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## **Meromorphic solutions of Benjamin-Ono equation**

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S E M I N A I R E   S U R  
LES EQUATIONS NON-LINEAIRES

- I -

MEROMORPHIC SOLUTIONS OF BENJAMIN-ONO EQUATION

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Meromorphic solutions of Benjamin-Ono equation.

We are sure that the existence of good meromorphic solutions that can be at least asymptotically exactly investigated give us good a good test for complete integrability of non-linear system. In fact most of the completely integrable systems are connected with linear eigenvalue problem and so with inverse scattering method. However it is too strong to claim that all completely integrable systems arise from some inverse scattering problem or condition of commutativity of two linear operators. In fact there are some examples of equations possessing multisoliton solutions, laws of conservation and even Bäcklund transformation but being unidentified with some linear problem and Gelfand-Levitan-Marchenko equation. For some of these equations [9], [1], [10] there exists an artificial Hirota's procedure reducing equation to so called bilinear form (so called dependent variable transformation (DVT) [7], [1]). On the other hand physical phenomena, especially wave propagation need analytical examinations of nonlinear equations corresponding to different dispersion relations. Especially it is connected with Stoke's waves phenomena and instability arising sometimes in the propagation of waves in shallow water. In this situation there has been

written a few model equations known by the general name of Benjamin-Ono type equations. These equations in general correspond to arbitrary dispersion relation

$$\frac{\partial u(x,t)}{\partial t} + c u(x,t) \frac{\partial u(x,t)}{\partial x}$$

(B01)

$$= - \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} dk dx' c(k) \frac{\partial u(x',t)}{\partial x'} \exp[ik(x' - x)]$$

for any dispersion relation  $c(k)$ . For  $c(k) = c_0(1 - \beta k^2)$  we obtain Korteweg-de Vries equation. However for bigger depth dispersion relation must be a little different

$$c(k) = c_0(1 - \gamma|k|).$$

This is very important especially because of the nonanalytical character of  $c(k)$  in  $k = 0$ . In this situation the Benjamin-Ono equation reduces to one involving Hilbert transforms. Such Benjamin-Ono equations can be written in the form

$$\frac{df}{dt} + \alpha f \frac{df}{dx} - \beta \frac{d^2}{dx^2} H(f) = 0,$$

where

$$H(f) = \frac{1}{\pi} \text{PV} \int_{-\infty}^{\infty} \frac{f(z)}{\xi - z} dz.$$

This equation can be written after a scalar transformation in the natural form

$$(B02) \quad u_t = 2uu_x - \int_{-\infty}^{\infty} \frac{u_{xx}}{x-y} dy$$

or

$$u_t = 2uu_x - H[u_{xx}].$$

In the first papers of Benjamin [2] and Ono [3] it was predicted that (B02) must have good multisoliton solutions and laws of conservations but there were no analytical indications. Later multisoliton like solutions were observed by R. I. Joseph. [Multisoliton-like solutions to the Benjamin-Ono equation, J. Math. Phys. 18 (1977), No. 12, 2251-2258.] These multisolitons were nonsingular rational functions with elastic interaction between one solitons. Recently M. Case has described the same type of multisoliton solutions and moreover has found new algebraic conservation laws for (B02) similar to those of KdV [8], [4].

The explanation of these excellent properties of (B02) lies in the fact that this equation has very good meromorphic solutions like the first Burgers-Hopf equation [5]. The poles of [B02] evolutionate like poles of  $BH_2$  [5] and thus like particles with potential  $x^{-2}$ . However the main difference is in the fact that unlike the Burgers-Hopf equation we

have particles with two charges  $\pm 1$  or  $\pm\sqrt{-1}$ . We have the following precise result[8].

Proposition 1: Any meromorphic solution  $u(x,t)$  of (B02) has the form  $u(x,t) = \sum_{i \in I} c_i (x - a_i)^{-1}$ ,  $a_i = a_i(t)$  and  $c_i = \sqrt{-1}$ , if  $\text{Im}(a_i) > 0$  and  $c_i = -\sqrt{-1}$  if  $\text{Im}(a_i) < 0$ :  $i \in I$ .

Thus

$$(3) \quad c_i = \sqrt{-1} \text{ sgn } \text{Im} (a_i).$$

If

$$(S) \quad u(x,t) = \sum_{i \in I} c_i (x - a_i)^{-1}$$

where  $c_i$  is defined in (3). Then equation (B02) is satisfied if and only if

$$(4) \quad a_{i,t} = 2 \sum_{j \neq i} c_j (a_i - a_j)^{-1}: \quad i \in I.$$

Now we consider a system (4) with arbitrary

$$c_i = \pm\sqrt{-1}: \quad i \in I$$

not necessarily satisfying (3). With all  $c_i$  equal we come simply to the Burgers-Hopf system of poles. However, the properties of the system remain the same for arbitrary  $c_i$  of two signs.

Proposition 2: If  $c_i = \pm c$  for  $i \in I$  and

$$(5) \quad a_{it} = -2 \sum_{j \neq i} c_j (a_i - a_j)^{-1}; \quad i \in I,$$

then the particles  $a_i$  with given sign of  $c_i$  interact separately via potential  $x^{-2}$ :

$$(6) \quad a_{itt} = -8c^2 \sum_{j \neq i, e_j = c_i} (a_i - a_j)^{-3}.$$

Formulae (6) are obtained from (5) by differentiation and in this way we obtain the following system equivalent to (6):

$$(7) \quad a_{itt} = 4 \sum_{j \neq i} c_j (c_i - c_j) (a_i - a_j)^{-3}; \quad i \in I.$$

However the system (7) is true, of course, for arbitrary  $c_i$ .

As a consequence we find that for solution (S) of (B02) poles  $a_i$  in upper and lower half-planes interact separately as particles with potential  $Gx^{-2}$  for  $G = -4$  analogically to Burgers and Hopf and Schrödinger operators. This separate behavior to the point show the conservation of charge in (3)

Real solutions  $u(x,t)$  we obtain considering symmetric configuration, when together with  $a_i$  we have also conjugate



poles  $a_i^*$ .

Solutions being rational functions in  $x$  or  $\sin x$  can be solved using our information about matrix  $L$  for  $\theta(x) = x^{-2}$  or  $\theta(x) = \sin^{-2} x$  with  $G = -4$ .

Benjamin-Ono equation

$$(B02)' \quad v_t + 2vv_x + H[v_{xx}] = 0$$

for

$$H(f) = \frac{1}{\pi} \text{p.v.} \int_{-\infty}^{\infty} \frac{f(z)}{\xi - z} dz$$

is also solvable by Schrödinger equation.

We consider time-dependent Schrödinger

$$\left(\frac{d}{dx^2} + u\right)\psi = i \frac{d}{dt} \psi$$

for  $u = u(z, t)$  and  $\psi = \psi(z, t)$  for function  $\psi(z, t)$  holomorphic in  $\text{Im } z \geq 0$ . Let's suppose that

$$|\psi(z, t)|^2$$

is independent of  $x$  (e.g. constant) for  $z \rightarrow x + i.0$ .

Then

$$\chi = i \cdot \frac{d \log \psi}{\psi}$$

is real for  $z \rightarrow x + i.0$  and is the solution of (B02)'. In fact

$$\begin{aligned} u_x &= \left( \frac{i\psi_t - \psi_{xx}}{\psi} \right)_x = \chi_t + (i\chi_x + \chi^2)_x = \\ &= \chi_t + 2\chi\chi_x + i\chi_{xx} \end{aligned}$$

Then if  $u_z(z, t)$  is regular for  $\text{Im } z > 0$  then

$$\chi_{xx} = H(\chi_t + 2\chi\chi_x)$$

what is equivalent to (B02)'.

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