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ELASTOSTATICS PROBLEMS WITH UNILATERAL CONSTRAINTS

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(Abstract of the lectures)

Let A be a bounded region (open set) of the space occupied by an elastic body in its natural configuration. We assume that this natural configuration exists, i.e. we suppose that there exists a configuration of the body such that the stress-tensor is identically zero.

The term elasticity will be understood in the sense of classical elasticity, however not necessarily restricted to homogenous isotropic bodies.

It is convenient, in order to include in our study both the cases of 3-dimensional and 2-dimensional elasticity, to consider A as a domain of the r-dimensional cartesian space X^r. The treatment in the general case proceeds exactly as in the cases of physical interest $r = 2,3$.

The elastic nature of A is determined as soon as the elastic potential is given. The elastic potential is a function $W(x,\varepsilon)$ depending on the point x of A and on the strain tensor, i.e. the tensor ε, whose rectangular components ε_{ik} are given by

$$\varepsilon_{ik} = 2^{-1}(u_{i/k} + u_{k/i})$$

u is the displacement vector and $u_1,\dots,u_r$ its rectangular components.

If we wish to consider elasticity for an inhomogeneous and anisotropic body, we assume

$$W(x,\varepsilon) = \sum_{ik,jh} \tfrac{1}{2} a_{ik,jh}(x)\, \varepsilon_{ik}\, \varepsilon_{jh}$$

with $a_{ik,jh}(x) \equiv a_{jh,ik}(x)$.

We shall assume that the real functions $a_{ik,jh}(x)$ are very smooth, for instance C^∞, and defined in the whole space.

The quadratic form $W(x,\varepsilon)$ is supposed positive definite for any $x \epsilon X^r$.

Let ∂A be the boundary of A and Σ a part of ∂A. We assume Σ to be a rigid and frictionless surface and the body resting on Σ in its natural configuration. Let $\Sigma^* = \partial A - \Sigma$ and suppose that the body is acted upon by a system of body-forces and by a system of surface forces, which act on $\partial A - \Sigma$.

The equilibrium condition and the stress-strain relation are expressed in the interior points of A by the following equations, as is well-known

(1) $$\sigma_{ik/k} - f_i = 0$$

(2) $$W_{\varepsilon_{ik}} + \sigma_{ik} = 0 \;;$$

σ_{ik} are the rectangular components of the stress tensor ; the f_i are known functions as soon as the body forces are assigned. By convention we will call the f_i the components of the body forces. The equilibrium condition on $\partial A - \Sigma$ is the following

$$\sigma_{ik} \nu_k - \varphi_i = 0 , \tag{3}$$

where the ν_k are the components of the unit vector normal to ∂A directed towards the inside of A, and the φ_i are functions assigned on $\partial A - \Sigma$ which by convention we call the components of the surface forces.

For that which regards the boundary conditions on Σ , one or the other of the two following systems of equations must be satisfied at every point of such a surface

$$(4) \begin{cases} u_i \nu_i = 0 \\ \sigma_{ik} \nu_i \nu_k \geqq 0 \\ \sigma_{ik} \nu_i \tau_k = 0 \end{cases} \qquad (5) \begin{cases} u_i \nu_i > 0 \\ \sigma_{ik} \nu_i \nu_k = 0 \\ \sigma_{ik} \nu_i \tau_k = 0 , \end{cases}$$

where we have denoted by τ any vector tangent to Σ in the point considered.

The conditions (4) express the fact that, in the point under consideration, the elastic body in its equilibrium configuration rests on Σ , and therefore, that the reaction of the constraints has a non-negative component along the inward normal. Any tangential component of such a reaction is null since the surface Σ is frictionless.

On the other hand, if the system (5) is satisfied, then in coming to equilibrium the body has left the supporting surface Σ, which therefore no longer reacts on the body.

Problem (1), (2), (3), (4)-(5) was firstly formulated by Signorini [15] (numbers in brackets refer to the References at the end), who proposed for the boundary conditions (4)-(5) the name of ambiguous boundary conditions, since it is not known "a priori" if conditions (4) or (5) are satisfied in a given point of Σ.

The present lectures are concerned with the work I have done in connection with the Signorini problem. More precisely, results of paper [3] are expounded.

As far as I know, my paper [3], published in 1964, was the first on the subject of existence theorems for partial differential equations with inequalities as side conditions. Since then, several mathematicians got interested in the subject and many papers have appeared during this last year ([1], [2], [7], [8], [9], [10], [11], [12], [13], [14]).

In [3] a variational approach to the Signorini problem (1), (2), (3), (4)-(5) is considered.

A proper functional space S is considered and the subclass $\mathcal{U}_\Sigma$ of S, defined by the condition

$$u_i \nu_i \geqq 0 \quad \text{on} \quad \Sigma ,$$

is introduced.

Set

$$B(u,u) = \int_A W(x,\varepsilon)dx$$

$$F(u) = \int_A f_i u_i \, dx + \int_{\Sigma^*} \varphi_i u_i \, d\sigma .$$

It is shown that the "energy integral"

$$I(u) = B(u,u) - F(u)$$

has a minimum in $\mathcal{U}_\Sigma$ if one of the following conditions is satisfied.

a) $$F(v) \leqq 0$$

for any rigid displacement(*) belonging to $\mathcal{U}_\Sigma$ and the further assumption that the equality sign holds only when v and -v both belong to $\mathcal{U}_\Sigma$.

b) $$F(v) = 0$$

for any rigid displacement v and some further assumption on Σ (for instance Σ to be planar)(**).

In [3] it is shown that if Σ is planar, conditions a) or b) not only are sufficient for the existence of the minimum of I(u) in $\mathcal{U}_\Sigma$, but also necessary.

In [3] the regularity properties of the minimizing function are fully investigated and it is shown the equivalence between the variational problem and a proper generalisation of the Signorini problem.

(*) The term <u>rigid displacement</u> means any vector v such that

$$v = b + Ax ,$$

where b is a constant r-vector and A a skew-symmetric $r \times r$-matrix with constant entries.

(**) Condition a) has been also introduced in paper [12], (see teor. 5.1), where an abstract setting of problems with unilateral constraints is exhibited including non self-adjoint cases.

The method used for the Signorini problem applies to the simpler situation of the analogous problems for a second order self-adjoint elliptic operator, i.e. the problem where

$$B(u,u) = \frac{1}{2} \int_A a_{hk}(x)\, u_{/h}\, u_{/k}\, dx \,,$$

$$F(u) = \int_A f\, u\, dx + \int_{\Sigma^*} \varphi\, u\, d\sigma \,;$$

the a_{hk}, u, f, φ are real-valued functions and $a_{hk}(x)\, \lambda_h\, \lambda_k > 0$. In this case the existence conditions a) and b) reduce to the unique condition

$$F(1) \leqq 0 \,.$$

For details and proofs we refer the reader to our papers [3],[4], [5], [6].

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