $f,g : X \to U$ be two compact fields. If the inequality

$$\|f(x) - g(x)\| \leq \operatorname{dist}(f(x), E - U)$$

holds for each $x \in X$, then the fields f and g are compactly homotopic.

Proof. The above inequality implies that for each $x \in X$ the segment $[f(x),g(x)]$ joining $f(x)$ and $g(x)$ in E is contained in U, hence the formula

$$\begin{aligned} h_t(x) &= tf(x) + (1-t)g(x) \\ &= x - [tF(x) + (1-t)G(x)] \qquad (x \in X,\ t \in I) \end{aligned}$$

defines a required compact homotopy between f and g.

(6.4) Definition. Let X and Y be two subsets of E and α be an element of the directed set $\mathfrak{I}$. A family of α-fields $h_t : X \to Y$ is called an α-homotopy, provided $H : X \times I \to E$ is an α-mapping. Two α-fields $f,g : X \to Y$ are called α-homotopic if there is an α-homotopy $h_t : X \to Y$ such that $h_0 = f$ and $h_1 = g$.

We shall write $f \overset{\sim}{\alpha} g$ to mean that α-fields f and g are α-fields f and g are α-homotopic.

The relation of α-homotopy is an equivalence relation in $\mathfrak{S}_\alpha(X,Y)$ and therefore decomposes the above set into disjoint α-homotopy classes. If $f \in \mathfrak{S}_\alpha(X,Y)$ we let $[f]_\alpha$ denote the α-homotopy class which contains f. The set of these classes will be denoted by $\pi_\alpha(X,Y)$. We note further that the relation $\overset{\sim}{\alpha}$ satisfies the compositive property in the definition of an h-category and consequently it converts $\mathfrak{S}_\alpha$ into an h-subcategory $(\mathfrak{S}_\alpha,\overset{\sim}{\alpha})$ of $(\mathfrak{S},\sim)$.

(6.5) Let $h_t : X \to E$ be a compact homotopy. Then for every $\varepsilon > 0$ there exists an α-homotopy $h'_t : X \to E$ such that

$$\|h'_t(x) - h_t(x)\| < \varepsilon \quad \text{for all} \quad x \in X .$$

Proof. This clearly follows from the Approximation Lemma.

7. The extension problem for compact fields.

Let $h_t : X \to Y$ be a compact homotopy (respectively an α-homotopy) and A be a subset of X. We let $h_t|A = h'_t$ denote the partial compact homotopy (respectively α-homotopy); in this case, we shall write also $h'_t \subset h_t$ and say that h_t is an extension (respectively an α-extension) of h'_t over X.

Given a pair $(X,A) \subset E$ with A clased in X and a field (respectively an α-field) $f : A \to U$ we may consider the extension problem for f, i.e., the problem of extending f over X in $\mathfrak{S}$ (respectively in $\mathfrak{S}_\alpha$).

The following important lemma asserts that under some hypotheses this problem depends only on the homotopy (respectively α-homotopy) class of a given field f.

(7.1) (Homotopy Extension Lemma). Let (X,A) be a pair in E, A be closed in X and U an open set in E. Let $h'_t : A \to U$ $(0 \leq t \leq 1)$ be a compact homotopy such that $h'_0 \subset h_0 \in \mathfrak{S}(X,U)$. If either (i) E is complete or (ii) h_t is an α-homotopy, then there exists a compact homotopy (α-homotopy) $h_t : X \to U$ such that $h'_t \subset h_t$.

Proof. Let us put $T = (X \times \{0\}) \cup (A \times I)$. By the assumption, there is a compact mapping (α-mapping) $H^*_0 : T \to E$ such that

$$H^*_0(x,t) = \begin{cases} H'(x,t) & \text{for } x \in A , \ 0 \leq t \leq 1 \\ H_0(x) & \text{for } x \in X , \ t = 0 \end{cases}$$

and

$$x - H^*_0(x,t) \in U \quad \text{for all} \quad (x,t) \in T .$$

Since T is closed in $X \times I$, there is, in view of Lemma 4.3, a compact extension $H_0 : X \times I \to E$ of H^*_0 over $X \times I$. Putting

$$B = \{x \in E \ ; \ x - H_0(x,t) \in E - U \ \text{ for some } \ t \in I\} ,$$

we may suppose that the closed set B is not empty. We note further that A and B are evidently disjoint. Now take a real-valued function $\lambda : X \to I$ such that $\lambda(B) = 0$ and $\lambda(A) = 1$ and put

$$H(x,t) = H_0(x,\lambda(x)t) \quad , \quad (x \in X \ , \ t \in I)$$

$$h_t(x) = x - H(x,t) \quad , \quad (x \in X \ , \ t \in I) .$$

It is easily seen that $h_t : X \to U$ is a required compact homotopy. The proof is completed.

(7.2) COROLLARY. Let (X,A) be a pair in E with A closed in X and $f_o, g_o : A \to U$ two α-homotopic α-fields. If there exists an α-extension $f : X \to U$ of f_o over X, then there exists also an α-extension g of g_o over X and such that f and g are α-homotopic. If the space E is complete, the above is true for arbitrary compact fields f_o and g_o.

Remark. For some pairs (X,A) in E Lemma (7.1) and its corollary hold without assuming E to be complete. It is not, however, known whether the above lemma is true for arbitrary-closed pairs without the above additional hypothesis.

8. The generalized suspension and the cone functors.

Notation. Given a linear (closed) subspace M of E we let S_M denote the unit sphere in M. We assume that we are given a direct sum decomposition $E = M \oplus N$ where M and N are complementary linear subspaces of E.

(8.1) Definition. Given a subset X of N we let

$$S_M(X) = \{z = tx + (1-t)y \in E; x \in X, y \in S_M, 0 \leq t \leq 1\}$$

be the union of all segments in E joining points x in X with points y in the unit sphere S_M. Given two subsets X and Y of N and a mapping $f : X \to Y$ we let $S_M(f) : S_M(X) \to S_M(Y)$ be the mapping defined for $x \in X$, $y \in S_M$ and $0 \leq t \leq 1$ by

$$S_M(f)(tx + (1-t)y) = tf(x) + (1-t)y .$$

We say that $S_M(X)$ and $S_M(f)$ are the M-suspensions of X and f respectively.

For a linear subspace N of E denote by $\mathfrak{P}(N)$ the category whose objects are subsets of N and whose maps are continuous transformations between the objects.

Note that for any two composable mappings f and g in $\mathfrak{P}(N)$ we have

$$S_M(g \circ f) = S_M(g) \circ S_M(f) .$$

It follows that the assignments $X \to S_M(X)$, $f \to S_M(f)$ define a covariant functor S_M from $\mathfrak{P}(N)$ to $\mathfrak{P}(E)$ and called the generalized M-suspension functor.

(8.2) The M-suspension functor S_M has the following properties :

(i) if X is closed (respectively bounded) in N then so is $S_M(X)$ in E ;

(ii) if f is a compact field (respectively an α-field) then so is $S_M(f)$;

(iii) if the fields (respectively α-fields) f and g are compactly homotopic (respectively α-homotopic) then so are their M-suspensions $S_M(f)$ and $S_M(g)$.

Proof. (i) is evident. To prove (ii), write $F(x) = x - f(x)$ and take a compact set $C \subset N$ such that $F(X) \subset C$. Note that the set C_1 given by

$$C_1 = \{z \in N;\ z = tw\ ,\ 0 \leqq t \leqq 1\ ,\ w \in C\}$$

is also a compact subset of N. Since for an arbitrary point $z = tx + (1-t)y$ of $S_M(X)$ we have

$$z - S_M f(z) = t(x-f(x)) \in C_1\ ,$$

it follows that $S_M(f)$ is a compact field. The proof of (ii) is completed. The proof of other assertions is similar.

(8.3) Let N be a finite dimension and X be a compact subset of N. Let us put $U_o = N-X$ and $U = E-S_M(X)$. Then the inclusion map $i : U_o \to U$ induces an isomorphism

$$i_* : \pi_n(U_o) \to \pi_n(U)$$

of the homotopy groups for all $n < \dim N-1$.

Proof. By assumption, $N = L_{\alpha_o}$ for some $\alpha_o \in \mathfrak{L}$. Let us put $\mathfrak{L}_o = \{\alpha \in \mathfrak{L}, \alpha \geqq \alpha_o\}$. Clearly, $\mathfrak{L}_o$ is a cofinal subset of $\mathfrak{L}$. Now, for any relation $\alpha \leqq \beta$ in $\mathfrak{L}_o$, let

$$i_{\alpha\beta} : U_\alpha \to U_\beta \qquad \text{and} \qquad i_\alpha : U_\alpha \to U$$

denote the inclusions. Consider the corresponding direct system over $\mathfrak{L}_o$ of homotopy groups $\{\pi_n(U_\alpha); (i_{\alpha\beta})_*\}$ and the direct family $\{(i_\alpha)_*\}$ of homomorphisms

$$(i_\alpha)_* : \pi_n(U_\alpha) \to \pi_n(U)\ .$$

It follows from Lemma 4.2 that

$$\operatorname*{Lim}_{\alpha} \{(i_\alpha)_*\} : \operatorname*{Lim}_{\alpha} \{\pi_n(U_\alpha)\ ;\ (i_{\alpha\beta})_*\} \to \pi_n(U)$$

is an isomorphism. On the other hand, if $n < \dim N-1$, then, by finite dimensional argument, it is clear that

$$(i_{\alpha_o\alpha})_* : \pi_n(U_{\alpha_o}) \to \pi_n(U_\alpha)$$

is an isomorphism and our assertion follows.

(8.4) Definition. Let y^+ be a fixed point in S_M. We define the cone functor $C : \mathfrak{P}(N) \to \mathfrak{P}(E)$ by putting for $X \subset N$

$$C(X) = \{z \in E;\ z = tx + (1-t)y^+;\ x \in X\ ,\ 0 \leqq t \leqq 1\}$$

and

$$Cf(z) = tf(x) + (1-t)y^+$$

for any mapping $f : X \to Y$ with $X\ ,\ Y \subset E$.

(8.5) The cone functor C has the following properties :

(i) if X is closed (resp. bounded) in N then so is $C(X)$ in E ;

(ii) if f is a compact field (resp. α-field) then so is Cf ;

(iii) if the fields (resp. α-fields) f and g are compactly homotopic (resp. α-homotopic) then so are Cf and Cg .

9. The Leray-Schauder category $\mathfrak{H}$.

Denote by $\mathfrak{H}(E)$ or simply by $\mathfrak{H}$ the h-subcategory of $\mathfrak{S}(E)$ generated by closed bounded subsets of E , $\mathfrak{H}(E)$ will be called the Leray-Schauder category corresponding to the linear space E.

In what follows, the category $\mathfrak{H}$ being of primary interest, we shall be concerned with such geometrical properties of its objects that remain invariant under the equivalences or homotopy equivalences in $\mathfrak{H}$.

Remark. In all that follows the objects of the Leray-Schauder category $\mathfrak{H}(E)$ will be simply called the objects.

(9.1) There exist two equivalent objects X_1 and X_2 such that $\pi_1(E-X_1) = 0$ and $\pi_1(E-X_2) \neq 0$.

Proof. Let $E = M \oplus N$ be the direct sum decomposition of E such that $\dim N = 3$. Let Y_1 be the unit interval in N and $Y_2 \subset N$ be the Artin-Fox example (cf. [1]) i.e. the set homeomorphic to Y_1 with $\pi_1(N-Y_2) \neq 0$; let $f : Y_1 \to Y_2$ denote the corresponding homeomorphism.

Now we let

$$X_1 = S_M(Y_1) \quad , \quad X_2 = S_M(Y_2) \ ,$$

Clearly, by (8.2),

$$S_M f : X_1 \to X_2$$

is an invertible compact field and thus the objects X_1 and X_2 are equivalent in $\mathfrak{H}(E)$. On the other hand by (8.3) $\pi_1(E-X_2) \approx \pi_1(N-Y_2) \neq 0$ and $\pi_1(E-X_1) = 0$. The proof is completed.

Remark. It can be shown that all the homology groups $H_q(E-X_1)$ and $H_q(E-X_2)$ vanish. By taking, instead of Y_2 , the Alexander horned sphere in N and by repeating the above construction, one obtains two equivalent objects X_1 and X_2 in $\mathfrak{H}(E)$ such that

$$\pi_1(E-X_1) = 0 \quad , \quad \pi_1(E-X_2) \neq 0$$

$$H_0(E-X_1) \approx H_0(E-X_2) \approx Z \ .$$

II. CONTINUOUS FUNCTORS

In what follows our basic constructions depends largely on the continuity property of the functors under consideration. This chapter is devoted to the above property and its main result can be briefly stated as follows : every continuous functor defined on the subcategory $\mathfrak{X}_0$ of $\mathfrak{X}$ admits the unique extension over $\mathfrak{X}$. Throughout the chapter λ_0 stands for a contravariant functor from $\mathfrak{X}_0$ to the category A.

1. Approximating families and the carriers.

Notation. $\mathfrak{X}_0$ being dense in $\mathfrak{X}$, we let for X in $\mathfrak{X}_0$

$$\lambda(X) = \lambda_0(X)$$

and for a field $f : X \to Y$ in $\mathfrak{X}_0$ we denote by

$$f^* = \lambda_0(f) : \lambda(Y) \to \lambda(X)$$

the map in A induced by f.

(1.1) Definition. Let Y be an object. A family $\{Y_k\}$ of objects indexed by a directed set N is said to be an approximative family for Y provided

(i) $Y_n \subset Y_k$ for any relation $k \leqq n$ in N,

(ii) $Y = \bigcap_{k \in N} Y_k$.

In case $N = \{1,2,\ldots\}$ such a family will be refered to as an approximating sequence for Y.

We note the following evident proposition :

(1.2) Let Y be an object and α be an arbitrary element of $\mathfrak{L}_Y$. If $\{Y_k\}_{k \in N}$ is an approximating family for Y, then so is the family $\{Y_k \cap L_\alpha\}_{k \in N}$ for Y_α.

Let Y be an object and $\{Y_k\}_{k \in N}$ an approximating family for Y. Denote by

$$i_{kn} : Y_n \to Y_n \qquad k \leqq n$$

$$j_k : Y \to Y_k \qquad k \in N$$

the corresponding inclusions, all of them being finite dimensional fields.

It is not difficult to see that the objects $\lambda(Y_k)$ together with the maps i^*_{kn} given for every relation $k \leqq n$ in N form a direct system $\{\lambda(Y_k); i^*_{kn}\}$ in A over N and the family $\{j^*_k\}$ of maps

$$j^*_k : \lambda(Y_k) \to \lambda(Y)$$

is a direct family of maps in A.

Consequently, we have the direct limit map in A

$$\varinjlim_{k} \{j_k^*\} : \varinjlim_{k} \{\lambda(Y_k); i_{kn}^*\} \to \lambda(Y)$$

(1.3) Definition. We shall say that a functor $\lambda_o : \mathfrak{o}_o \to A$ is continuous provided for every object Y and an approximating family $\{Y_k\}_{k\in N}$ for Y, the map $\varinjlim_{k} \{j_k^*\}$ is invertible in the category A.

Given a pair of objects (X,A) let $j_{AX} : A \to X$ be the corresponding inclusion.

In the following definition we assume that λ_o is a functor from $\mathfrak{o}_o$ to the category of abelian groups.

(1.4) Definition. Let A be an object, x be a point in A and ξ a non-trivial element of $\lambda(A)$. An object $S_x(\xi) = Y$ contained in A and containing x is called a carrier of ξ (with respect to x) provided $j_{YA}^*(\xi) \neq 0$. A carrier $S_x(\xi)$ of the element ξ is said to be essential (with respect to x) provided for any object $X \subset S_x(\xi)$ containing x we have $j_{XA}^*(\xi) = 0$.

The following lemma expresses an important property of continuous functors :

(1.5) Lemma. Let A be an object and $x\in A$. If the functor $\lambda_o : \mathfrak{o}_o \to Ab$ is continuous, then for any non-trivial element ξ of $\lambda(A)$ there exists at least one essential carrier $S_x(\xi)$ of ξ with respect to the point x.

Proof. For an element $\xi \neq 0$ consider the set $\mathfrak{m}_x$ of all carriers $S_x(\xi) \subset A$ of ξ partially ordered downward by inclusion. If $\{Y_k\}_{k\in N}$ is a totally ordered subset of $\mathfrak{m}_x$ then the intersection $Y = \bigcap_{k\in N} Y_k$ is a non-empty object and $\{Y_k\}_{k\in N}$ is an approximating family for Y. From the continuity of the functor λ_o we infer that Y is a carrier and thus $\{Y_k\}_{k\in N}$ has a lower bound in $\mathfrak{m}_x$. By the Zorn Lemma, the set $\mathfrak{m}_x$ contains a minimal element which is a required essential carrier for ξ.

2. Approximating systems.

Notation. For an object Y and a natural number k we let

$$Y^{(k)} = \{x\in E \ , \ \rho(x,Y) \leq \frac{1}{k}\} .$$

To a sequence $\{Y_k\}$ we assign the enlarged sequence $\{\tilde{Y}_k\}$ by putting

$$\tilde{Y}_k = \{x\in E \ , \ \rho(x,Y_k) \leq \frac{1}{k}\} .$$

We begin with a proposition concerning approximating sequences.

(2.1) Let $\{X_k\}$ and $\{Y_k\}$ be two approximating sequences for X and Y respectively and let $f : X_1 \to E$ be a compact field. Then

(i) $\{X_k \cup Y_k\}$ *is an approximating sequence for* $X \cup Y$;

(ii) $\{\tilde{X}_k\}$ *is an approximating sequence for* X ;

(iii) $f(X_k)$ *is an approximating sequence for* $f(X)$.

Proof. Properties (i) and (ii) are evident. In order to establish (iii) it is sufficient to prove the inclusion

$$\bigcap_{k=1}^{\infty} f(X_k) \subset f\Big(\bigcap_{k=1}^{\infty} X_k\Big) .$$

Let $y \in \bigcap_{k=1}^{\infty} f(X_k)$; we have $y = f(x_k)$, where $x_k \in X_k$ and thus $y = x_k - F(x_k)$. Since F is compact we may assume without loss of generality that $\lim_{k=\infty} x_k = x$. Consequently, $y = \lim_{k=\infty} f(x_k) = f(x)$. Since $x \in \bigcap_{k=1}^{\infty} X_k$, this completes the proof.

(2.2) *Definition.* Let X and Y be two objects and let $f : X \to Y$ be a compact field. A sequence $\{Y_k, f_k\}$ of objects Y_k and α_k-fields $f_k : X \to Y_k$ is said to be an *approximating system* for f , provided

(i) $\{Y_k\}$ is an approximating sequence for Y ;

(ii) $f_k \sim j_k f$ in $\mathcal{J}$ where $j_k : Y \to Y_k$ is the inclusion ;

(iii) $f_k \approx i_{kn} f_n$ in $\mathcal{J}_0$, where $i_{kn} : Y_n \to Y_k$ is the inclusion $(k \leq n)$.

(2.3) *Let* $f : X \to Y$ *be a compact field. Then, for each* k , *there is an* α_k*-field* $f_k : X \to Y_k$, *where* $Y_k = Y^{(k)}$, *such that*

$$\|f(x) - f_k(x)\| < \frac{1}{k} \quad \text{for all} \quad x \in X$$

moreover, $\{Y_k, f_k\}$ *is an approximating system for* f .

In what follows any system $\{Y_k, f_k\}$ as in Proposition (2.3) will be called a *standard* approximating system for f .

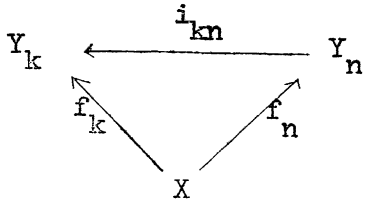
We note also the following evident proposition :

(2.4) *Let* $\{Y_k, f_k\}$ *be an approximating system for a field* $f : X \to Y$ *and* $\{\bar{Y}_k\}$ *be an approximating sequence for* y *such that for each* k *we have* $Y_k \subset \bar{Y}_k$. *Denote by* $\lambda_k : Y_k \to \bar{Y}_k$ *the corresponding inclusion and put* $\bar{f}_k = \lambda_k f_k$. *Then* $\{\bar{Y}_k, \bar{f}_k\}$ *is again an approximating system for* f .

In what follows we assume that $\lambda_0 ; \mathcal{J}_0 \to A$ is a continuous h-functor from $\mathcal{J}_0$ to A .

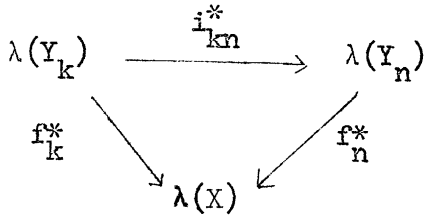
Let X and Y be two objects and $f : X \to Y$ be a compact vector field. Let $\{Y_k, f_k\}$ be an arbitrary approximating system for f .

In view of the definition of an approximating system to an h-commutative diagram in $\mathcal{S}_0$



where $k \leq n$

corresponds the commutative diagram in A



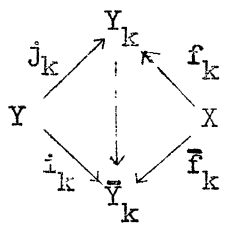
Consequently $\{f^*_k\}$ is a direct sequence of maps and therefore

$$\operatorname{Lim}_{\vec{k}} \{f^*_k\} : \operatorname{Lim}_{\vec{k}} \{\lambda(Y_k) , i^*_{kn}\} \to \lambda(X) .$$

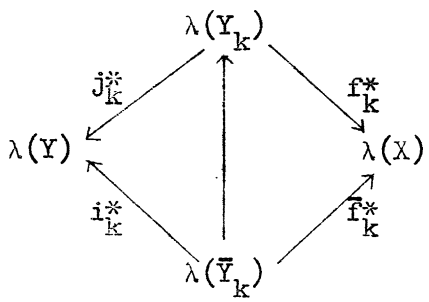
(2.5) Let $\{Y_k, f_k\}$ and $\{\bar{Y}_k, \bar{f}_k\}$ be two approximating systems for f as in (2.4) and let $j_k : Y \to \bar{Y}_k, i_k : Y \to \bar{Y}_k$ denote the corresponding inclusions. Then we have

$$\operatorname{Lim}_{\vec{k}} \{f^*_k\} \, (\operatorname{Lim}_{\vec{k}} \{j^*_k\})^{-1} = \operatorname{Lim}_{\vec{k}} \{f^*_k\} \, (\operatorname{Lim}_{\vec{k}} \{i^*_k\})^{-1} .$$

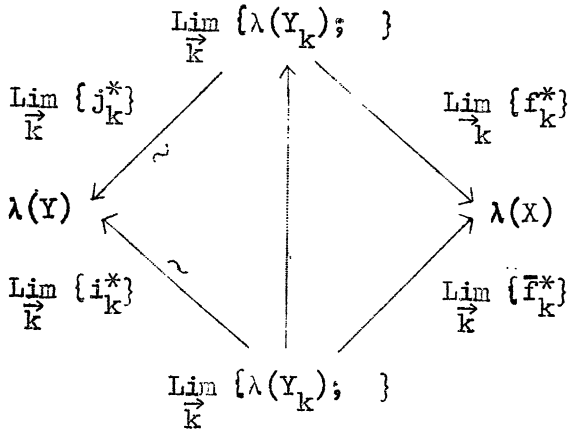
Proof. Since for each k the diagram



is commutative in $\mathcal{S}_0$, it follows that its image under λ_0 in A is also commutative.



By considering the corresponding commutative diagram in the category of direct systems of objects in A and applying to it the direct limit functor we obtain the following commutative diagram



This, in view of the continuity of λ_o, implies our assertion.

(2.6) Let $\{Y_k, f_k\}$, $\{Y_k, \bar{f}_k\}$ be two approximating systems for a field $f : X \to Y$ with the same sequence $\{Y_k\}$. Consider the enlarged sequence $W_k = \{\tilde{Y}_k\}$ denote by $i_k : Y \to W_k$, $\ell_k : Y_k \to W_k$ the corresponding inclusions and put $g_k = \ell_k f_k$, $\bar{g}_k = \ell_k \bar{f}_k$. Then $\{W_k, g_k\}$ and $\{W_k, \bar{g}_k\}$ are again approximating systems for f and we have :

$$\varinjlim_k \{g_k^*\} \quad (\varinjlim_k \{j_k^*\})^{-1} = \varinjlim_k \{\bar{g}_k^*\} \quad (\varinjlim_k \{j_k^*\})^{-1} :$$

Proof. In view of the definition of an approximating system, the fields $f_k, \bar{f}_k : X \to Y_k$ are compactly homotopic for every k. Let $h_t^{(k)} : X \to Y_k$ be a corresponding compact homotopy joining f_k and $\bar{f}_k$. In view of the Approximation Lemma there exists an α_k-homotopy $\bar{h}_t^{(k)} : X \to E$ such that

$$\|h_t^{(k)}(x) - \bar{h}_t^{(k)}(x)\| < \frac{1}{k} \quad \text{for all} \quad (x,t) \in X \times I .$$

Clearly for each point $(x,t) \in X \times I$ we have $\bar{h}_t^{(k)}(x) \in W_k$ and consequently $\bar{h}_t^{(k)}$ may be viewed as an α_k-homotopy $\bar{h}_t^{(k)} : X \to W_k$. Assuming without loss of generality that f_k and $\bar{f}_k$ are α_k-fields, we evidently have $g_k \simeq h_o^{(k)}$, $\bar{g}_k \simeq h_1^{(k)}$ and therefore $g_k \simeq \bar{g}_k$.

This implies $g_k^* = \bar{g}_k^*$ for each k, and the proof is completed.

(2.7) Assume that $\{Y_k, f_k\}$ and $\{\bar{Y}_k, \bar{f}_k\}$ are two arbitrarily given approximating systems for a field $f : X \to Y$. Denote by $j_k : Y \to Y_k$, $i_k : Y \to \bar{Y}_k$ the corresponding inclusions. Then we have

$$\varinjlim_k \{f_k^*\} \circ (\varinjlim_k \{j_k^*\})^{-1} = \varinjlim_k \{\bar{f}_k^*\} \circ (\varinjlim_k \{i_k^*\})^{-1}$$

Proof. Let us put for every positive integer k

$$W_k = (Y_k \cup \bar{Y}_k)^{(k)} .$$

In view of (2.1), $\{W_k\}$ is an approximating sequence for Y. Denote by

$$\ell_k : Y_k \to W_k \quad \text{and} \quad \bar{\ell}_k : \bar{Y}_k \to W_k$$

the corresponding inclusions and define the fields

$$g_k , \bar{g}_k : X \to W_k$$

by putting

$$g_k = \ell_k f_k \quad \text{and} \quad \bar{g}_k = \bar{\ell}_k \bar{f}_k .$$

Note that $\{W_k, g_k\}$ and $\{W_k, \bar{g}_k\}$ are both approximating systems for f. It is clear that the pairs $\{W_k, g_k\}$, $\{Y_k, f_k\}$ and $\{W_k, \bar{g}_k\}$, $\{Y_k, f_k\}$ satisfy the assumptions of (2.5). Now, our assertion follows from (2.6).

3. The Extension Theorem for continuous functors.

(3.1) Definition. Given a compact field $f : X \to Y$, let $\{Y_k, f_k\}$ be an approximatin g system for f and let $j_k : Y \to Y_k$ be the inclusion. We define the induced map:

$$f^* : \lambda(Y) \to \lambda(X)$$

by the following formula :

$$f^* = \operatorname{Lim}_{\vec{k}} \{f_k^*\} \, (\operatorname{Lim}_{\vec{k}} \{j_k^*\})^{-1} ,$$

(3.2) The definition of f^* does not depend on the choice of an approximating system $\{Y_k, f_k\}$ for f.

Proof. This clearly is a reformulation of (2.7).

(3.3) Definition. Define the function $\lambda : \mathcal{Q} \to A$ by putting for X in $\mathcal{Q}$

$$\lambda(f) = f^* .$$

and for a compact field f in $\mathcal{Q}$

$$\lambda(f) = f^* .$$

(3.4) If $f \in \mathcal{Q}_o$, then $\lambda_o(f) = \lambda(f)$. In other words, λ is an extension of the functor λ_o from $\mathcal{Q}_o$ over $\mathcal{Q}$.

Proof. This follows from the definition (3.1) by taking for f an approximating system $\{Y_k, f_k\}$ with $Y_k = Y$ and $f_k = f$ for all $k = 1,2,\ldots$

(3.5) λ is an h-functor from the Leray-Schauder category $\mathcal{S}$ to the category A. In other words, the induced map f^* satisfies the following two properties:

(a) if the fields f and g are compactly homotopic, then $f^* = g^*$;

(b) for any two composable compact fields f and g we have $(gf)^* = f^* \circ g^*$.

Proof of the property (a): Assume that the fields $f,g : X \to Y$ are compactly homotopic and denote by $h_t : X \to Y$ $(0 \leq t \leq 1)$ a compact homotopy such that

$$h_0 = f \quad \text{and} \quad h_1 = g .$$

By the Approximation Lemma there exists an α_k-homotopy $h_t^{(k)} : X \to Y^{(k)}$ $(0 \leq t \leq 1)$ satisfying

$$\|h_t^{(k)}(x) - h_t(x)\| < \frac{1}{k} \quad \text{for all} \quad (x,t) \in X \times I .$$

Let us put $f_k = h_0^{(k)}$ and $g_k = h_1^{(k)}$ for every k. It is easily seen that $\{Y^{(k)}, f_k\}$ and $\{Y^{(k)}, g_k\}$ are approximating systems for f and g respectively. Since for every k the α_k-fields $f_k, g_k : X \to Y^{(k)}$ are homotopic in $\mathcal{S}_0$, it follows that $\varinjlim \{f_k^*\} = \varinjlim \{g_k^*\}$.

Consequently, we have $f^* = g^*$ and the proof is completed.

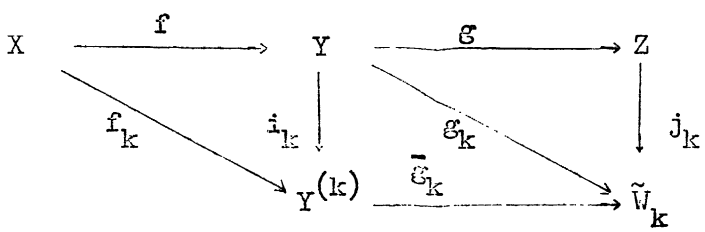
Proof of property (b). (Special case: for finite dimensional g). Given two compact fields $f : X \to Y$ and $g : Y \to Z$, assume that g is an α_0-field and let $h = gf$. We shall prove that $h^* = f^* \circ g^*$.

Take a standard approximating system $\{Y^{(k)}, f_k\}$ for f; Lemma I.4.3 implies that there exists an α_0-field $\bar{g} : Y^{(1)} \to E$ such that g is the contraction of $\bar{g}$ to the pair (Y,Z).

Let us put for every $k = 1,2,\ldots$

$$W_k = g(Y^{(k)}) \cup Z .$$

By (2.1) both W_k and $\tilde{W}_k$ are approximating sequences for Z. Now consider the following diagram



in which i_k and j_k are the inclusions, $\bar{g}_k$ is defined by

$$\bar{g}_k(y) = g(y) \quad \text{for} \quad y \in Y^{(k)}$$

and

$$g_k = \bar{g}_k \circ i_k .$$

Since for every k both f_k and g_k are finite dimensional we may apply to the fields

$$h_k = \bar{g}_k \circ f_k \quad \text{and} \quad g_k = \bar{g}_k \circ i_k$$

the functor λ_0 and therefore we have

$$h_k^* = f_k^* \bar{g}_k^* \quad \text{and} \quad g_k^* = i_k^* \bar{g}_k^* .$$

Consequently,

$$\operatorname*{Lim}_{\vec{k}} \{h_k^*\} = \operatorname*{Lim}_{\vec{k}} \{f_k^*\} \operatorname*{Lim}_{\vec{k}} \{\bar{g}_k^*\} = \operatorname*{Lim}_{\vec{k}} \{f_k^*\} (\operatorname*{Lim}_{\vec{k}} i_k^*)^{-1} \operatorname*{Lim}_{\vec{k}} \{g_k^*\}$$

and thus

$$\operatorname*{Lim}_{\vec{k}} \{h_k^*\} (\operatorname*{Lim}_{k} \{j_k^*\})^{-1} = f^* \operatorname*{Lim}_{\vec{k}} \{g_k^*\} (\operatorname*{Lim}_{k} \{j_k^*\})^{-1} .$$

Further, it is clear that $\{\tilde{W}_k, h_k\}$ and $\{\tilde{W}_k, g_k\}$ are approximating systems for h and g respectively. Therefore, it follows from the last formula and Definition (3.1) that $h^* = f^* \circ g^*$, and the proof is completed.

Proof of the property (b) (general case). Let $f : X \to Y$ and $g : Y \to Z$ be two compact fields and let $h = gf$.

Let $\{Z^{(k)}, h_k\}$ and $\{Z^{(k)}, g_k\}$ be two standard approximating systems for h and g , respectively. From the inequalities

$$\|h_k(x) - h(x)\| < \frac{1}{k} \quad \text{for all} \quad x \in X ,$$

$$\|g_k f(x) - h(x)\| < \frac{1}{k} \quad \text{for all} \quad x \in X ,$$

it follows that for every integer k the fields $g_k f, h_k : X \to Z^{(k)}$ are homotopic in $\mathcal{Y}$. This implies, in view of property (a), that $h_k^* = (g_k f)^*$. Since each g_k is finite dimensional we have, by the proved special case of property (b) ,

$$(g_k f)^* = f^* \circ g_k^*$$

and thus we obtain

$$\operatorname*{Lim}_{\vec{k}} \{h_k^*\} = \operatorname*{Lim}_{\vec{k}} \{f^* g_k\} = f^* \operatorname*{Lim}_{\vec{k}} \{g_k^*\} .$$

This implies $h^* = f^* \circ g^*$ and the proof is completed.

We now summarize the proceeding discussion in the following

(3.6) THEOREM. Let $\lambda_0 : \mathcal{Y}_0 \to A$ be a continuous contravariant h-functor. Then λ_0 can be uniquely extended over $\mathcal{Y}$ to an h-functor $\lambda : \mathcal{Y} \to A$.

Proof. In view of (3.4) and (3.5) it is sufficient to prove only the uniqueness of an extension. But this, in view of the definition of an approximating system, follows clearly from the formula (*). The proof is completed.

III. THE FUNCTOR $H^{\infty-n}$.

Now given a cohomology theory H^* on the category K_E, we shall construct for every n a contravariant h-functor $H^{\infty-n}$ from the Leray-Schauder category $\mathfrak{H}$ to the category of abelian groups. First, we define a functor $H_o^{\infty-n}$ on $\mathfrak{H}_o$. Then, using the continuity of H^* on K_E and an algebraic lemma on "interchanging double limits", we show that the functor $H_o^{\infty-n}$ is continuous and get a unique extension $H^{\infty-n}$ of $H^{\infty-n}$ over $\mathfrak{H}$.

1. Preliminaries on the Mayer-Vietoris homomorphism.

Notation. We denote by $\mathfrak{h}$ any of the following h-categories :

$\mathfrak{H}$ = the Leray-Schauder category on E ;

K_E = the full subcategory of $\mathfrak{H}$ generated by the compact subsets of finite dimensional subspaces of E ;

In what follows, by a triad in $\mathfrak{h}$ we shall understand an ordered triple $T = (X;X_1,X_2)$ of objects in $\mathfrak{h}$ such that $X = X_1 \cup X_2$ and, by a map

$$f : (X;X_1,X_2) \to (Y;Y_1,Y_2)$$

between the triads, a map $f' : X \to Y$ in $\mathfrak{h}$ wich carries X_i into Y_i for $i = 1, 2$.

Let $\mathfrak{h}^2$ be the h-category of pairs in $\mathfrak{h}$ and let $\vartheta : \mathfrak{h}^2 \to \mathfrak{h}^2$ be the covariant functor defined by

$\vartheta(X,A) = A = (A,\theta)$ for any $(X,A) \in \mathfrak{h}^2$,

$\vartheta(f) = f|_A : A \to B$ for any map $f : (X,A) \to (Y,B)$ in $\mathfrak{h}^2$.

(1.1) Definition. A cohomology theory $H^* = \{H^n,\delta^n\}$ on $\mathfrak{h}$ is a sequence of contravariant h-functors

$$H^n : \mathfrak{h}^2 \to Ab \quad (-\infty < n < +\infty)$$

together with a sequence of natural transformations

$$\delta^n : H^n \circ \vartheta \to H^{n+1} \quad (-\infty < n < +\infty)$$

satisfying the following conditions :

(a) (Strong Excision). If $(X;A,B)$ is a triad in $\mathfrak{h}$ and $k : (A,A \cap B) \to (X,B)$ is the inclusion, then $H^n(k) : H^n(X,B) \approx H^n(A,A \cap B)$ for all n.

(b) (Exactness). If (X,A) is a pair in $\mathfrak{h}$ and $i : A \to X$, $j : X \to (X,A)$ are the inclusion maps, then the cohomology sequence

$$\ldots \to H^{n-1}(A) \to H^n(X,A) \xrightarrow{j^*} H^n(X) \xrightarrow{i^*} H^n(A) \to H^{n+1}(X,A) \to \ldots$$

of (X,A) is exact.

(c) (Continuity). Given an approximating family $\{Y_k\}_{k\in N}$ of objects in $\mathfrak{h}$ for $Y = \bigcap Y_k$ we have an isomorphism for each n :

$$H^n(Y) \approx \varinjlim_{k} \{H^n(Y_k), i_{k\ell}^*\} ,$$

where $i_{k\ell} : Y_\ell \to Y_k$, $(k \leq \ell)$ stands for the inclusion.

Convention. Cohomology theories on $\mathfrak{h}$ are denoted by

$$H^{\infty-*} = \{H^{\infty-n}, \delta^{\infty-n}\} ,$$

where $H^{\infty-n}$ and $\delta^{\infty-n}$ play the role of H^{-n} and δ^{-n} in Definition (1.1).

If $H^{\infty-*} = \{H^{\infty-n}, \delta^{\infty-n}\}$ is a cohomology theory on $\mathfrak{h}$, then the graded group $H^{\infty-n}(S)$, where S is the unit sphere in E is called the group of coefficients of the theory $H^{\infty-*}$.

The aim of this and the next chapter is to show that any cohomology theory on K_E gives rise to a geometrically meaningful cohomology theory on $\mathfrak{h}$. The rest of this section is devoted to general remarks on the Mayer-Vietoris homomorphism which are applicable both for cohomology theories on K_E and on $\mathfrak{h}$.

Let H^* be a cohomology theory on $\mathfrak{h}$. Given a triad $(X;X_1,X_2)$ in $\mathfrak{h}$ with $A = X_1 \cap X_2$, denote by

$$j_{o_1} : A \to X_1 \qquad j_{o_2} : A \to X_2$$

$$i_1 : X_1 \to X \qquad i_2 : X_2 \to X$$

$$j_1 : X \to (X,X_1) \qquad j_2 : X \to (X,X_2)$$

the corresponding inclusions.

Definition. The Mayer-Vietoris cohomology sequence of a triad $(X;X_1,X_2)$ with $A = X_1 \cap X_2$ is the sequence of abelian groups

$$\cdots H^{n-1}(A) \xrightarrow{\Delta^n} H^n(X) \xrightarrow{\vartheta} H^n(X_1) \oplus H^n(X_2) \xrightarrow{\psi} H^n(A) \cdots$$

in which ϑ and ψ are given by

$$\vartheta(\xi) = (i_1^*(\xi), i_2^*(\xi)) \quad \text{for} \quad \xi \in H^n(X) ,$$

$$\psi(\xi_1+\xi_2) = j_{o_1}^*(\xi_1) - j_{o_2}^*(\xi_2) \quad \text{for} \quad \xi_i \in H^n(X_i) \quad (i = 1,2)$$

and the Mayer-Vietoris homomorphism Δ^n is defined by

$$\Delta^n = j_1^* \circ (k^*)^{-1} \circ \delta^{n-1}$$

where k is the isomorphism induced by the excision $k : (X_2,A) \to (X,X_1)$. We shall often drop the superscript n on Δ^n, when there is no danger of confusion.

Definition. The cohomology sequence of a triple $B \subset A \subset X$ with inclusions

$$A \xrightarrow{k} (A,B) \xrightarrow{i} (X,B) \xrightarrow{j} (X,A)$$

is the sequence of abelian groups

$$H^{n-1}(A,B) \xrightarrow{\partial^{n-1}} H^n(X,A) \xrightarrow{j^*} H^n(X,B) \xrightarrow{i^*} H^n(A,B)$$

in which the coboundary homomorphism ∂^{n-1} is defined as the composite

$$H^{n-1}(A,B) \xrightarrow{k^*} H^{n-1}(A) \xrightarrow{\delta^{n-1}} H^n(X,A)$$

By purely argument we deduce from the axioms

(1.2) The Mayer-Vietoris sequence of a triad $(X;X_1,X_2)$ is exact. If $f : (X;X_1,X_2) \to (Y;Y_1,Y_2)$ is a map of one triad into another, then f induces a homomorphism of the Mayer-Vietoris sequence of the second triad into that of the first.

(1.3) The cohomology sequence of a triple is exact. If $f : (X,A,B) \to (X',A',B')$ is a map between two triples in $\mathfrak{h}$, then f induces a homomorphism of the cohomology sequence of the second triple into that of the first.

Let $T_0 = (A;A_1,A_2)$ and $T_1 = (X;X_1,X_2)$ be two triads. Then, T_0 is a subtriad of T, written $T_0 \subset T$, provided $A \subset X$ and $A_i \subset X_i$ for $i = 1,2$; T_0 is said to be a proper subtriad of T, written $T_0 \Subset T$, provided $A_i = A \cap X_i$ for $i = 1,2$.

If $T_0 \Subset T$, then clearly $T_0 \subset T$ and $A_0 = A_1 \cap A_2 = A \cap X_0$, where $X_0 = X_1 \cap X_2$; moreover, the inclusions

$$q : (X_0,A_0) \to (X_0 \cap A_2, A_2)$$
$$k : (X_2, X_0 \cap A_2) \to (X, X_1 \cap A)$$

are excisions.

Definition. Given $(A;A_1,A_2) \Subset (X;X_1,X_2)$ we define the relative Mayer-Vietoris homomorphism

$$\Delta^n : H^{n-1}(X_0,A_0) \to H^n(X,A)$$

by

$$\Delta^n = j^* \circ (k^*)^{-1} \circ \delta \circ (q^*)^{-1},$$

where

$$\delta : H^{n-1}(X_0 \cap A_2, A_2) \to H^n(X_2, X_0 \cap A_2)$$

is the coboundary homomorphism of the triple $(X_2; X_0 \cap A_2, A_2)$

and

$$j : (X,A) \to (X, X_1 \cap A)$$

is the inclusion.

The following proposition is an immediate consequence of the definitions involved :

(1.4) To a commutative diagram of triads

$$\begin{array}{ccc} (B;B_1,B_2) & \subset & (Y;Y_1,Y_2) \\ g \uparrow & & \uparrow f \\ (A;A_1,A_2) & \subset & (X;X_1,X_2) \end{array}$$

corresponds the following commutative diagram of abelian groups

$$\begin{array}{ccc} H^{n-1}(Y_o,B_o) & \xrightarrow{\Delta} & H^n(Y,B) \\ g^* \downarrow & & \downarrow f^* \\ H^{n-1}(X_o,A_o) & \xrightarrow{\Delta} & H^n(X,A) \end{array}$$

2. Orientation in E .

We begin by defining an orientation in L_α . To this end, consider the set of all linear isomorphisms from L_α to the euclidean space $R^{d(\alpha)}$ of dimension $d(\alpha)$ (we recall that $d(\alpha) = L_\alpha$

Call two linear isomorphisms $\ell_1, \ell_2 : L_\alpha \to R^{d(\alpha)}$ equivalent, $\ell_1 \sim \ell_2$, provided $\ell_1 \ell_2^{-1} \in GL_+(d(\alpha))$, i.e., the determinant of the corresponding matrix is positive. With respect to this equivalence relation, the set of linear isomorphisms from L_α to $R^{d(\alpha)}$ decomposes into exactly two equivalence classes. An arbitrary choice ω_α of one of these classes will be called an orientation in L_α .

Let us choose now for each $\alpha \in \mathfrak{A}$ an orientation ω_α in L_α and call the family $\omega = \{\omega_\alpha\}$ to be an orientation in E .

Given an elementary relation $\alpha < \beta$ in $\mathfrak{A}$ and $\ell_\alpha \in \omega_\alpha$, there exists $\ell_\beta \in \omega_\beta$ such that $\ell_\beta(x) = \ell_\alpha(x)$ for all $x \in L_\alpha$.

We let

$$L_\beta^+ = \ell_\beta^{-1}(R_+^{d(\beta)}) \quad \text{and} \quad L_\beta^- = \ell_\beta^{-1}(R_-^{d(\beta)}) .$$

Clearly, the definition of L_β^+ and L_β^- depends only on the orientations of L_β and L_α .

As a consequence, given an object X and an elementary relation $\alpha < \beta$ in $\mathcal{S}_X$, the orientations of L_α and L_β determine the triad $(X_\beta; X_\beta^+, X_\beta^-)$ where

$$X_\beta^+ = X \cap L_\beta^+ \qquad \text{and} \qquad X_\beta^- = X \cap L_\beta^-$$

and such that

$$X_\alpha = X_\beta^+ \cap X_\beta^- .$$

(2.1) *Let X and Y be two objects, $f : X \to Y$ be an α_o-field and let $\alpha < \beta$ be an elementary relation in $\mathcal{S}_X$ such that $\alpha_o \leqq \alpha < \beta$. Then $f(X_\beta) \subset Y_\beta$ and $f_\beta : X_\beta \to Y_\beta$ induces a map, also denoted by f_β of the triad $(X_\beta; X_\beta^+, X_\beta^-)$, into the triad $(Y_\beta; Y_\beta^+, Y_\beta^-)$.*

3. Definition of the group $H^{\infty-n}(X)$.

Let $H^* = \{H^q, \delta^q\}$ be a cohomology theory on K_E and X be an arbitrarily-given object of the Leray-Schauder category $\mathcal{S}$. Now starting with H^*, we shall define for an integer n the group $H^{\infty-n}(X)$.

First, we fix an orientation $\omega = \{\omega_\alpha\}$ in the space E. Next, for any relation $\alpha \leqq \beta$ in $\mathcal{S}_X$ we define a homomorphism

$$\gamma_{\alpha\beta}^{(n)} : H^{d(\alpha)-n}(X_\alpha) \to H^{d(\beta)-n}(X_\beta)$$

as follows : if $\alpha = \beta$ we let $\gamma_{\alpha\beta}^{(n)}$ be the identity. If $\alpha < \beta$ is elementary, we let $\gamma_{\alpha\beta}^{(n)}$ be the Mayer-Vietoris homomorphism of the triad $(X_\beta; X_\beta^+, X_\beta^-)$ with $X_\beta^+ \cap X_\beta^- = X_\alpha$.

In order to extend this definition to an arbitrary relation $\alpha < \beta$ in $\mathcal{S}_X$ we shall need the following lemma :

(3.1) LEMMA. *Let X be an object and $\alpha < \beta$ be a relation in $\mathcal{S}_X$ such that $d(\beta) = d(\alpha) + 2$. Assume that $\alpha < \gamma < \beta$ and $\alpha < \tilde{\gamma} < \beta$ are two different chains in $\mathcal{S}_X$ joining α and β. Then*

$$\gamma_{\gamma\beta} \circ \gamma_{\alpha\gamma} = \gamma_{\tilde{\gamma}\beta} \circ \gamma_{\alpha\tilde{\gamma}} .$$

The proof of Lemma 3.1 is given in section 7.

(3.2) *Definition.* Let $\alpha < \beta$ be an arbitrary relation in $\mathcal{S}_X$ and let $\alpha = \alpha_o < \alpha_1 < \ldots < \alpha_{k+1} = \beta$ be a chain of elementary relations in $\mathcal{S}_X$ joining α and β. We define

$$\gamma_{\alpha\beta}^{(n)} = \gamma_{\alpha_k\beta}^{(n)} \circ \ldots \gamma_{\alpha_1\alpha_2}^{(n)} \circ \gamma_{\alpha\alpha_1}^{(n)}$$

as the composition of the corresponding Mayer-Vietoris homomorphisms.

It follows from Lemma (3.1) that the definition of $\gamma_{\alpha\beta}^{(n)}$ does not depend on the choice of the chain $\alpha_1, \ldots, \alpha_k$ joining α and β.

Consider now the abelian groups $H^{d(\alpha)-n}(X_\alpha)$ together with the homomorphisms $\Delta^{(n)}_{\alpha\beta}$ given for each relation $\alpha < \beta$ in $\mathcal{E}_X$. The family $\{H^{d(\alpha)-n}(X_\alpha); \Delta^{(n)}_{\alpha\beta}\}$ indexed by $\mathcal{E}_X$ will be called the $(\infty-n)$-th cohomology system of X corresponding to the theory H^* and the orientation ω in E.

(3.3) $\{H^{d(\alpha)-n}(X_\alpha); \Delta^{(n)}_{\alpha\beta}\}$ is a direct system of abelian groups over $\mathcal{E}_X$.

Proof. This follows clearly from Lemma (3.1).

(3.4) Definition. For an object X we define an abelian group

$$H^{\infty-n}(X) = \varinjlim_{\alpha} \{H^{d(\alpha)-n}(X_\alpha); \Delta^{(n)}_{\alpha\beta}\}$$

to be the direct limit over $\mathcal{E}_X$ of the $(\infty-n)$th cohomology system of X.

Remark. We note that the group $H^{\infty-n}(X)$ depends only up to an isomorphism, on the orientation in E used for its definition. In fact, suppose that $\{\omega_\alpha\}$ and $\{\bar{\omega}_\alpha\}$ are two orientations in E. These determine two direct systems of abelian groups $\{H^{d(\alpha)-n}(X_\alpha), \Delta^{(n)}_{\alpha\beta}\}$ and $\{H^{d(\alpha)-n}(X_\alpha), \bar{\Delta}^{(n)}_{\alpha\beta}\}$ respectively. For each $\alpha \in \mathcal{E}_X$ define $\varphi_\alpha : H^{d(\alpha)-n}(X_\alpha) \to H^{d(\alpha)-n}(X_\alpha)$ by

$$\varphi_\alpha = \begin{cases} 1 & \text{if} \quad \omega_\alpha = \bar{\omega}_\alpha, \\ -1 & \text{if} \quad \omega_\alpha = -\bar{\omega}_\alpha. \end{cases}$$

Since $\{\varphi_\alpha\}$ is clearly an isomorphism of the above direct systems, it follows that the corresponding limit groups are isomorphic.

4. Definition of f^* for finite dimensional field f.

(4.1) Let X and Y be two objects and let $f : X \to Y$ be an α_o-field, where $\alpha_o \in \mathcal{E}_X$. Then, for every relation $\alpha < \beta$ in $\mathcal{E}_X$ such that $\alpha_o < \alpha < \beta$, the following diagram commutes

$$\begin{array}{ccc} H^{d(\alpha)-n}(Y_\alpha) & \xrightarrow{f^*_\alpha} & H^{d(\alpha)-n}(X_\alpha) \\ \downarrow{\scriptstyle \Delta^{(n)}_{\alpha\beta}} & & \downarrow{\scriptstyle \Delta^{(n)}_{\alpha\beta}} \\ H^{d(\beta)-n}(Y_\beta) & \xrightarrow{f^*_\beta} & H^{d(\beta)-n}(X_\beta) \end{array}$$

Proof. If $\alpha < \beta$ is elementary, this follows from (1.2). Our assertion for an arbitrary relation in $\mathcal{E}_X$ follows then from the definition of the homomorphism $\Delta^{(n)}_{\alpha\beta}$.

(4.1) implies that $f : X \to Y$ induces a map

$$\{f_\alpha^*\} : \{H^{d(\alpha)-n}(Y_\alpha);\Delta_{\alpha\beta}^{(n)}\} \to \{H^{d(\alpha)-n}(X_\alpha);\Delta_{\alpha\beta}^{n}\}$$

from the $(\infty-n)$-th cohomology system of Y into that of the object X.

(4.2) Definition. Given a finite dimensional field $f : X \to Y$ we define the induced homomorphism

$$f^* = \varinjlim_\alpha \{f_\alpha^*\} : H^{\infty-n}(Y) \to H^{\infty-n}(X)$$

to be the direct limit over $\mathfrak{L}_X$ of the family $\{f_\alpha^*\}$.

(4.3) The induced homomorphism f^* satisfies the following properties :

(a) if 1 is the identity on X, then 1^* is the identity on $H^{\infty-n}(X)$;

(b) for any two composable finite dimensional fields f and g we have

$$(gf)^* = f^* \circ g^* ;$$

(c) if the finite dimensional fields f and g are homotopic in $\mathfrak{S}_0$ then $f^* = g^*$.

Proof. This clearly follows from (1.4) and the definition of $\Delta_{\alpha\beta}^{(n)}$.

We summarize the preceeding discussion in the following :

(4.4) THEOREM. The assignments $X \to H^{\infty-n}(X)$ for $X \in \mathfrak{S}_0$ and $f \to f^*$ for $f \in \mathfrak{S}_0$ define a contravariant h-functor $H_0^{\infty-n}$ from the h-category $(\mathfrak{S}_0, \simeq)$ to the category of abelian groups.

5. An algebraic lemma.

Given a directed set $\mathfrak{L} = \{\alpha,\beta,\gamma,\ldots\}$, denote by the same letter the category having as objects the elements of $\mathfrak{L}$ and as maps the relations $\alpha \leqq \beta$ in $\mathfrak{L}$. For a small category $\mathfrak{b}$, denote by $(\mathfrak{L},\mathfrak{b})$ the category of covariant functors from $\mathfrak{L}$ to $\mathfrak{b}$, i.e. the category of direct systems of objects of $\mathfrak{b}$ over $\mathfrak{L}$.

By $\varinjlim : (\mathfrak{L},\mathfrak{b}) \to \mathfrak{b}$ we shall denote the "direct limit" functor, i.e. the left-adjoint to the constant functor from $\mathfrak{b}$ to $(\mathfrak{L},\mathfrak{b})$.

Let $\mathscr{N} = \{k,\ell,m,\ldots\}$ and $\mathfrak{L} = \{\alpha,\beta,\gamma,\ldots\}$ be two directed sets. Denote by $\mathfrak{L} \times \mathscr{N}$ the corresponding product category.

Given a direct system of Abelian groups

$$u \in (\mathfrak{L} \times \mathscr{N}, \mathrm{Ab})$$

let us put

$$\varkappa(\alpha,k) = H_\alpha^k ,$$

$$\varkappa(\alpha < \beta,\ k \leqq k) = \Delta_{\alpha\beta}^k$$

$$\varkappa(\alpha,\ k \leqq \ell) = i_{k,\ell}^\alpha .$$

For any relations $k \leqq \ell$ and $\alpha \leqq \beta$ we have the commutative diagram

$$\begin{array}{ccc} H_\alpha^k & \xrightarrow{\quad i_{k\ell}^\alpha \quad} & H_\alpha^\ell \\ \Big\downarrow{\scriptstyle \Delta_{\alpha\beta}^k} & & \Big\downarrow{\scriptstyle \Delta_{\alpha\beta}^\ell} \\ H_\beta^k & \xrightarrow{\quad i_{k\ell}^\alpha \quad} & H_\beta^\ell \end{array} .$$

Clearly, each double system of Abelian groups $\{H_\alpha^k\}$, indexed by $\mathcal{L} \times \mathcal{N}$ together with the maps $\{i_{k\ell}^\alpha\}$, $\{\Delta_{\alpha\beta}^k\}$, (satisfying the natural functorial properties), may be identified with a functor $\varkappa \in (\mathcal{L} \times \mathcal{N}, \mathrm{Ab})$. We shall write simply $\varkappa = \{H_\alpha^k\ ;\ .\}$.

We shall make use of the following algebraic lemma on interchanging double limits.

(5.1) LEMMA. *For any double direct system of Abelian groups* $\varkappa = \{H_\alpha^k, .\}$ *indexed by* $\mathcal{L} \times \mathcal{N}$, *we have a natural isomorphism*

$$\tau_\varkappa : \varinjlim_\alpha \varinjlim_k \{H^k; .\} \to \varinjlim_k \circ \varinjlim_\alpha \{H_\alpha^k; .\}$$

between the limit groups ; more precisely, if

$$\{f_k^\alpha\} : \{H_\alpha^k; .\} \to \{\bar{H}_\alpha^k; .\}$$

is a map between two double direct systems of Abelian groups, then the following diagram commutes :

$$\begin{array}{ccc} \varinjlim_\alpha \varinjlim_k \{H^k; .\} & \xrightarrow{\quad\sim\quad} & \varinjlim_k \varinjlim_\alpha \{H_\alpha^k; .\} \\ \Big\downarrow{\scriptstyle \varinjlim_\alpha \varinjlim_k \{f_k^\alpha\}} & & \Big\downarrow{\scriptstyle \varinjlim_k \varinjlim_\alpha \{f_k^\alpha\}} \\ \varinjlim_\alpha \varinjlim_k \{\bar{H}_\alpha^k; .\} & \xrightarrow{\quad\sim\quad} & \varinjlim_k \varinjlim_\alpha \{\bar{H}_\alpha^k; .\} \end{array}$$

6. Continuity of the functor $H_o^{\infty-n}$.

Now we are prepared to show that the functor $H_o^{\infty-n}$ is continuous. To this end take an object Y and let $\{Y_k\}_{k\in\mathscr{N}}$ be an approximating family for Y. Denote by

$$j_k : Y \to Y_k \ , \ i_{k\ell} : Y_\ell \to Y_k \qquad (k \leqq \ell)$$

the corresponding inclusions and consider the direct system of abelian groups $\{H^{\infty-n}(Y_k) \ , \ i_{k\ell}^*\}$ over $\mathscr{N}$, together with the direct family $\{j_k^*\}$ of homomorphisms

$$j_k^* : H^{\infty-n}(Y_k) \to H^{\infty-n}(Y) \ .$$

(6.1) THEOREM. *The map*

$$\varinjlim_k \{j_k^*\} : \varinjlim_k \{H^{\infty-n}(Y_k) \ ; \ i_{k\ell}^*\} \to H^{\infty-n}(Y)$$

is an isomorphism. In other words, the functor $H_o^{\infty-n} : \mathfrak{H}_o \to \mathrm{Ab}$ *is continuous.*

Proof. For an arbitrary element α in $\mathscr{L}_Y$ and $k\in\mathscr{N}$, let us put

$$Y_\alpha^k = Y_k \cap L_\alpha$$

and denote by

$$j_{k\alpha} : Y_\alpha \to Y_\alpha^k \qquad , \qquad i_{k\ell}^\alpha : Y_\alpha^\ell \to Y_\alpha^k \qquad (k \leqq \ell)$$

the corresponding inclusions.

Now, for any relations $k \leqq \ell$ and $\alpha \leqq \beta$ in $\mathscr{N}$ and $\mathscr{L}_Y$ respectively, consider the following diagram

$$\begin{array}{ccc} H^{d(\alpha)-n}(Y_\alpha^k) & \xrightarrow{(i_{k\ell}^\alpha)^*} & H^{d(\alpha)-n}(Y_\alpha^\ell) \\ \Big\downarrow{\scriptstyle \Delta_{\alpha\beta}^k} & & \Big\downarrow{\scriptstyle \Delta_{\alpha\beta}^\ell} \\ H^{d(\beta)-n}(Y_\beta^k) & \xrightarrow{(i_{k\ell}^\beta)^*} & H^{d(\beta)-n}(Y^\ell) \ . \end{array}$$

It follows from Definition (3.2) that the above diagram is commutative. Consequently, the groups $H^{d(\alpha)-n}(Y_\alpha^k)$, together with the homomorphisms $(i_{k\ell}^\alpha)^*$ and $\Delta_{\alpha\beta}^k$, determine a double direct system of abelian groups over $\mathscr{N}\times\mathscr{L}_Y$ which we denote simply by $\varkappa = \{H^{d(\alpha)-n}(Y_\alpha^k);.\}$.

Let $\bar{\varkappa} = \{H^{d(\alpha)-n}(Y_\alpha);.\}$ be the $(\infty-n)$-cohomology system of Y. We shall treat $\bar{\varkappa}$ as a double direct system over $\mathscr{N}\times\mathscr{L}_Y$.

Now let us consider the double family of homomorphisms $\{j_{k\alpha}^*\}$. Taking into account the various commutativity relations between the inclusions, it follows from Definition (3.2) that $\{j_{k\alpha}^*\}$ is a map from $\varkappa$ to $\bar{\varkappa}$.

In view of the continuity of the cohomology theory $\{H^q, \delta^q\}$, the map $\varinjlim_k \{j^*_{k\alpha}\}$ is an isomorphism for each $\alpha \in \mathcal{L}_Y$, and therefore so is the map $\varinjlim_\alpha \varinjlim_k \{j^*_{k\alpha}\}$. Consequently, in view of Lemma (5.1) the map

$$\varinjlim_k \{j^*_k\} = \varinjlim_k \varinjlim_\alpha \{j^*_{k\alpha}\}$$

is also an isomorphism and the proof of the theorem is completed.

Now, Theorem (6.1), in view of Theorem (II.3.6), gives us the final result of this chapter :

(6.2) THEOREM. The functor $H_0^{\infty-n}$ extends uniquely from $\mathfrak{H}_0$ over $\mathfrak{H}$ to an h-functor $H^{\infty-n} : \mathfrak{H} \to Ab$.

7. Consecutive pairs of triads and Proof of Lemma (3.1).

Notation : The following symbols denote the subsets of R^{k+1} :

$$\begin{aligned}
Q^{k+1} &= \{x \in R^{k+2} \; ; \; x_{k+1} = 0\} , \\
Q^{k+1}_+ &= \{x \in Q^{k+1} \; ; \; x_{k+2} \geqq 0\} , \\
Q^{k+1}_- &= \{x \in Q^{k+1} \; ; \; x_{k+2} \leqq 0\} , \\
P^{k+2}_+ &= \{x \in R^{k+2} \; ; \; x_{k+1} \geqq 0\} , \\
P^{k+2}_- &= \{x \in R^{k+2} \; ; \; x_{k+1} \leqq 0\} .
\end{aligned}$$

The proof of Lemma (3.1) will be preceded by a preliminary discussion about the triads. We assume in this section that $H^* = \{H^q, \delta^q\}$ is a cohomology theory on the category K_E. By a triad we understand an additive triad in K_E.

For a triad $T = (X; X_1, X_2)$ we let $-T = (X; X_2, X_1)$ and denote by $\Delta^n(T)$, or simply by $\Delta(T)$, the Mayer-Vietoris homomorphism :

$$\Delta(T) : H^n(X_1 \cap X_2) \to H^{n+1}(X)$$

of the triad T. We note that

$$\Delta(T) = -\Delta(-T) . \tag{1}$$

Let $T_0 = (Y; Y_1, Y_2)$ and $T = (X; X_1, X_2)$ be two triads. A pair (T_0, T) is a consecutive pair of triads, written $T_0 \Rightarrow T$, provided $Y_1 \cup Y_2 = Y = X_1 \cap X_2$; we say in this case that (T_0, T) starts at $Y_1 \cap Y_2$ and ends at $X_1 \cup X_2$.

We observe that, if (T_0, T) is a consecutive pair of triads, then we may form the composite

$$\Delta(T)\,\Delta(T_0) = \Delta^{n+1}(T) \circ \Delta^n(T_0) : H^n(Y_1 \cap Y_2) \to H^{n+2}(X)$$

of the corresponding Mayer-Vietoris homomorphisms.

(7.1) LEMMA. Let us assume that in the following diagram of triads

$$\begin{array}{ccc} T_o & \Longrightarrow & T \\ \cup\uparrow & & \uparrow\cup \\ T_o' & \Longrightarrow & T' \end{array}$$

the consecutive pairs

$$(T_o,T) = ((Y;Y_1,Y_2),\ (X;X_1,X_2))$$

$$(T_o',T') = ((Y';Y_1',Y_2'),\ (X';X_1',X_2'))$$

both start at $Y_1 \cap Y_2 = Y_1' \cap Y_2'$ and both end at $X = X'$. Then, for the composites of the corresponding Mayer-Vietoris homomorphisms we have

$$\Delta(T) \circ \Delta(T_o) = \Delta(T') \circ \Delta(T_o') .$$

Proof. This is an immediate consequence of (1.4).

(7.2) LEMMA. Let us assume that

$$((Y;Y_1,Y_2),\ (X;X_1,X_2))$$

and

$$(Z;Z_1,Z_2),\ (X;W_1,W_2))$$

are two consecutive pairs of triads both starting at

$$Y \cap Z = Y_1 \cap Y_2 = Z_1 \cap Z_2$$

and both ending at X. Assume further that

$$Z_i = Z \cap X_i \quad \text{and} \quad Y_i = Y \cap W_i \quad \text{for} \quad i = 1,2 .$$

Then, we have

$$\Delta(X;X_1,X_2) \circ \Delta(Y;Y_1,Y_2) = -\Delta(X;W_1,W_2) \circ \Delta(Z;Z_1,Z_2) .$$

Proof. Let us consider the following triads :

$$T_1 = (Y;Y_1,Y_2) , \qquad T_2 = (X;X_1,X_2) ,$$

$$T_3 = ((W_1 \cap X_2) \cup Y_2);W_1 \cap X_2,Y_2) , \qquad T_4 = (X;X_1 \cup W_1,X_2) ,$$

$$T_5 = (Y_2 \cup Z_2;Z_2,Y_2) , \qquad T_6 = (X;X_1 \cup W_1,X_2 \cap W_2),$$

$$T_7 = ((X_1 \cap W_2) \cup Z_2;X_1 \cap W_2,Z_2) , \qquad T_8 = (X;X_1 \cup W_1,W_2) ,$$

$$T_9 = (Z;Z_1,Z_2) , \qquad T_{10} = (X;W_1,W_2) .$$

We claim that every pair (T_{2i-1},T_{2i}) for $i = 1,2,3,4,5$ is a consecutive pair of triads starting at $Y \cap Z$ and ending at X.

For $i = 1$ and $i = 5$, this is true by assumption.

Assume now that $i = 2$. Taking into account the inclusions

$$Y_2 = Y \cap W_2 \subset X_2 \quad \text{and} \quad Y_1 \subset W_1$$

we have respectively

$$(W_1 \cap X_2) \cap Y_2 = W_1 \cap Y_2 = W_1 \cap W_2 \cap Y = Z \cap Y ,$$

$$(X_1 \cup W_1) \cap X_2 = (X_1 \cap X_2) \cup (W_1 \cap X_2) = Y \cup (W_1 \cap X_2)$$
$$= Y_2 \cup (W_1 \cap X_2) ,$$

and thus the statement holds for $i = 2$.

Next, we suppose that $i = 4$. In this case, the proof is strictly analogous to that for $i = 2$.

Assuming finally that $i = 3$, we have

$$Z_2 \cap Y_2 = (Z \cap X_2) \cap (Y \cap W_2) = (Z \cap W_2) \cap (Y \cap X_2) = Z \cap Y ,$$

and

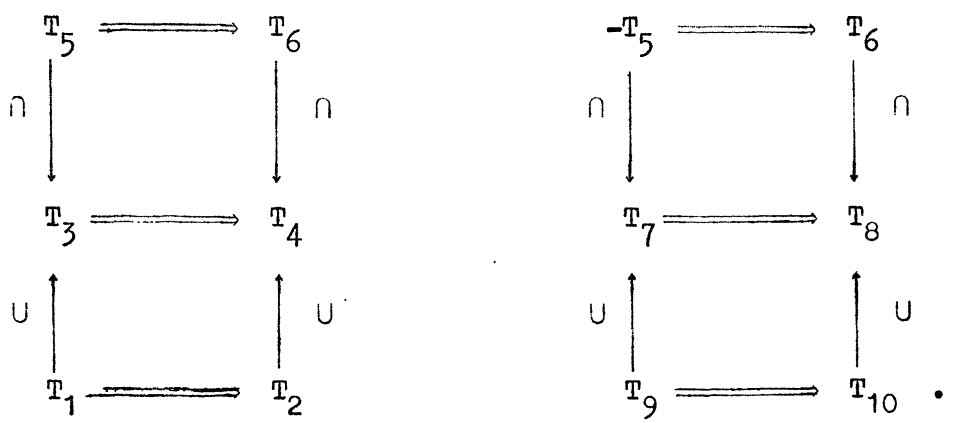
$$(X_1 \cup W_1) \cap (X_2 \cap W_2) = (X_1 \cap X_2 \cap W_2) \cup (W_1 \cap X_2 \cap W_2)$$
$$= (Y \cap W_2) \cup (Z \cap X_2) = Y_2 \cup Z_2 .$$

Thus, the proof of our statement is completed.

Further, we note the following inclusions between the triads

$$T_1, T_5 \subset T_3 \quad , \quad T_2, T_6 \subset T_4 ,$$
$$-T_5, T_9 \subset T_7 \quad , \quad T_6, T_{10} \subset T_8 .$$

The various established interrelations between the triads may be displayed as follows :



Now, we let

$$\Delta_i = \Delta(T_i) \qquad \text{for} \qquad i = 1,2,\ldots,10$$

and apply Lemma (7.1) and property (1) to our situation. We obtain :

$$\Delta_2 \Delta_1 = \Delta_4 \Delta_3 = \Delta_6 \Delta_5 = -\Delta_8 \Delta_7 = -\Delta_{10} \Delta_9$$

and the proof of the lemma is completed.

Proof of Lemma (3.1).

We shall use the notation given at the beginning of this section. Letting $k = d(\alpha)$ we have $d(\gamma) = d(\tilde{\gamma}) = k+1$ and $d(\beta) = k+2$. Define a linear isomorphism $\varphi : R^{k+2} \to R^{k+2}$ by putting

$$\varphi(x_1, \ldots, x_k, x_{k+1}, x_{k+2}) = (x_1, \ldots, x_k, x_{k+2}, x_{k+1}) .$$

Now take linear isomorphisms

$$\ell_\alpha : L_\alpha \to R^k , \quad \ell_\gamma : L_\gamma \to R^{k+1} , \quad \ell_{\tilde{\gamma}} : L_{\tilde{\gamma}} \to R^{k+1}$$

such that

$$\ell_\alpha \in \omega_\alpha , \quad \ell_\gamma \in \omega_\gamma , \quad \ell_{\tilde{\gamma}} \in \omega_{\tilde{\gamma}}$$

and

$$\ell_\gamma(x) = \ell_{\tilde{\gamma}}(x) = \ell_\alpha(x) \quad \text{for all} \quad x \in L_\alpha .$$

There is a unique isomorphism $\ell : L_\beta \to R^{k+2}$ such that

$$\ell(x) = \ell_\gamma(x) \quad \text{for all} \quad x \in L_\gamma$$

$$\ell(x) = \varphi \circ \ell_{\tilde{\gamma}}(x) \quad \text{for all} \quad x \in L_{\tilde{\gamma}} .$$

Consider the following triads

$$T_1 = X \cap \ell^{-1}(R^{k+1}; R^{k+1}_+ , R^{k+1}_-)$$
$$T_2 = X \cap \ell^{-1}(R^{k+2}; R^{k+2}_+ , R^{k+2}_-)$$
$$T_3 = X \cap \ell^{-1}(Q^{k+1}; Q^{k+1}_+ , Q^{k+1}_-)$$
$$T_4 = X \cap \ell^{-1}(R^{k+2}; P^{k+2}_+ , P^{k+2}_-)$$

By straightforward computation, one easily verifies that (T_1, T_2) and (T_3, T_4) are consecutive pairs of triads satisfying the assumption of Lemma (7.2).

We have therefore

$$\Delta(T_2)\, \Delta(T_1) = -\Delta(T_4)\, \Delta(T_3) . \tag{2}$$

There are two alternatives : either $\ell \in \omega_\beta$ or $\ell \in -\omega_\beta$. Now, we shall show that, in any of the above cases, we obtain the desired conclusion

$$\Delta_{\gamma\beta} \circ \Delta_{\alpha\beta} = \Delta_{\tilde{\gamma}\beta}\, \Delta_{\alpha\tilde{\gamma}} . \tag{*}$$

If $\ell \in \omega_\beta$, then $\varphi_o \ell \in -\omega_\beta$ and we have

$$\Delta_{\alpha\gamma} = \Delta(T_1) \quad , \quad \Delta_{\gamma\beta} = \Delta(T_2) \quad ,$$
$$\Delta_{\alpha\tilde{\gamma}} = \Delta(T_3) \quad , \quad \Delta_{\tilde{\gamma}\beta} = -\Delta(T_4) \quad .$$

Thus, in view of (2), we obtain (*).

If $\ell \in -\omega_\beta$, then $\varphi \circ \ell \in \omega_\beta$ and we have

$$\Delta_{\alpha\gamma} = \Delta(T_1) \quad , \quad \Delta_{\gamma\beta} = -\Delta(T_2) \quad ,$$
$$\Delta_{\alpha\tilde{\gamma}} = \Delta(T_3) \quad , \quad \Delta_{\tilde{\gamma}\beta} = \Delta(T_4) \quad .$$

Consequently, again by (2), we get the desired formula (*). The proof of Lemma (3.1) is completed.

IV. COHOMOLOGY THEORIES ON $\mathfrak{H}$

Having defined the absolute cohomology we turn now to the relative case and show that to any cohomology theory H^* on K_E corresponds certain "infinite dimensional" cohomology theory $H^{*-\infty}$ on the Leray-Schauder category $\mathfrak{H}$. More specifically, for every n, we construct the relative cohomology functor $(X,A) \to H^{\infty-n}(X,A)$, the coboundary transformation $\delta^{\infty-n} : H^{\infty-n}(A) \to H^{\infty-n+1}(X,A)$, and then we prove that $H^{\infty-*} = \{H^{\infty-n}, \delta^{\infty-n}\}$ is a cohomology theory on $\mathfrak{H}$ in the sense of Definition III.1.1 .

1. The relative cohomology functor $H^{\infty-n}$.

Notation : $F = E \oplus R$ stands for the direct product of E and the real line R ; we consider E as a 1-codimensional linear subspace of F. We fix a point in F not lying in E by putting $y^+ = (0,1)$, where $0 \in E$, $1 \in R$.

We begin by fixing an orientation $\{\omega_\alpha\}$ in the space E. For technical reasons, we shall consider also an orientation in the space F ; this will be defined in a specified way as follows. Let $\mathfrak{L}_E$ and $\mathfrak{L}_F$ be the directed sets of finite dimensional linear subspaces of E and F, respectively, and $\mathfrak{L}_o$ be a subset of $\mathfrak{L}_F$ consisting of those linear subspaces $\alpha \in \mathfrak{L}_F$ which contain the point y^+ ; clearly, $\mathfrak{L}_o$ is cofinal in $\mathfrak{L}_F$. For $\alpha \in \mathfrak{L}_E$ denote by α' the element of $\mathfrak{L}_o$ given by $L_{\alpha'} = L_\alpha \oplus Ry^+$.

To the orientation $\{\omega_\alpha\}$ in E we assign an orientation $\{\bar{\omega}_\alpha\}$ in F by the following rule : If $\alpha \in \mathfrak{L}_E$, we let $\bar{\omega}_\alpha = \omega_\alpha$. If $\alpha \notin \mathfrak{L}_E$ and $\alpha \notin \mathfrak{L}_o$, we define $\bar{\omega}_\alpha$ arbitrarily. Assuming that $\alpha \in \mathfrak{L}_o$, there is a $\beta \in \mathfrak{L}_E$ such that $\beta' = \alpha$. We take a representative $\ell : L_\beta \to R^k$ in ω_β, where $k = d(\beta)$, and put

$$\ell(x) = (\ell_1(x), \ldots, \ell_k(x)) \qquad \text{for} \qquad x \in L_\beta .$$

Now let $\bar{\ell} : L_\alpha \to R^{k+1}$ be a linear map such that

$$\bar{\ell}(x) = (0, \ell_1(x), \ldots, \ell_k(x)) \qquad \text{for} \qquad x \in L_\beta ,$$

$$\bar{\ell}(y^+) = (1,0,\ldots,0)$$

and let $\bar{\omega}_\alpha$ be the orientation of L_α determined by $\bar{\ell}$. Thus, we have defined an orientation $\{\bar{\omega}_\alpha\}$ in F ; we call $\{\bar{\omega}_\alpha\}$ an extension of $\{\omega_\alpha\}$ from E over F.

From now on, we assume that such an orientation $\{\bar{\omega}_\alpha\}$ in F is fixed.

Next, consider the categories

$$\mathfrak{H}_E = \mathfrak{H}(E) \quad , \quad \mathfrak{H}_F = \mathfrak{H}(F)$$

and observe that $\mathfrak{H}_E$ and $\mathfrak{H}_0(E)$ are h-subcategories of $\mathfrak{H}_F$ and $\mathfrak{H}_0(F)$, respectively. We will denote by $\mathfrak{H}_E^2$ and $\mathfrak{H}_F^2$ the corresponding h-categories of pairs.

In what follows, we shall reduce certain facts in the relative case to those in the absolute case. This will be done with the aid of a functor ρ from $\mathfrak{H}_E^2$ to $\mathfrak{H}_F$ which will now be defined in terms of the cone functor as follows : Let $C : \mathfrak{H}_E \to \mathfrak{H}_F$ be the cone functor corresponding to the point y^+. We recall that for $A \subset E$

$$C(A) = \{x \in F ; x = t\,a + (1-t)y^+ , \quad a \in A , \quad 0 \leq t \leq 1\}$$

and for $f : A \to B$ in $\mathfrak{H}_E$ the field $C(f)$ is given by

$$C(f)(x) = t\,f(a) + (1-t)\,y^* \quad \text{for all} \quad x \in C(A) .$$

Now, given a pair (X,A) in $\mathfrak{H}_E$, let us put

$$\rho(X,A) = \begin{cases} X \cup CA & \text{if } A \neq \emptyset , \\ X & \text{if } A = \emptyset , \end{cases}$$

and for a map $f : (X,A) \to (Y,B)$ in $\mathfrak{H}_E$ define

$$\rho(f) = \tilde{f} : X \cup CA \to Y \cup CB$$

by

$$\tilde{f}(x) = \begin{cases} Cf(x) & \text{for all } x \in CA \\ f(x) & \text{for all } x \in X . \end{cases}$$

(1.1) <u>The assignments</u> $(X,A) \to X \cup CA$ <u>and</u> $f \to \tilde{f}$ <u>define a covariant</u> h-<u>functor</u> ρ <u>from the category</u> $\mathfrak{H}_E^2$ <u>to the category</u> $\mathfrak{H}_F$. <u>Moreover, we have</u> $\rho(\mathfrak{K}_E^2(E)) \subset \mathfrak{K}_0(F)$ <u>and</u> $\rho(K_E^2) \subset K_F$.

Now let H^* be a fixed cohomology theory on K_E and (X,A) be a pair in $\mathfrak{H}_E$. We turn to the definition of the relative groups $H^{\infty-n}(X,A)$. To this end, for an $\alpha \in \mathfrak{L}_E$ such that $X_\alpha \neq \emptyset$, let

$$e_\alpha : (X_\alpha, A_\alpha) \to (X_\alpha \cup CA_\alpha, CA_\alpha)$$

$$j_\alpha : (X_\alpha \cup CA_\alpha) \to (X_\alpha \cup CA_\alpha)$$

denote the corresponding inclusions. Since e_α is an excision and CA_α has all cohomology groups trivial, the induced maps

$$e_\alpha^* : H^n(X_\alpha \cup CA_\alpha, CA_\alpha) \to H^n(X_\alpha, A_\alpha) ,$$

$$j_\alpha^* : H^n(X_\alpha \cup CA_\alpha, CA_\alpha) \to H^n(X_\alpha \cup CA_\alpha)$$

are isomorphisms. Define an isomorphism

$$\eta_\alpha : H^n(X_\alpha, A_\alpha) \to H^n(X_\alpha \cup CA_\alpha)$$

by

$$\eta_\alpha = j_\alpha^* \circ (e_\alpha^*)^{-1} .$$

Let $\alpha < \beta$ be an elementary relation in $\mathcal{L}_E$ and suppose that X_α is non-empty. Then $(A_\beta; A_\beta^+, A_\beta^-)$ is a proper subtriad of $(X_\beta; X_\beta^+, X_\beta^-)$ and we denote the corresponding relative Mayer-Vietoris homomorphism by

$$\Delta_{\alpha\beta}^{(n)} : H^{d(\alpha)-n}(X_\alpha, A_\alpha) \to H^{d(\beta)-n}(X_\beta, A_\beta) .$$

Note that, in this case, $(X \cup CA)_{\alpha'} = X_\alpha \cup CA_\alpha$ and we have the following :

(1.2) Lemma. The diagram

$$\begin{array}{ccc} H^{d(\alpha)-n}(X_\alpha, A_\alpha) & \xrightarrow{\Delta_{\alpha\beta}} & H^{d(\beta)-n}(X_\beta, A_\beta) \\ \downarrow \eta_\alpha & & \downarrow \eta_\beta \\ H^{d(\alpha)-n}(X_\alpha \cup CA_\alpha) & \xrightarrow{\Delta_{\alpha'\beta'}} & H^{d(\beta)-n}(X_\beta \cup CA_\beta) \end{array}$$

in which $\Delta_{\alpha'\beta'}$ is the Mayer-Vietoris homomorphism of the triad

$$((X \cup CA)_{\beta'}; (X \cup CA)_{\beta'}^+, (X \cup CA)_{\beta'}^-)$$

is commutative.

Now let $\alpha \leqq \beta$ be an arbitrary relation in $\mathcal{L}_E$ and $\alpha = \alpha_0 < \alpha_1 < \ldots < \alpha_{k+1} = \beta$ be a chain of elementary relations joining α and β. We define

$$\Delta_{\alpha\beta}^{(n)} = \Delta_{\alpha_k\beta}^{(n)} \circ \Delta_{\alpha_k\alpha_{k-1}}^{(n)} \cdots \Delta_{\alpha\alpha_1}^{(n)}$$

as the composition of the corresponding relative Mayer-Vietoris homomorphisms. In view of Lemma (III.3.1) and Lemma (1.2), this definition does not depend on the choice of a chain. Furthermore, the groups $H^{d(\alpha)-n}(X_\alpha, A_\alpha)$ together with the homomorphisms $\Delta_{\alpha\beta}^{(n)}$ form a direct system of abelian groups over $\mathcal{L}_X$ which we will call the $(\infty-n)$-th cohomology system of the pair (X,A) (corresponding to the theory H^* and the orientation $\{\omega_\alpha\}$ in E).

(1.3) Definition. For an integer n we define the relative cohomology group

$$H^{\infty-n}(X,A) = \varinjlim_\alpha \{H^{d(\ )-n}(X,A); \Delta_{\alpha\beta}^{(n)}\}$$

as the direct limit of the $(\infty-n)$-th cohomology system of the pair (X,A).

Evidently, this definition extends that of the absolute group $H^{\infty-n}(X)$.

Now, for the orientation $\{\bar{\omega}_\alpha\}$ in F, apply the construction of the previous chapter to the space F and denote by $\bar{H}^{\infty-n} : \mathfrak{A}_F \to Ab$ the functor corresponding to H and the orientation $\{\bar{\omega}_\alpha\}$.

Observe that, by Lemma (1.2), the family $\{\eta_\alpha\}$ is a direct family of maps between $\{H^{d(\alpha)-n}(X_\alpha, A_\alpha); \Delta^{(n)}_{\alpha\beta}\}$ and $\{H^{d(\alpha')-n-1}((X \cup CA)_{\alpha'}); \Delta^{(n+1)}_{\alpha'\beta'}\}$. Moreover, since for all α with a sufficiently large dimension $d(\alpha)$ the η_α is an isomorphism, we conclude that the direct limit map

$$\eta = \varinjlim_{\alpha} \{\eta_\alpha\} : H^{\infty-n}(X,A) \to \bar{H}^{\infty-(n+1)}(X \cup CA)$$

is also an isomorphism.

(1.4) Definition. For $f : (X,A) \to (Y,B)$ in $\mathfrak{A}_E$ we define the induced map $f^* = H^{\infty-n}(f)$ by imposing commutativity on the following diagram :

$$\begin{array}{ccc} H^{\infty-n}(Y,B) & \xrightarrow{\quad H^{\infty-n}(f) \quad} & H^{\infty-n}(X,A) \\ \eta \downarrow & & \downarrow \eta \\ \bar{H}^{\infty-n-1}(Y \cup CB) & \xrightarrow{\quad \bar{H}^{\infty-n-1}(f) \quad} & \bar{H}^{\infty-n-1}(X \cup CA) \end{array} .$$

Thus, $H^{\infty-n}(f) = \eta^{-1} \circ \bar{H}^{\infty-n-1}(f) \circ \eta$.

(1.5) The assignments $(X,A) \to H^{\infty-n}(X,A)$ and $f \to f^*$ define a contravariant h-functor $H^{\infty-n}$ from the category $\mathfrak{A}^2_E$ to the category of abelian groups. Moreover, η is a natural equivalence between the functors $H^{\infty-n}$ and $\bar{H}^{\infty-(n+1)} \circ \rho$.

Proof. (1.5) follows clearly from the definitions involved and Theorem III.6.2.

2. The homomorphism δ^n_α.

Next some lemmas which will be used in defining the coboundary transformation $\delta^{\infty-n}$. For an object A in $\mathfrak{L}_E$ we let

$$\mathfrak{L}_A = \{\alpha \in \mathfrak{L}_E ; A_\alpha \neq \emptyset\} .$$

(2.1) Definition. For a pair (X,A) in $\mathfrak{A}^2_E$ and $\alpha \in \mathfrak{L}_A$ define the homomorphism

$$\delta^n_\alpha : H^{d(\alpha)-n-1}(A_\alpha) \to H^{d(\alpha)-n}(X_\alpha, A_\alpha)$$

by putting

$$\delta^n_\alpha = (-1)^{d(\alpha)} \delta_{(X_\alpha, A_\alpha)}$$

where $\delta_{(X_\alpha, A_\alpha)}$ is the coboundary homomorphism of the pair (X_α, A_α).

(2.2) Lemma. Let (X,A) be a pair in $\mathfrak{D}_E$. Then, for every relation $\alpha < \beta$ in $\mathcal{S}_A$ the following diagram commutes:

$$\begin{array}{ccc} H^{d(\alpha)-n-1}(A_\alpha) & \xrightarrow{\delta^n_\alpha} & H^{d(\alpha)-n}(X_\alpha, A_\alpha) \\ \downarrow{\scriptstyle \Delta^{n+1}_{\alpha\beta}} & & \downarrow{\scriptstyle \Delta^n_{\alpha\beta}} \\ H^{d(\beta)-n-1}(A_\beta) & \xrightarrow{\delta^n_\beta} & H^{d(\beta)-n}(X_\beta, A_\beta)\,. \end{array}$$

Proof. Assume first that $\alpha < \beta$ is elementary. Let

$$\tilde{\delta}^n_\alpha : H^{d(\alpha)-n-1}(A_\alpha) \to H^{d(\alpha)-n}(X_\alpha \cup CA_\alpha) = H^{d(\alpha')-n-1}(X \cup CA_{\alpha'})$$

denote the Mayer-Vietoris homomorphism of the triad $(X \cup CA_\alpha, CA_\alpha, X_\alpha)$. Evidently, we have

$$\tilde{\delta}^n_\alpha = (-1)^{d(\alpha)} \circ \eta_\alpha \circ \delta^n_\alpha\,.$$

Next, we observe that the consecutive pairs of triads

$$(X_\alpha \cup CA_\alpha; CA_\alpha, X_\alpha)\,, \quad ((X \cup CA)_{\beta'}; (X \cup CA)^+_{\beta'}, (X \cup CA)^-_{\beta'})$$

$$(A_\beta; A^+_\beta, A^-_\beta)\,, \quad (X_\beta \cup CA_\beta; CA_\beta, X_\beta)$$

satisfy the assumptions of Lemma III.7.2. Consequently,

$$\Delta^n_{\alpha'\beta'} \circ \tilde{\delta}^n_\alpha = -\tilde{\delta}^n_\beta \circ \Delta^{n+1}_{\alpha\beta}\,.$$

Now we consider the diagram:

$$\begin{array}{ccccc} H^{d(\alpha)-n-1}(A_\alpha) & \xrightarrow{\tilde{\delta}^n_\alpha} & H^{d(\alpha)-n}(X_\alpha \cup CA_\alpha) & \xrightarrow{\eta_\alpha^{-1}} & H^{d(\alpha)-n}(X_\alpha, A_\alpha) \\ \downarrow{\scriptstyle \Delta^{n+1}_{\alpha\beta}} & & \downarrow{\scriptstyle \Delta^n_{\alpha'\beta'}} & & \downarrow{\scriptstyle \Delta^n_{\alpha\beta}} \\ H^{d(\beta)-n-1}(A_\beta) & \xrightarrow{\tilde{\delta}^n_\beta} & H^{d(\beta)-n}(X_\beta \cup CA_\beta) & \xrightarrow{\eta_\beta^{-1}} & H^{d(\beta)-n}(X_\alpha, A_\alpha)\,. \end{array}$$

The composition of the top row homomorphisms equals $(-1)^{d(\alpha)}\delta^n_\alpha$ and the composition of the bottom row homomorphisms equals $(-1)^{d(\beta)}\delta^n_\beta$. Since the left-hand square is anticommutative and the right-hand square is, by Lemma (1.2), commutative, the assertion follows. This, in turn, implies that $\Delta^n_{\alpha\beta} \circ \delta^n_\alpha = \delta^n_\beta \circ \Delta^{n+1}_{\alpha'\beta}$, for any $\alpha \leqq \beta$ and the proof is completed.

The following two propositions are immediate consequences of the definition of δ^n_α.

(2.3) Let (X,A) be a pair in $\mathfrak{S}_E^2$ and let $i : A \to X$, $j : X \to (X,A)$ denote the inclusion maps. Then, for every α in $\mathcal{L}_A$, the following sequence is exact:

$$\cdots \xrightarrow{j^*_\alpha} H^{d(\alpha)-n}(X_\alpha) \xrightarrow{i^*_\alpha} H^{d(\alpha)-n}(A_\alpha) \xrightarrow{\delta^{n+1}_\alpha} H^{d(\alpha)-n+1}(X_\alpha, A_\alpha) \xrightarrow{j^*_\alpha} \cdots$$

(2.4) Let (X,A) and (Y,B) be two pairs in $\mathfrak{S}_E^2$ and let $f : (X,A) \to (Y,B)$ be an α_0-field. Then, for any α in $\mathcal{L}_A$ such that $\alpha_0 < \alpha$, the following diagram commutes

$$\begin{array}{ccc} H^{d(\alpha)-n-1}(A) & \xrightarrow{\delta^n_\alpha} & H^{d(\alpha)-n}(X_\alpha, A_\alpha) \\ \Big\downarrow{\scriptstyle (f_\alpha|_A)^*} & & \Big\downarrow{\scriptstyle (f_\alpha)^*} \\ H^{d(\alpha)-n-1} & \xrightarrow{\delta^n_\alpha} & H^{d(\alpha)-n}(Y_\alpha, B_\alpha) \end{array}$$

3. Definition of the co-boundary transformation $\delta^{\infty-n}$.

Let (X,A) be a pair in $\mathfrak{S}_E$. It follows from Lemma (2.2) that the family $\{\delta^n_\alpha\}$ is a direct family of homomorphisms. The coboundary homomorphism

$$\delta^{\infty-n}_{(X,A)} : H^{\infty-n-1}(A) \to H^{\infty-n}(X,A)$$

is defined by

$$\delta^{\infty-n}_{(X,A)} = \varinjlim \{\delta^n_\alpha\} .$$

Similarly, we let

$$\tilde{\delta}^{\infty-n}_{(X,A)} = \varinjlim_{\alpha} \{(-1)^{d(\alpha)}\tilde{\delta}^n_\alpha\} : H^{\infty-n-1}(A) \to \bar{H}^{\infty-n-1}(X \cup CA) .$$

(3.1) The following diagrams commutes:

$$\begin{array}{ccc} H^{\infty-n-1}(A) & \xrightarrow{\delta^{\infty-n}} & H^{\infty-n}(X,A) \\ & \searrow^{\tilde{\delta}^{\infty-n}} & \Big\downarrow{\scriptstyle \wr\ \eta} \\ & & \bar{H}^{\infty-n-1}(X \cup CA) \end{array}$$

Proof. This clearly follows from the definitions involved.

(3.2) The family $\delta^{\infty-n} = \{\delta^{-n}_{(X,A)}\}$ indexed by the pairs (X,A) in $\mathfrak{S}_E$ is a natural transformation from $H^{\infty-n-1} \circ \partial$ to $H^{\infty-n}$.

Proof. In view of (3.1), it suffices to prove that, for $f : (X,A) \to (Y,B)$ in $\mathfrak{S}_E$, the following diagram is commutative

$$\begin{array}{ccc} H^{\infty-n-1}(B) & \xrightarrow{\tilde{\delta}^{\infty-n}} & H^{\infty-n-1}(Y \cup CB) \\ \Big\downarrow (f|_A)^* & & \Big\downarrow \tilde{f}^* \\ H^{\infty-n-1}(A) & \xrightarrow{\tilde{\delta}^{\infty-n}} & H^{\infty-n-1}(X \cup CA) \end{array}$$

Assume first that the field f is finite dimensional. In this special case, we apply a straightforward passage to the limit in the commutative diagram of (2.2) and (2.4) and the desired conclusion follows by (3.1).

Consider now the general case and take an approximating system

$$f^{(k)} : X,A) \to (Y_k,B_k)$$

for f. The definition and the proof of the existence of such a system is similar to that in the absolute case. It follows from (1.1) that the sequence

$$\tilde{f}^{(k)} : X \cup CA \to Y_k \cup CB_k$$

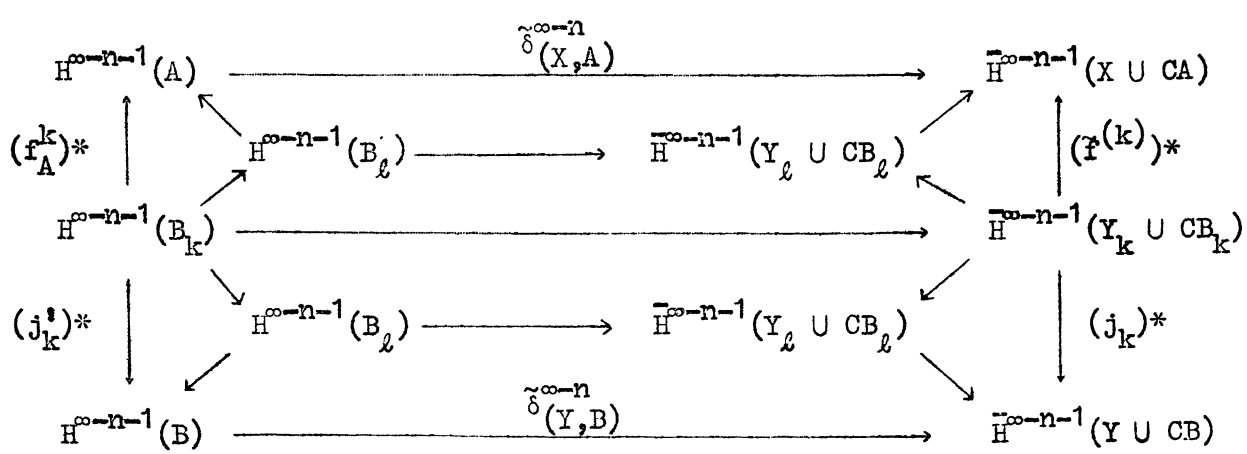
forms an approximating system for $\tilde{f} : X \cup CA \to Y \cup CB$.

Consider the inclusions

$$j_k : Y \cup CB \to Y_k \cup CB_k \ , \ j_k' : B \to B_k$$

$$i_{k\ell} : Y_\ell \cup CB_\ell \to Y_k \cup CB_k \ , \ i_{k\ell}' : B_\ell \to B_k \quad (k \leq \ell) .$$

By the special case of our assertion, the following diagram commutes for each pair $k \leq \ell$



Applying the direct limit functor to the corresponding commutative diagram in the category of direct systems of abelian groups, we obtain the following commutative diagram :

$$\begin{array}{ccc}
H^{\infty-n-1}(A) & \xrightarrow{\tilde{\delta}^{\infty-n}(X,A)} & \bar{H}^{\infty-n-1}(X \cup CA) \\
\Big\uparrow \operatorname{Lim}_{\vec{k}}(f_A^{(k)})^* & & \Big\uparrow \operatorname{Lim}_{\vec{k}}(\bar{f}^{(k)})^* \\
\operatorname{Lim}_{\vec{k}} \{H^{\infty-n-1}(B_k);.\} & \longrightarrow & \operatorname{Lim}_{\vec{k}} \{\bar{H}^{\infty-n-1}(Y_k \cup CB_k);.\} \\
\Big\downarrow \operatorname{Lim}_{\vec{k}} \{(j_k')^*\} & & \Big\downarrow \operatorname{Lim}_{\vec{k}} \{j_k^*\} \\
H^{\infty-n-1}(B) & \xrightarrow{\tilde{\delta}^{\infty-n}(Y,B)} & \bar{H}^{\infty-n-1}(Y \cup CB) \ .
\end{array}$$

By Theorem (III.6.1) the homomorphisms $\operatorname{Lim}_{\vec{k}} \{(j_k')^*\}$ and $\operatorname{Lim}_{\vec{k}} \{j_k^*\}$ are invertible. This, in view of (3.1) and the definition of the induced map, implies our assertion and thus the proof is completed.

(3.3) THEOREM. $H^{\infty-*} = \{H^{\infty-n}, \delta^{\infty-n}\}$ *is a cohomology theory on* $\mathfrak{H}$. *Moreover for each* n *we have*

$$H^{\infty-n-1}(S) \approx H^n \text{ (point)}$$

i.e. the coefficients of the theory $H^{\infty-*}$ *coincide with those of the theory* H^*.

Proof. The Exactness Axiom follows from (2.3) and the definition of $H^{\infty-n}$ by passing to the limit with α.

To show that the Excision Axiom is fullfilled, let $(X;A,B)$ be a triad in $\mathfrak{H}$ with $A \cup B = X$; if $k : (A, A \cap B) \to (X,B)$ is the inclusion then so is $\tilde{k} : A \cup C(A \cap B) \to X \cup CB$. Since $(\tilde{k}_\alpha)^*$ is an isomorphism for each α and $(\tilde{k})^* = \operatorname{Lim}_{\vec{\alpha}} \{\tilde{k}_\alpha^*\}$, it follows that $H^{\infty-n}(k)$ is an isomorphism.

To show that the last assertion of Theorem (3.3) is satisfied, take the $(\infty-n-1)$-cohomology system

$$\{H^{d(\alpha)-n-1}(S_\alpha) \ ; \ \Delta_{\alpha\beta}\}$$

of the unit sphere S in E ; note that, if $\alpha < \beta$ is elementary, then $\Delta_{\alpha\beta}$ coincides with the suspension isomorphism. Consequently, for sufficiently large α, we have

$$H^{\infty-n-1}(S) \approx H^{d(\alpha)-n-1}(S_\alpha)$$

and our assertion follows.

V. GENERALIZED COHOMOLOGY THEORIES AND DUALITY

Assume that the starting point of our discussion in Chapter III is not a cohomology theory H^* but the stable cohomotopy Σ^* on K_E. Then, by applying previous constructions to this case, we are led to the infinite dimensional stable cohomotopy theory $\Sigma^{\infty-*}$ on $\mathfrak{H}$. It turns out that (more generally) to every spectrum A (in the sense of G. Whitehead [14]) corresponds a (generalized) cohomology theory $\{H^{\infty-n}(\ ;A)\}$ on $\mathfrak{H}$. This being established, our main concern is the Alexander type of duality in infinite dimensional normed space E:

$$D : H^{\infty-n}(X,Y;A) \approx H_{n-1}(E-Y,E-X;A) ,$$

where $H_{n-1}(\ ;A)$ is the generalized homology with coefficients in A. Although the above duality holds for an arbitrary spectrum A, we shall confine ourselves to the case when A is either the Eilenberg-MacLane spectrum $K(\Pi)$ (the Alexander-Pontriagin duality) or the spectrum of spheres S (the Spanier-Whitehead duality).

1. Generalized homology and cohomology theories and the spectra.

Notation :

P = the category of finite polyhedra ;

T = the category of all topological spaces ;

W = the category of CW complexes ;

D = any of the above categories ;

D^* = the category of pointed objects in D ;

D^2 = the category of pairs in D ;

S and Σ stand for the suspension and the reduced suspension functors.

Let $\vartheta : D^2 \to D^2$ be the covariant functor defined by

$\vartheta(X,A) = A \ (= (A,\emptyset)$ for any (X,A) in D^2,

$\vartheta(f) = f'_A : A \to B$ for any map $f : (X,A) \to (Y,B)$ in D^2 .

A generalized homology theory H_* on D is a sequence of covariant h-functors $H_n : D^2 \to Ab$ $(-\infty < n < \infty)$ together with a sequence of natural transformations $\partial_n : H_n \to H_{n-1} \circ \vartheta$ satisfying the Excision and Exactness axioms for homology. The graded group $\{H_n(p_o)\}$, where p_o is a point, is called the group of coefficients of the theory H_*.

Similarly, a generalized cohomology theory H^* on D is a sequence of contravariant h-functors $H^n : D^2 \to Ab$ $(-\infty < n < \infty)$ together with a sequence of natural transformations $\delta_n : H^{n-1} \circ \vartheta \to H^n$ satisfying the analogous Excision and Exactness axioms for cohomology ; the graded group $\{H^n(p_o)\}$ is the group of coefficients of the theory H^*.

Thus, a generalized homology theory (respectively cohomology theory) satisfies the Eilenberg-Steenrod axioms, except for the Dimensions axiom. The important examples of generalized theories are provided by the stable homotopy and cohomotopy.

Various generalized homology and cohomology theories can be treated in a unified manner within the framework of homotopy theory with the aid of spectra.

A spectrum A is a sequence $\{A_n\}$ of objects of W^* together with a sequence of maps $\alpha_n : \Sigma A_n \to A_{n+1}$ in W^*. If $A = \{A_n, \alpha_n\}$, $B = \{B_n, \beta_n\}$ are spectra, a map $f : A \to B$ is a sequence of maps $f_n : A_n \to B_n$ in W such that the diagrams

$$\begin{array}{ccc} \Sigma A_n & \xrightarrow{\alpha_n} & A_{n+1} \\ {\scriptstyle \Sigma f_n}\downarrow & & \downarrow{\scriptstyle f_{n+1}} \\ \Sigma B_n & \xrightarrow{\beta_n} & B_{n+1} \end{array}$$

are homotopy commutative. Two such maps f and g are homotopic if and only if, for each n, f_n is homotopic to g_n. Clearly, the spectra form an h-category.

The simplest examples are provided by the spectrum of spheres $S = \{S^n, \sigma_n\}$ in which $\sigma_n : \Sigma S^n \to S^{n+1}$ is the natural identification and by the Eilenberg-MacLane spectrum $K(\Pi)$ defined for an abelian group Π. An important example of maps of spectra is provided by the Hopf-Hurewicz map $h : S \to K(Z)$.

In what follows we shall consider various homology and cohomology theories on various categories with coefficients in a spectrum.

We recall first some basic facts due to G. Whitehead [14] :

(1.1) For any spectrum A there is on P (or more generally on W) a homology $H_*(\ ;A)$ and a cohomology $H^*(\ ,A)$ with coefficients in A [1].

(1.2) $H_*(\ ,A)$ and $H^*(\ ,A)$ are functors of the second variable ; thus, given a map $f : A \to B$ of spectra, we have natural transformations.

$$f_* : H_*(\ ;A) \to H_*(\ ;B)$$

$$f^* : H^*(\ ;A) \to H^*(\ ;B)$$

between corresponding theories.

(1.3) If $A = K(\Pi)$, then the corresponding homology and cohomology theories are naturally isomorphic to the ordinary singular homology and cohomology with coefficient group Π . The homology and cohomology theory with coefficients in S are isomorphic to the stable homotopy and cohomotopy theory, respectively.

(1.4) If $h : S \to K(Z)$ is the Hopf-Hurezicz map then

$$h_* : H_*(\ ;S) \to H_*(\ ;K(Z))$$

is the Hurewicz map (from the stable homotopy to the singular theory over Z) and

$$h^* : H^*(\ ;S) \to H^*(\ ;K(Z))$$

is the Hopf map (from the stable cohomotopy to the singular cohomology over Z).

---

1) We recall briefly how $H^*(\ ;A)$ is defined. First, for $X \in P$ we define the reduced cohomology $\tilde{H}^n(X;A)$ with coefficients in A . We take $f : \Sigma^k X \to A_{n+k}$ representing an element $\gamma \in \pi(\Sigma^k X, A_{n+k})$; ($\pi(\Sigma^k X, A_{n+k})$ is an abelian group for $k \geq 2$). Then the composition

$$\Sigma^{k+1}X \xrightarrow{\Sigma f} \Sigma A_{n+k} \xrightarrow{\alpha_{n+k}} A_{n+k+1}$$

represents an element $\lambda_k(\gamma) \in \pi(\Sigma^{k+1}X, A_{n+k+1})$. The assignment $\gamma \to \lambda_k(\gamma)$ defines a homomorphism $\lambda_k : \pi(\Sigma^k X, A_{n+k}) \to \pi(\Sigma^{k+1}X, A_{n+k+1})$,

and we put

$$\tilde{H}^n(X;A) = \varinjlim_k \{\pi(\Sigma^k X, A_{n+k}) ; \lambda_k\} .$$

Now, for a polyhedral pair (Y, Y_0) we let

$$H^n(Y, Y_0; A) = \tilde{H}^n(Y/Y_0; A) .$$

For further development, we shall need an appropriate extension of the cohomology theory $H(\ ;A)$ on P over the category K_E. This is done by means of the Čech limiting process and we obtain the following :

(1.5) To every spectrum A corresponds a theory $H^*(\ ;A)$ on K_E called the cohomology theory with coefficients in A. Every such theory is continuous and satisfies the strong excision axiom. Moreover, the assignment $A \to H^*(\ ;A)$ is natural with respect to maps of spectra.

On the other hand, in the treatment of duality in the infinite dimensional case, we shall need our homology groups to be defined for open subsets of E. We must have therefore an appropriate extension of $H(\ ;A)$ over T.

(1.6) To every spectrum A corresponds a homology theory $H_*(\ ;A)$ on T (which extends $H_*(\ ;A)$ on P over T) and satisfies the compact carriers axiom. Moreover, the assignment $A \to H_*(\ ;A)$ is natural with respect to maps of spectra.

Let (V,U) be a pair of open subsets in E. Given a relation $\alpha \leqq \beta$ in $\mathfrak{L}$ denote by

$$i_\alpha : (V_\alpha, U_\alpha) \to (V,U)$$
$$i_{\alpha\beta} : (V_\alpha, U_\alpha) \to (V_\beta, U_\beta)$$

the corresponding inclusion maps. Let $H_n(\ ;A)$ be a homology theory on T as in (1.6) and consider the direct system $\{H_n(V_\alpha,U_\alpha;A)\ ;\ (i_{\alpha\beta})_*\}$ together with the direct family of homomorphisms $\{(i_\alpha)_*\}$.

From the Approximation Theorem one deduces the following

(1.7) Lemma. The map

$$\varinjlim_\alpha \{(i_\alpha)_*\} : \varinjlim_\alpha \{H_n(V_\alpha,U_\alpha;A)\ ,\ (i_{\alpha\beta})_*\} \to H_n(U,V)$$

is an isomorphism.

2. Cohomology theory $\{H^{\infty-n}(\ ,A)\}$.

The following result is a generalization of Theorem IV.3.3.

(2.1.) Theorem. To every spectrum A there corresponds a cohomology theory $H^{\infty-*}(\ ,A)$ on $\mathfrak{P}$ with the same group of coefficients as $H^*(\ ;A)$ on P. The theory $H^{\infty-*}(\ ;A)$ is continuous and satisfies the strong excision axiom. Moreover, the assignment $A \to H^{\infty-*}(\ ;A)$ is natural with respect to maps of spectra.

When $A = S$, the proof (which uses (1.5)) is strictly analagous to that of IV.3.3. For the proof for an arbitrary A we refer to [5].

We let

$$\Sigma^{\infty-*} = \{\Sigma^{\infty-n}\} = H^{\infty-*}(\ ;S)$$

$$H^{\infty-*}(\ ;\Pi) = H^{\infty-*}(\ ;K(\Pi))$$

and call $\Sigma^{\infty-*}$ the <u>stable cohomotopy</u> on $\mathfrak{H}$.

The Hopf-Hurewicz map $h : S \to K(Z)$ induces a natural transformation h^* from $\Sigma^{\infty-*}$ to $H^{\infty-*}(\ ;Z)$. More precisely, for any field $f : (X,A) \to (Y,B)$ in $\mathfrak{H}^2$ the diagrams

$$\begin{array}{ccc} \Sigma^{\infty-n-1}(A) & \xrightarrow{\delta} & \Sigma^{\infty-n}(X,A) \\ \downarrow{\scriptstyle h^*} & & \downarrow{\scriptstyle h^*} \\ H^{\infty-n-1}(A;Z) & \xrightarrow{\delta} & H^{\infty-n}(X,A;Z) \end{array}$$

and

$$\begin{array}{ccc} \Sigma^{\infty-n}(Y,B) & \xrightarrow{\Sigma^{\infty-n}(f)} & \Sigma^{\infty-n}(X,A) \\ \downarrow{\scriptstyle h^*} & & \downarrow{\scriptstyle h^*} \\ H^{\infty-n}(Y,B;Z) & \xrightarrow{H^{\infty-n}(f)} & H^{\infty-n}(X,A;Z) \end{array}$$

are commutative. The map h^* will be called the <u>Hopf-map</u> from $\Sigma^{\infty-*}$ to $H^{\infty-*}(\ ;Z)$.

3. <u>Duality in</u> S^n <u>for polyhedra</u>.

Next we indicate the consecutive steps of the proof of the duality in E. We begin by recalling the definition and basic properties of the Alexander duality isomorphism D_n for polyhedra in S^n; these were established by G. Whitehead with the aid of the theory of products for arbitrary spectra.

For our purposes it will be of importance to specify D_n (by selecting for each n an orientation of S^n) and to exhibit an appropriate relation between D_n and D_{n+1} in terms of the Mayer-Vietoris homomorphism.

Assume that we are given a spectrum A and let $(S,A) \to A$ be the natural pairing. We choose a generator $z_1 \in H_1(S^1;S) \approx H_1(S^1;Z)$ and define inductively

$$z_n = (-1)^n \Delta z_{n-1} \in H_n(S^n;S) \approx H_n(S^n;Z) .$$

For a pair (L,M) of polyhedra in S^n (in some triangulation) we denote (M^*,L^*) its dual pair and define the Alexander duality map.

$$D_n : H^q(L,M;A) \to H_{n-q}(M^*,L^*;A)$$

by putting $D_n(w) = z_n \cap w$ for $w \in H^q(L,M;A)$, where the cap product

$\cap : H_n(S^n;S) \otimes H^q(L,M;A) \to H_{n-q}(M^*,L^*;A)$ corresponds to the natural pairing $(S,A) \to A$ of spectra.

We have the following important properties [14]

(3.1.) The duality map D_n is an isomorphism.

(3.2.) Let $N \subset M \subset L$ be subcomplexes of a triangulation of S^n.

Then the diagram

$$\begin{array}{ccccccccc}
\cdots \to & H^q(L,M;A) & \to & H^q(L,N;A) & \to & H^q(M,N;A) & \to & H^{q+1}(L,M;A) & \cdots \\
& \downarrow D_n & & \downarrow D_n & & \downarrow D_n & & \downarrow D_n & \\
\cdots \to & H_{n-q}(M^*,L^*;A) & \to & H_{n-q}(N^*,L^*;A) & \to & H_{n-q}(N^*,M^*;A) & \to & H_{n-q-1}(M^*,L^*;A) & \cdots
\end{array}$$

in which the upper row is the cohomology sequence of the triple (L,M,N) and the lower row is the homology sequence of the triple (N^*,M^*,L^*) , has two left-hand squares commutative and the third commutative up to the sign $(-1)^{n+1}$.

(3.3) Let $f : A \to B$ be a map of spectra. Then the following diagram commutes :

$$\begin{array}{ccc}
H^q(L,M;A) & \xrightarrow{H^q(L,M;f)} & H^q(L,M;B) \\
\downarrow D_n & & \downarrow D_n \\
H_{n-q}(M^*,L^*;A) & \xrightarrow{H_{n-q}(M^*,L^*;f)} & H_{n-q}(M^*,L^*;B)
\end{array}$$

Now let $M \subset L$ be subcomplexes of some triangulation of S^{n+1} and let (M^*,L^*) denote the corresponding dual pair in S^{n+1}. Putting $M_o = M \cap S^n$ and $L_o = L \cap S^n$, let (M_o^*,L_o^*) be the corresponding dual pair in S^n. Denote by Δ the relative Mayer-Vietoris homomorphism corresponding to the proper inclusion of the triads.

$$(M, C_+S^n \cap M, C_-S^n \cap M) \subset (L, C_+S^n \cap L, C_-S^n \cap L)$$

and let $i : (M_o^*,L_o^*) \to (M^*,L^*)$ be the inclusion.

Now, the lemma which is the main tool in extending the Alexander type of duality to the infinite dimensional case.

(3.4) Lemma. The following diagram is commutative :

$$\begin{array}{ccc} H^q(L_o,M_o;A) & \xrightarrow{\Delta} & H^{q+1}(L,M;A) \\ \downarrow D_n & & \downarrow D_{n+1} \\ H_{n-q}(M^*_o,L^*_o;A) & \xrightarrow{i^*} & H_{n-q}(M^*,L^*;A) \end{array}$$

For the proof of (3.4) (based on the results in [14]) we refer to [5].

4. Duality in S^n for compacta

Let A be a spectrum $H^* = \{H^n(\ ;A)\}$ the continuous cohomology theory on K_R^∞ and $H_* = \{H_n(\ ;A)\}$ the homology theory with compact supports on T.

Now let $Y \subset X$ be a pair of compact subsets of S^n. Let $\{M_k\}$, $\{L_k\}$, $M_k \supset L_k$ be approximative sequences for Y and X, respectively, consisting of subcomplexes of triangulations of S^n. Let $i_k : (L_{k+1},M_{k+1}) \to (L_k,M_k)$ denote the inclusion.

Then, by continuity of M^*, we have

$$H^q(X,Y;A) \approx \varinjlim\{H^q(L_k,M_k;A) \ ; \ i_k^*\}$$

Without any loss of generality, we may assume that

$$M_k^* \subset M_{k+1}^*, L_k^* \subset L_{k+1}^*, \bigcup_{k=1}^{\infty} M_k^* = S^n - Y, \bigcup_{k=1}^{\infty} L_k^* = S^n - X .$$

Let $j_k : (M_k,L_k) \to (M_{k+1},L_{k+1})$ be the inclusion.

Since H_* has compact supports,

$$H_m(S^n-Y,S^n-X;A) \approx \varinjlim\{H_m(M_k,L_k;A);(j_k)_*\} .$$

By the straightforward passage to the limit, D_n extends uniquely to an isomorphism (still denoted by D_n)

$$D_n : H^q(X,Y;A) \to H_{n-q}(S^n-Y, S^n-X;A) \quad .$$

defined for all compact subsets $Y \subset X$ of S^n.

Now as a consequence of (3.2) and (3.3) we obtain

(4.1) The duality isomorphism D_n satisfies properties similar to those in (3.2) and (3.3).

Let (X,A) be a pair of compacta in S^{n+1} and $(X_o,A_o) = (X \cap S^n, A \cap S^n)$. Denote by Δ the relative Mayer-Vietoris homomorphism corresponding to the proper inclusion of the triads

$$(A, A \cap E_+^{n+1}, A \cap E^{n+1}) \subset (X, X \cap E_+^{n+1}, X \cap E_-^{n+1})$$

and by $i : (S^n-A_o, S^n-X_o) \to (S^{n+1}-A, S^{n+1}-X)$ the inclusion.

(4.2) The following diagram is commutative :

$$\begin{array}{ccc} H^q(X_o,A_o;A) & \xrightarrow{\Delta} & H^{q+1}(X,A;A) \\ \downarrow{\scriptstyle D_n} & & \downarrow{\scriptstyle D_{n+1}} \\ H_{n-q}(S^n-A_o, S^n-X;A) & \xrightarrow{i_*} & H_{n-q}(S^{n+1}-A, S^{n+1}-X;A) \end{array}$$

5. Duality in R^n.

Notation : We choose a sequence $\{\omega_n\}$ of continuous maps $\omega_n : R^n \to S^n$ with the following properties :

(i) $\omega_{n+1}(x) = \omega_n(x)$ for all $x \in R^n$,

(ii) ω_n maps R^n homeomorphically onto S^n-q,

where $q = (-1,0,\ldots,0) \in S^n \subset R^{n+1}$,

(iii) $\omega_n(R_+^n) \subset C_+S^{n-1}$, $\omega_n(R_-^n) \subset C_-S^{n-1}$.

If X is a subset of R^n we let $\omega_X : X \to S^n$ denote the map defined by

$\omega_X(x) = \omega_n(x)$.

From now on we assume that A is either the Eilenberg-MacLane spectrum $K(\Pi)$ or the spectrum S of spheres. We note that, in either case, the homomorphism

$$(\omega_{R^n-X})_* : H_q(R^n-X;A) \to H_q(S^n-\omega_X(X))$$

is an isomorphism for all $q \leqq n-2$.

Assume that (X,Y) is a pair of compacta in R^n and $q \leqq n-2$. We define the duality map

$$D_n : H^{n-q}(X,Y;A) \to H_q(R^n-Y,R^n-X;A)$$

by putting

$$D_n = (\omega_{R^n-X})_*^{-1} \circ D_n \circ (\omega_X^*)^{-1} .$$

(5.1) <u>The duality map D_n is an isomorphism and whenever defined satisfies properties similar to those in the previous section</u>.

6. <u>Duality isomorphism</u> D_α

We pass now to the infinite dimensional case. By (X,Y,Z) we denote a triple in $\mathfrak{H}$ and we let $U = E-X$, $V = E-Y$, $W = E-Z$.

Let $\{\mathcal{O}_\alpha\}$ be a fixed orientation in E . For each $\alpha \in \mathcal{O}_\alpha$ let $\mathfrak{L}_\alpha : L_\alpha \to R^{d(\alpha)}$ be in $\mathcal{O}_\alpha$. If $X \subset E$ we let $X'_\alpha = \mathfrak{L}_\alpha(X_\alpha) \subset R^{d(\alpha)}$ and we denote by the same letter $\mathfrak{L}_\alpha$ the homeomorphism from X_α onto X'_α given by $x \to \mathfrak{L}_\alpha(x)$.

Now with the aid of $\mathfrak{L}_\alpha$, we "transfer" the duality map (defined in the previous section) from $R^{d(\alpha)}$ to L_α .

(6.1) <u>Definition</u>. Assume that $\alpha \in \mathfrak{L}$ and $d(\alpha) \geqq n+2$.
We define the duality isomorphism

$$D_\alpha : H^{d(\alpha)-n}(X_\alpha,Y_\alpha;A) \to H_n(V_\alpha,U_\alpha;A)$$

by imposing the commutativity on the diagram

$$\begin{array}{ccc} H^{d(\alpha)-n}(X_\alpha,Y_\alpha;A) & \xrightarrow{\mathfrak{L}_\alpha^*} & H^{d(\alpha)-n}(X'_\alpha,Y'_\alpha;A) \\ \Big\downarrow D_\alpha & & \Big\downarrow D_{d(\alpha)} \\ H_n(V_\alpha,U_\alpha;A) & \xrightarrow{(\mathfrak{L}_\alpha)_*} & H_n(V'_\alpha,U'_\alpha;A) \end{array}$$

Clearly D_α depends only on the orientations σ_α of L_α.

From (5.1) we obtain

(6.2) Let $\alpha\in\mathcal{C}$ and $d(\alpha)\geq n+2$. Then the diagram

$$\begin{array}{ccccccc}
H^{d(\alpha)-n}(X_\alpha,Y_\alpha;A) & \to & H^{d(\alpha)-n}(X_\alpha,Z_\alpha;A) & \to & H^{d(\alpha)-n}(Y_\alpha,Z_\alpha;A) & \to & H^{d(\alpha)-n+1}(X_\alpha,Y_\alpha;A) \\
\downarrow D_\alpha & & \downarrow D_\alpha & & \downarrow D_\alpha & & \downarrow D_\alpha \\
H_n(V_\alpha,U_\alpha;A) & \to & H_n(W_\alpha,V_\alpha;A) & \to & H_n(W_\alpha,V_\alpha;A) & \to & H_{n-1}(V_\alpha,U_\alpha;A)
\end{array}$$

in which the upper row is part of the cohomology sequence of $(X_\alpha,Y_\alpha,Z_\alpha)$ and the lower row is a part of the homology sequence of $(W_\alpha,V_\alpha,U_\alpha)$ has two left-hand squares commutative and the third square commutative up to $(-1)^{d(\alpha)+1}$.

(6.3) If $\alpha\leqq\beta$ is a relation in $\mathcal{C}$ with $d(\alpha)\geq n+2$ then the following diagram commutes

$$\begin{array}{ccc}
H^{d(\alpha)-n}(X_\alpha,Y_\alpha;A) & \xrightarrow{\Delta_{\alpha\beta}} & H^{d(\beta)-n}(X_\beta,Y_\beta;A) \\
\downarrow D_\alpha & & \downarrow D_\beta \\
H_n(V_\alpha,U_\alpha;A) & \xrightarrow{(i_{\alpha\beta})_*} & H_n(V_\beta,U_\beta;A)
\end{array}$$

Proof. If $\alpha\leqq\beta$ is elementary, this follows from the property of the duality map in R^n, which is analogous to that in (4.2). The assertion in the general case follows then from the definition of $\Delta_{\alpha\beta}$.

(6.4) If $\alpha\in\mathcal{C}$ and $d(\alpha)\geq n+2$ then the following diagram commutes

$$\begin{array}{ccc}
\Sigma^{d(\alpha)-n}(X_\alpha,Y_\alpha) & \xrightarrow{h^*} & H^{d(\alpha)-n}(X_\alpha,Y_\alpha;Z) \\
\downarrow D_\alpha & & \downarrow D_\alpha \\
\Sigma_n(V_\alpha,U_\alpha) & \xrightarrow{h^*} & H_n(V_\alpha,U_\alpha;Z)
\end{array}$$

7. The duality in E and the Hopf Theorem

By passing to the limit in the diagram of (6.3) we get an isomorphism

$$\operatorname*{Lim}_{\alpha} \{D_\alpha\} : H^{\infty-n}(X,Y;A) \longrightarrow \operatorname*{Lim}_{\alpha} \{H_n(V_\alpha,U_\alpha;A) \ ; \ (i_{\alpha\beta})_*\} \ .$$

Now we define the duality map

$$D : H^{\infty-n}(X,Y;A) \ \to \ H_n(V,U;A)$$

by putting

$$D = \operatorname*{Lim}_{\alpha} \{(i_\alpha)_*\} \ \circ \ \operatorname*{Lim}_{\alpha} \{D_\alpha\}$$

where

$$\operatorname*{Lim}_{\alpha} \{(i_\alpha)_*\} \ : \ \{H_n(V_\alpha,U_\alpha;A) \ , \ (i_{\alpha\beta})_*\} \longrightarrow H_n(V,U;A)$$

Now we are ready to state the main result (cf. [14], [10], [13]) .

(7.1) Theorem. (The Alexander Duality in E). Let A be either the spectrum of spheres S or the Eilenberg-MacLane spectrum $K(\Pi)$. Then

(i) the duality map $D : H^{\infty-n}(X,Y;A) \longrightarrow H_n(V,U;A)$ is an isomorphism ;

(ii) D maps the sequence of a triple $(X,Y,Z) \in \mathfrak{H}$ into the homology sequence of the complementary triple (W,V,U) , i.e., the following diagram commutes :

$$\begin{array}{ccccccccccc}
\cdots & \to & H^{\infty-n}(X,Y;A) & \to & H^{\infty-n}(X,Z;A) & \to & H^{\infty-n}(Y,Z;A) & \xrightarrow{\delta} & H^{\infty-n+1}(X,Y;A) & \to & \cdots \\
 & & \downarrow D & & \downarrow D & & \downarrow D & & \downarrow D & & \\
 & & H_n(V,U;A) & \to & H_n(W,U;A) & \to & H_n(W,V;A) & \xrightarrow{\delta} & H_{n-1}(U,V;A) & \to & \cdots
\end{array}$$

(iii) D is natural with respect to the Hopf-Hurewicz map of spectra $h : S \to K(Z)$, i.e., the following diagram commutes :

$$\begin{array}{ccc}
\Sigma^{\infty-n}(X,Y) & \xrightarrow{h_*} & H^{\infty-n}(X,Y;Z) \\
\downarrow D & & \downarrow D \\
\Sigma_n(V,U) & \xrightarrow{h_*} & H_n(V,U;Z)
\end{array}$$

Proof. (i) follows, in view of the definition of D , from Lemma (1.7) and (iii) is an evident consequence of (6.4) .

To prove (ii), we recall that δ is (by the definition) the composite

$$H^{\infty-n}(Y,Z;A) \xrightarrow{j^*} H^{\infty-n}(Y;A) \xrightarrow{\delta_1} H^{\infty-n+1}(X,Y;A) ,$$

where j^* is induced by the inclusion and δ_1 is the coboundary homomorphism of the pair (X,Y).

According to (IV.2.1) we have

$$\delta_1 = \varinjlim \{\delta_\alpha^{n+1}\} , \quad \delta_\alpha^{n+1} = (-1)^{d(\alpha)+1} \delta_{(X_\alpha,Y_\alpha)}$$

where $\delta_{(X_\alpha,Y_\alpha)} : H^{d(\alpha)-n}(Y_\alpha;A) \to H^{d(\alpha)-n+1}(X_\alpha,Y_\alpha;A)$ is the coboundary homomorphism of (X_α,Y_α).

Hence, in view of (6.2), the diagram

$$\begin{array}{ccccc} H^{d(\alpha)-n}(Y_\alpha,Z_\alpha;A) & \xrightarrow{(j_\alpha)^*} & H^{d(\alpha)-n}(Y_\alpha;A) & \xrightarrow{\delta_\alpha^{n+1}} & H^{d(\alpha)-n}(X_\alpha,Y_\alpha;A) \\ & \searrow^{D} & & & \swarrow_{D} \\ & & H_n(W_\alpha,U_\alpha;A) & \xrightarrow{\partial_\alpha} & H_{n-1}(V_\alpha,U_\alpha;A) \end{array}$$

is commutative for each $\alpha \in \mathfrak{L}$. This implies the commutativity of the right hand square. The commutativity of the two left hand squares follows clearly from (5.2) and thus the proof of the theorem is completed.

Next, some corollaries of Theorem (7.1).

(7.2) (The Alexander-Pontriagin Invariance in E). The relation $X \underset{h}{\sim} Y$ in $\mathfrak{H}$ implies that

$$H_{n-1}(E-X;\Pi) \approx H_{n-1}(E-Y;\Pi) .$$

for any $n \geq 1$ and any group of coefficients Π.

(7.3) (The Spanier-Whitehead Invariance in E). The relation $X \underset{h}{\sim} Y$ in $\mathfrak{H}$ implies that for any $n \geq 1$

$$\Sigma_{n-1}(E-X) \approx \Sigma_{n-1}(E-Y) .$$

The following important corollary is an immediate consequence of Theorem (7.1) and the Hurewicz Theorem in S-theory (cf.[12] and [13]).

(7.4) (The Hopf Theorem). For any pair in $\mathfrak{H}$ the first non-vanishing stable cohomotopy group is isomorphic to the first non-vanishing cohomology group over Z. More precisely, we have

(i) $\Sigma^{\infty-q}(X,Y) = 0 \Leftrightarrow H^{\infty-q}(X,Y;Z) = 0$ for any $0 \leqq q < n$.

(ii) if $\Sigma^{\infty-q}(X,Y) = 0$ for $0 \leqq q < n$; then the Hopf map

$h^* : \Sigma^{\infty-n}(X,Y) \to H^{\infty-n}(X,Y;Z)$ is an isomorphism.

VI. REPRESENTABILITY OF THE STABLE COHOMOTOPY AND CODIMENSION

Consider compact fields from an object X to the open set $E - E_{n-1}$, $\dim E_{n-1} = n - 1$, and denote by $\pi^{\infty-n}(X)$ the corresponding set of homotopy classes. Our next aim is to show that there exists a natural isomorphism between $\pi^{\infty-n}(X)$ and the stable cohomotopy group $\Sigma^{\infty-n}(X)$.

Then some applications of this result to the notion of codimension are given.

1. Fields with admissible range U.

Call an objet $U \in \mathfrak{S}$ admissible provided U is open in E and its complement is contained in a finite dimensional subspace of E.

From now on U will stand for an arbitrary but fixed admissible set and $W = E-U$ for its complement.

We shall use the following abbreviations :

$$\mathfrak{S}(X) = \mathfrak{S}(X,U) \qquad \pi(X) = \pi(X,U)$$

$$\mathfrak{S}_\alpha(X) = \mathfrak{S}_\alpha(X,U) \qquad \pi_\alpha(X) = \pi_\alpha(X,U)$$

$$\mathfrak{S}(X_\alpha) = \mathfrak{S}(X_\alpha,U_\alpha) \qquad \pi(X_\alpha) = \pi(X_\alpha,U_\alpha)$$

Denote by $\mathfrak{L}_U$ the cofinal subset of $\mathfrak{L}$ defined by the condition

$$\alpha \in \mathfrak{L}_U \Leftrightarrow \begin{cases} \text{(i)} \quad W \subset L_\alpha \\ \text{(ii)} \quad U_\alpha = L_\alpha - W \text{ is connected} \end{cases}$$

and put $\mathfrak{L}_{X,U} = \mathfrak{L}_U \cap \mathfrak{L}_X$. The elements of $\mathfrak{L}_{X,U}$ are said to be admissible with respect to X. The elements of α which will appear in the sequel are assumed to be admissible with respect to the objects under consideration.

(1.1) *Let $X \in \mathfrak{h}$. Then, for each $\alpha \in \mathfrak{L}_{X,U}$ the set $\pi_\alpha(X)$ is non-empty and the restriction map*

$$\tau_\alpha : \pi_\alpha(X) \to \pi(X_\alpha)$$

given by the assignment

$$\lfloor f \rfloor_\alpha \to \lfloor f_\alpha \rfloor$$

is bijective.

The proof is straightforward and is omitted.

(1.2) Let $X \in \mathfrak{H}$ and $h_t : X \to U$ be a compact homotopy. Then $\operatorname{dist}(h(X \times I, W)) > 0$ and for each ε satisfying

$$0 < \varepsilon < \operatorname{dist}(h(X \times I, W))$$

there is an α-homotopy $h'_t : X \to U$ such that

$$\| h'_t(x) - h_t(x) \| < \varepsilon \quad \text{for all} \quad (x,t) \in X \times I .$$

The first assertion is evident and the second follows from the Approximation Lemma.

From (1.2) we obtain

(1.3) Let $f \in \mathfrak{S}_\alpha(X)$ be an α-field and $g \in \mathfrak{S}_\beta(X)$ be a β-field. If f and g are compactly homotopic, then for some γ satisfying $\alpha < \gamma$, $\beta < \gamma$, we have $f \overset{\sim}{\gamma} g$.

2. Homotopy systems $\{\pi_\alpha(X), i_{\alpha\beta}\}$ and $\{\pi(X_\alpha), j_{\alpha\beta}\}$.

Definition. For each relation $\alpha \leqq \beta$, let

$$i_{\alpha\beta} : \pi_\alpha(X) \to \pi_\beta(X)$$

be defined by the assignment $\lfloor f \rfloor_\alpha \to \lfloor f \rfloor_\beta$ and

$$i_\alpha : \pi_\alpha(X) \to \pi(X)$$

by $\lfloor f \rfloor_\alpha \to [f]$.

Clearly, the family $\{\pi_\alpha(X), i_{\alpha\beta}\}$ is a directed system of sets and $\{i_\alpha\}$ is a direct family of maps ; $\{\pi_\alpha(X), i_{\alpha\beta}\}$ will be called the homotopy system of X .

(2.1) Lemma. The map

$$\varinjlim_\alpha \{i_\alpha\} : \varinjlim_\alpha \{\pi_\alpha(X), i_{\alpha\beta}\} \to \pi(X)$$

is invertible in Ens .

Proof. This is a consequence of (1,2) and (1,3) .

Now for a relation $\alpha \leqq \beta$ in $\mathfrak{L}_{X,U}$ consider the map

$$\tau_\alpha^\beta : \pi_\alpha(X_\beta) \to \pi(X_\alpha)$$

defined by $\lfloor f \rfloor_\alpha \to \lfloor f_\alpha \rfloor$ and the map

$$i_\alpha^\beta : \pi_\alpha(X_\beta) \to \pi(X_\beta)$$

given by $\lfloor f \rfloor_\alpha \to \lfloor f \rfloor$.

It follows from (1.1) that τ_α^β is bijective.

Definition. For every $\alpha \leqq \beta$ define

$$j_{\alpha\beta} : \pi(X_\alpha) \to \pi(X_\beta)$$

by putting

$$j_{\alpha\beta} = i_{\alpha}^{\beta} \circ (\tau_{\alpha}^{\beta})^{-1}$$

$\{\pi(X_\alpha), j_{\alpha\beta}\}$ is called the restricted homotopy system of X .

Clearly, $\{\pi(X_\alpha), j_{\alpha\beta}\}$ is a directed system of sets over $\mathcal{L}_{X,U}$.

(2.2) Lemma. The family of restriction maps $\{\tau_\alpha\}$ defines an isomorphism from the $\{\pi_\alpha(X), i_{\alpha\beta}\}$ to $\{\pi(X_\alpha), j_{\alpha\beta}\}$, i.e., for each $\alpha \leqq \beta$ in $\mathcal{L}_{U,X}$ the following diagram

$$\begin{array}{ccc} \pi_\alpha(X) & \xrightarrow{i_{\alpha\beta}} & \pi_\beta(X) \\ \downarrow{\scriptstyle \tau_\alpha} & & \downarrow{\scriptstyle \tau_\beta} \\ \pi(X_\alpha) & \xrightarrow{j_{\alpha\beta}} & \pi(X_\beta) \end{array}$$

commutes.

3. Continuity of the functor $X \to \pi(X,U)$.

Notation : For a fixed admissible U denote by π an h-functor from $\mathfrak{S}$ to the category of sets which assigns to an object X in $\mathfrak{S}$ the set $\pi(X) = \pi(X,U)$ and to each field $f : X \to Y$ assigns

$$f^* : \pi(Y) \to \pi(X) .$$

We denote by $\pi_o : \mathfrak{S}_o \to \mathrm{Ens}$ the restriction of π to $\mathfrak{S}_o$.

Let $X \in \mathfrak{S}$ and $\varphi : X \to U$ be a field ; call φ inessential provided for any $Y \in \mathfrak{S}$ with $X \supset Y$ there is a field $\bar{\varphi} : Y \to U$ which extends φ over Y . Any two inessential fields are homotopic and the homotopy class which contains an inessential field is denoted by 0 and called the zero element of $\pi(X)$. One shows easily that $f^*(0) = 0$ for any $f \in \mathfrak{S}_o$ and thus π_o may be considered as an h-functor from $(\mathfrak{S}_o, \approx)$ to the category of based sets Ens^* .

Next we prove that π_o is continuous. We begin with two lemmas.

Let Y be an object and $\{Y_k\}$ an approximating family for Y .

(3.1) Lemma. Let $f : Y \to U$ be an α-field. There exists an index k and an α-field $f' : Y_k \to U$ such that $f = f' \mid Y$.

Proof. Let $\bar{F} : Y_{k_o} \to E$ be an α-map such that $\bar{F} \mid Y = F$. Clearly the set

$$C = \{x \in Y_{k_o} , \bar{f}(x) = x - \bar{F}(x) \in W\}$$

is compact ; consequently, for some $k \geqq k_o$, Y_k does not intersect C . Now putting $f' = f \mid Y_k$, we obtain a required α-field f' .

(3.2) Lemma. Let $f,g : Y_k \to U$ be to α-fields such that $f \mid Y \cong g \mid Y$. There exists an index $m \geq k$ such that $f \mid Y_m \cong g \mid Y_m$.

Proof. Let $f,g : Y_k \to U$ be an α-homotopy such that $h_0 = f$ and $h_1 = g$. On the set :

$$T = (Y_k \times \{0\}) \cup (Y \times I) \cup (Y_k \times \{1\})$$

define an α-mapping $H^* : T \to L_\alpha$ by

$$H^*(x,t) = \begin{cases} F(x) & \text{for } x \in Y_k, \ t = 0 ; \\ H(x,t) & \text{for } x \in Y, \ 0 \leq t \leq 1 ; \\ G(x) & \text{for } x \in Y_k, \ t = 1 . \end{cases}$$

Clearly, we have

$$x - H^*(x,t) \in U \text{ for all } (x,t) \in T .$$

By extending H^* from T over $Y_k \times I$, we obtain an α-mapping $\bar{H} : Y_k \times I \to L_\alpha$. Clearly, the set

$$C = \{(x,t) \in Y_k \times I ; \bar{h}_t(x) \in W\}$$

is compact ; therefore, for some $m > k$ the intersection $C \cap (Y_m \times I)$ is empty. Now, putting $h'_t = \bar{h}_t \mid Y_m$, we obtain an α-homotopy $h'_t : Y_m \to U$ such that $h'_0 = f \mid Y_m$ and $h'_1 = g \mid Y_m$. The proof is completed.

Consider the direct system $\{ \pi(Y_k) ; i^*_{k\ell} \}$ of based sets and the direct family $\{ i^*_k \}$ of based maps, where $i^*_{k\ell}$, i^*_k are induced by the inclusions

$$i_{k\ell} : Y_\ell \to Y_k \qquad k \leq \ell$$

$$i_k : Y \to Y_k$$

(3.3) Theorem. The map

$$\varinjlim_k \{i^*_k\} : \varinjlim_k \{\pi(Y_k) ; i^*_{k\ell}\} \to \pi(Y)$$

is a bijective based map. In other words, the functor $\pi_0 : \mathfrak{H}_0 \to \mathrm{Ens}$ is continuous.

(3.4) Corollary. π is an h-functor from $(\mathfrak{H}, \sim)$ to the category of based sets.

4. Natural group structure in $\pi(X,U)$.

Next, we establish that for some U (called algebraically admissible) the functor $\pi : \mathfrak{H} \to \mathrm{Ens}^*$ may be converted to a functor to the category of abelian groups.

Definition. An admissible object U is said to be algebraically admissible provided there is a cofinal subset $\mathfrak{L}'_U$ of $\mathfrak{L}_U$ such that

(i) for each $\alpha \in \mathfrak{L}'_U$ the assignment $X \to \pi(X_\alpha, U_\alpha)$ is an h functor from $(\mathfrak{H}_\alpha, \overset{\sim}{\alpha})$ to the category of abelian group ;

(ii) for every relation $\alpha \leq \beta$ in $\mathfrak{L}'_U$ the map $j_{\alpha\beta} : \pi(X_\alpha) \to \pi(X_\beta)$ is an isomorphism.

(4.1) Theorem. If U is algebraically admissible, $\pi(X)$ admits a structure of an abelian group for all objets X in $\mathfrak{H}$. Moreover, an abelian group structure in $\pi(X)$ is natural with respect to maps in $\mathfrak{H}$, i.e., the induced map $f^*: \pi(Y) \to \pi(X)$ is a homomorphism for any compact field $f : X \to Y$.

The proof of Theorem (4.1) uses (2,1), (2,2) and the continuity of the functor π.

5. The group $\pi^{\infty-n}(X)$.

Let $\{E^{\infty-n} \oplus E_n\}$ be a fixed sequence of direct sum decompositions of E as in the section 1.1. For $n \geq 1$ we let

$$U^{(n)} = U^{\infty-n} = E - E_{n-1} .$$

Clearly, $U^{\infty-n}$ is admissible and we denote by $\pi^{\infty-n}$ the corresponding functor from $\mathfrak{H}$ to Ens*.

Next, we let

$$\mathfrak{L}_n = \{ \alpha \in \mathfrak{L} \; ; \; E_{n-1} \subset L_\alpha \; ; \; d(\alpha) \geq 2n + 2 \} ,$$

$$\mathfrak{L}_{n,X} = \{ \alpha \in \mathfrak{L}_n \; ; \; X_\alpha \neq 0 \} .$$

Clearly, $\mathfrak{L}_{n,X}$ is cofinal in $\mathfrak{L}_n$ and $\mathfrak{L}_n$ is cofinal in $\mathfrak{L}$.

For $k > n > 0$, the map from S^{k-n} to $R^k - R^{n-1}$ given by the assignment

$$(x_1, \ldots, x_{k-n+1}) \to (0, \ldots, 0, x_1, \ldots, x_{k-n+1})$$

is a homotopy equivalence and we denote by

$$\gamma_{k,n} : R^k - R^{n-1} \to S^{k-n}$$

a homotopy inverse of this map.

Let $\{\omega_\alpha\}$ be a fixed orientation in E. For each $n \geq 1$ choose $\ell_n : E_n \to R^n$ which represents the orientation of E_n and $\ell_n(x) = \ell_{n+1}(x)$ for $x \in E_n$

For $\alpha \in \mathfrak{L}_{n,X}$ choose $\ell_\alpha \in \omega_\alpha$ such that $\ell_\alpha(x) = \ell_{n-1}(x)$ for all $x \in E_{n-1}$ and define a map

$$\gamma_{\alpha,n} : U_\alpha^{\infty-n} \to S^{d(\alpha)-n}$$

by

$$\gamma_{\alpha,n}(x) = \gamma_{d(\alpha),n} \circ \ell_\alpha(x) \quad \text{for } x \in U_\alpha^{\infty-n} ;$$

$\gamma_{\alpha,n}$ is a homotopy equivalence and therefore :

$$(\gamma_{\alpha,n})_* : \pi(X_\alpha, U_\alpha^{\infty-n}) \to \pi(X_\alpha, S^{d(\alpha)-n})$$

is bijective. Moreover, since

$$\dim X_\alpha \leq d(\alpha) \leq d(\alpha) + d(\alpha) - 2n - 2 = 2(d(\alpha)-n) - 2 ,$$

the set of homotopy classes $\pi(X_\alpha, S^{d(\alpha)-n})$ may be identified with the $(d(\alpha)-n)$-th cohomotopy group of X_α (and also with the $(d(\alpha)-n)$-th stable cohomotopy of X), i.e.

$$\pi(X_\alpha, S^{d(\alpha)-n}) = \pi^{d(\alpha)-n}(X) = \Sigma^{d(\alpha)-n}(X) .$$

Consequently, $\pi(X_\alpha, U_\alpha^{\infty-n})$ admits a unique abelian group structure such that $(\gamma_{\alpha,n})_*$ is an isomorphism; this structure is determined only by the orientation ω_α in L_α.

(5.1) Every $U^{\infty-n}$ is algebraically admissible $(n \geq 1)$. Moreover, for each $X \in \mathfrak{S}$ the family $(\gamma_{\alpha,n})_*$ where $\alpha \in \mathcal{L}_{n,X}$ defines an isomorphism from $\{\pi(X_\alpha, U_\alpha^{\infty-n}), j_{\alpha\beta}\}$ to $\{\Sigma^{d(\alpha)-n}(X_\alpha), \Delta_{\alpha\beta}\}$.

Proof. If $X, Y \in \mathfrak{S}$ and $f : X \to Y$ is an α-field, $\alpha \in \mathcal{L}_{n,X}$, then we have a commutative diagram

$$\begin{array}{ccc} \pi(Y_\alpha, U_\alpha^{(n)}) & \xrightarrow{(\gamma_{\alpha,n})_*} & \Sigma^{d(\alpha)-n}(Y_\alpha) \\ \Big\downarrow f_\alpha^* & & \Big\downarrow f_\alpha^* \\ \pi(X_\alpha, U_\alpha^{(n)}) & \xrightarrow{(\gamma_{\alpha,n})_*} & \Sigma^{d(\alpha)-n}(X_\alpha) . \end{array}$$

Therefore, the assignment $X \to \pi(X_\alpha, U_\alpha^{(n)})$ is an h-functor from $\mathfrak{S}_\alpha$ to the category of abelian groups. If $\alpha \leqq \beta$ is a relation in $\mathcal{L}_{n,X}$, then the following diagram commutes

$$\begin{array}{ccc} \pi(X_\alpha, U_\alpha^{(n)}) & \xrightarrow{j_{\alpha\beta}} & \pi(X_\beta, U_\beta^{(n)}) \\ \Big\uparrow (\gamma_{\alpha,n})_* & & \Big\downarrow (\gamma_{\beta,n})_* \\ \Sigma^{d(\alpha)-n}(X_\alpha) & \xrightarrow{\Delta_{\alpha\beta}} & \Sigma^{d(\beta)-n}(X_\beta) . \end{array}$$

Hence $j_{\alpha\beta}$ is a homomorphism and the proof is completed.

(5.2) Corollary. $\pi^{\infty-n}$ is an h-functor from $\mathfrak{S}$ to the category of abelian groups.*)

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*) The groups $\pi^{\infty-n}(X)$ were introduced and considered for the first time by K. Geba in [2].

We shall prove next that the functors $\pi^{\infty-n}$ and $\Sigma^{\infty-n}$ are naturally equivalent. To this end, recall that, in view of (2,1) and (2,2), we may identify $\pi^{\infty-n}(X)$ with $\varinjlim_{\alpha} \{\pi(X_\alpha, U_\alpha^{(n)}), j_{\alpha\beta}\}$. Using this identification we let

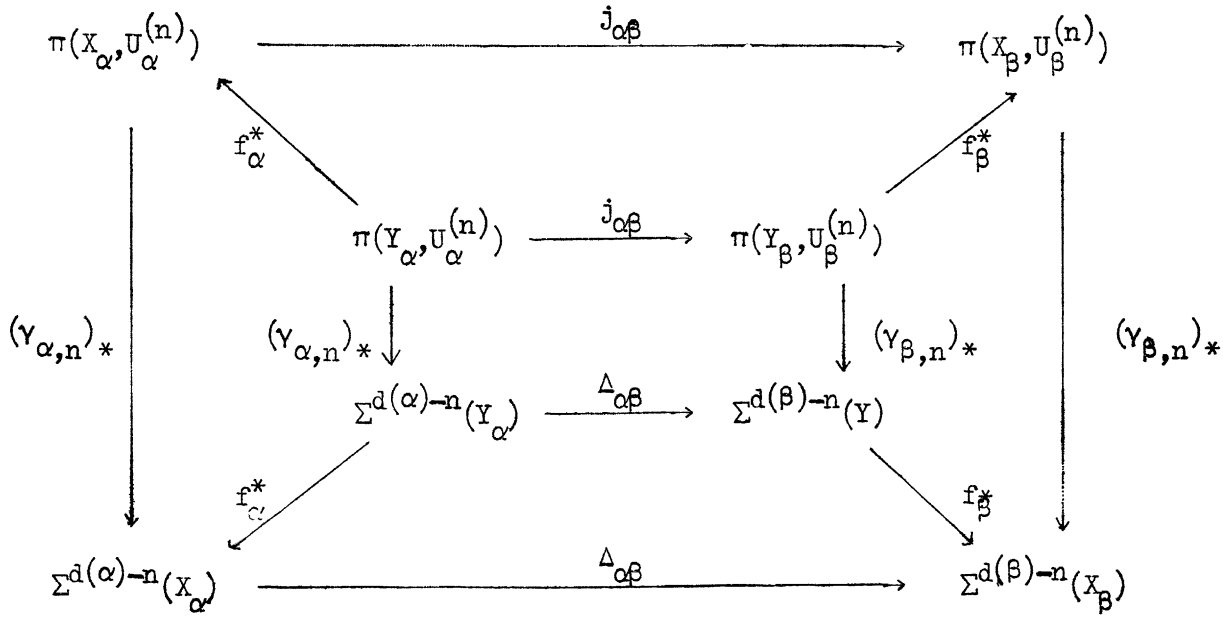
$$\gamma_X = \varinjlim_{\alpha} \{(\gamma_{\alpha,n})_*\} : \pi^{\infty-n}(X) \to \Sigma^{\infty-n}(X) .$$

(5.3) (Representation Theorem). <u>The family</u> $\gamma = \{\gamma_X\}$ <u>is a natural equivalence between the functors</u> $\pi^{\infty-n}$ and $\Sigma^{\infty-n}$.

<u>Proof</u>. By the definition γ_X is an isomorphism of abelian groups for each object X in $\mathfrak{S}$. It remains to prove that if $f : X \to Y$ is a map in $\mathfrak{S}$ then the following diagram commutes

$$\begin{array}{ccc} \pi^{\infty-n}(Y) & \xrightarrow{\gamma_Y} & \Sigma^{\infty-n}(Y) \\ \downarrow{\scriptstyle \pi^{\infty-n}(f)} & & \downarrow{\scriptstyle \pi^{\infty-n}(f)} \\ \pi^{\infty-n}(X) & \xrightarrow{\gamma_X} & \Sigma^{\infty-n}(X) \end{array}$$

In view of the continuity of $\pi^{\infty-n}$ and $\Sigma^{\infty-n}$ it suffices to prove it for $f \in \mathfrak{S}_0$. Suppose that f is an α_0-field. If $\alpha,\beta \in \mathfrak{L}_{n,X}$, $\alpha_0 \leqq \alpha \leqq \beta$ then the following diagram commutes :



Since we have identified $\varinjlim \{\pi(X_\alpha, U_\alpha^{(n)}), j_{\alpha\beta}\}$ with $\pi^{\infty-n}(X)$ and under this identification $\varinjlim_\alpha \{f_\alpha^*\} = \pi^{\infty-n}(f)$ the desired conclusion follows. This completes the proof.

Remark : The entire argument can be repeated in the relative case. Namely, letting $V^{\infty-n} = E - (\ell_{n+1})^{-1}(R_+^{n+1})$ we obtain a pair $(U^{\infty-n}, V^{\infty-n})$ of open subsets of E. One can prove that for a pair (X,A) in $\mathfrak{S}$ there exists a natural isomorphism

$$\gamma : \pi(X,A;U^{\infty-n}, V^{\infty-n}) \to \Sigma^{\infty-n}(X,A) \quad .$$

Thus Theorem (5.3) remains valied in the relative case.

An immediate consequence of (5.3) is the following :

(5.4) Corollary. For an object X in $\mathfrak{S}$, $E - X$ is connected if and only if any two compact fields $f,g : X \to E - \{0\}$ are compactly homotopic.

6. Codimension.

Let G be an abelian group and $H^{\infty-*}(\ ;G)$ the corresponding cohomology theory on $\mathfrak{S}$.

Definition. We define the codimension $\mathrm{Codim}_G(X)$ of an object X with respect to E as the smallest number n, such that $H^{\infty-n}(X,A) \neq 0$ for some object $A \subset X$.

Definition. Let X be an object in $\mathfrak{S}$ and $U \in \mathfrak{E}$. We say that U is an extension object for X provided that, given an object $A \subset X$ and a compact field $f_0 : A \to U$, there exists a field $f : X \to U$ being an extension of f_0 over X.

We denote by $\mathcal{E}(U)$ the set of objects defined by the condition :

$$X \in \mathcal{E}(U) \Leftrightarrow U \text{ is an extension object for } X \text{ and we let}$$

$$\mathcal{E}^{\infty-k} = \mathcal{E}(U^{\infty-k}) \quad .$$

Definition. For an object $X \in \mathfrak{S}$ we define the codimension $\mathrm{Codim}\, X$ of X with respect to E to be the smallest integer n for which $U^{\infty-(n+1)}$ is not an extension object for X.

Thus
$$\mathrm{Codim}\, X = n \Leftrightarrow \begin{cases} \text{(i)} & X \in \mathcal{E}^{\infty-(k+1)} \quad \text{for } k = 0,1,\ldots,n-1 \\ \text{(ii)} & X \notin \mathcal{E}^{\infty-(k+1)} \quad \text{for } k = n \end{cases} .$$

The following is an immediate consequence of the definitions :

(6.1) For any two equivalent objects X and Y in $\mathfrak{S}$ we have $\mathrm{Codim}\, X = \mathrm{Codim}\, Y$ and $\mathrm{Codim}_G(X) = \mathrm{Codim}_G(Y)$.

Now our aim is to prove that $\mathrm{Codim}\, X = \mathrm{Codim}_Z(X)$. This result(which is analogous to the "Fundamental Theorem in the dimension theory", due to Alexandroff [1]), will be established with the aid of the Hopf Theorem, and the Representation

Theorem after some preliminary lemmas.

First, as a consequence of the Homotopy Extension Lemma, we have the following :

(6.2) <u>If the space</u> E <u>is complete, then for an object</u> $X \in \mathfrak{H}$ <u>the following two conditions are equivalent</u> :

(i) $X \in \mathcal{E}^{\infty-(n+1)}$;

(ii) <u>For any pair of objects</u> $A \subset B \subset X$ <u>the restriction map</u>

$$j^*_{AB} : \pi^{\infty-n-1}(B) \to \pi^{\infty-n-1}(A)$$

<u>is an epimorphism</u>.

Next, two lemmas based on the continuity of the functors under consideration :

(6.3) <u>Let</u> X <u>be a given object. Assume that for any pair of objects</u> $A \subset B \subset X$ <u>the map</u>

(i) $$j^*_{AB} : H^{\infty-n-1}(B) \to H^{\infty-n-1}(A)$$

<u>is an epimorphism. Then for any object</u> $A \subset X$ <u>the group</u> $H^{\infty-n}(A)$ <u>is trivial</u>.

<u>Proof</u>. Assuming that our assertion is not true, take a nontrivial element ξ of the group $H^{\infty-n}(A)$.

For a point x in A , let $Y = S(\xi)$ be an essential carrier of ξ with respect to x . Now, take an additive triad (Y, Y_1, Y_2) in which both Y_1 , Y_2 are <u>proper</u> subsets of Y such that $x \in Y_0 = Y_1 \cap Y_2$ and consider the corresponding Mayer-Vietoris exact sequence :

$$\ldots \to H^{\infty-n-1}(Y_1) \oplus H^{\infty-n-1}(Y_2) \xrightarrow{\Psi} H^{\infty-n-1}(Y_0) \xrightarrow{\Delta} H^{\infty-n}(Y) \xrightarrow{\vartheta} H^{\infty-n}(Y_1) \oplus H^{\infty-n}(Y_2) \ldots$$

In view of the definition of the triad $(Y; Y_1, Y_2)$ we have

(ii) $$j^*_{AY}(\xi) \neq 0 \quad , \quad \vartheta j^*_{AY}(\xi) = 0$$

and therefore by exactness

(iii) $$j^*_{AY}(\xi) \in \operatorname{Im} \Delta \quad .$$

Further, by the assumtion (i) , the map Ψ is an epimorphism. From here, in view of (iii) , we infer that for some ξ'

$$j^*_{AY}(\xi) = \Delta \Psi(\xi') \quad .$$

Consequently, again by exactness, $j^*_{AY}(\xi) = 0$, contrary to (ii) .

(6.4) <u>For an object</u> X <u>the following two conditions are equivalent</u> :

1° $H^{\infty-n}(X, A) = 0$ <u>for all objects</u> $A \subset X$;

2° <u>For any pair of objects</u> $A \subset B \subset X$ <u>the map</u>

$$j^*_{AB} : H^{\infty-n-1}(B) \to H^{\infty-n-1}(A)$$

<u>induced by the inclusion</u> $j_{AB} : A \to B$ <u>is an epimorphism</u>.

Proof. $1^\circ \Rightarrow 2^\circ$: Assuming 1° we infer, in view of the exactness of the cohomology sequence of a pair, that both j^*_{AX} and j^*_{BX} are epimorphisms. Since $j^*_{AB} \circ j^*_{BX} = j^*_{AX}$; it follows that j^*_{AB} is also an epimorphism.

$2^\circ \Rightarrow 1^\circ$: Consider the cohomology sequence of the pair (X,A) . Assuming 2° , we infer by (6.3) that $H^{\infty-n}(X) = 0$ and, consequently, by exactness $H^{\infty-n}(X,A) = 0$. The proof is completed.

Now, from (6.1) , (6.2) , (6.3) , (V.7.4) , and (5.3) , we obtain the following :

(6.5) Theorem. If the space E is complete, then for every object X in $\mathfrak{H}$ we have $\operatorname{Codim} X = \operatorname{Codim}_Z(X)$.

Among other facts which follow easily from the proved theorems, we mention :

(6.6) If E is complete and $\operatorname{Codim} X \geq 2$, then $E - X$ is connected.

(6.7) (The Phragmen-Brouwer Theorem in E). For an object X , denote by $b_o(E - X)$ the number of bounded components of $E - X$. Let E be complete and $(Y;Y_1,Y_2)$ be an additive triad in $\mathfrak{H}$ such that $\operatorname{Codim} Y_1 \cap Y_2 > 2$. Then $b_o(E - Y) = b_o(E - Y_1) + b_o(E - Y_2)$.

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