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S E M I N A I R E   S U R

L'ANASYSE DIOPHANTIENNE ET SES APPLICATIONS

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COMPLETELY INTEGRABLE CLASS OF MECHANICAL SYSTEMS CONNECTED WITH KORTEWEG-de VRIES AND MULTICOMPONENT SCHRÖDINGER EQUATIONS - I -

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0. Introduction.

We consider the system of equations, describing the motion of a particle in n-dimensional space with central potential x^4 and added arbitrary non-central linear forces :

$$(1) \qquad x_i'' = \lambda_i x_i - x_i(\Sigma_{j=1}^n x_j^2) \ : \ i = 1,\dots,n$$

These equations arise in [3] as a special case of equations defining potential with the maximal number of bound states for n-dimensional Schrödinger equation and corresponding to $n = 4$. It was shown in [4] that system (1) is completely integrable for arbitrary $\lambda_1,\dots,\lambda_n$. In this paper we present the exact form of additional algebraic first integrals of (1) and give applications of existence of these integrals to Strurm-Liouville problem and algebraic geometry.

1. - First of all we'll give a proof of complete integrability of (1), based on the Lax'pair and the theory of stationary solutions of infinite-dimensional Hamiltonian systems [5], [6].

Theorem 1 [4] - The Hamiltonian system (1) with Hamiltonian

$$H = \Sigma_{i=1}^n p_i^2 + 1/2(\Sigma_{i=1}^n q_i^2)^2 - \Sigma_{i=1}^n \lambda_i q_i^2$$

is completely integrable for any $\lambda_1,\dots,\lambda_n$.

Let L be the following matrix differential operator of order 1 for $(n+1) \times (n+1)$ matrices

$$L = \begin{pmatrix} 1 & & & 0 \\ & -1 & & \\ & & \ddots & \\ 0 & & & -1 \end{pmatrix} \frac{d}{dx} + \begin{pmatrix} 0 & u_1 & \dots & u_n \\ v_1 & & & \\ \vdots & & 0 & \\ v_n & & & \end{pmatrix}$$

or

$$L = \mathrm{diag}(1,-1,\dots,-1). \frac{d}{dx} + U \quad ,$$

where $U_{1j} = u_{j-1}$, $U_{j1} = v_{j-1}$ for $j = 2,\dots,n+1$ and $U_{ij} = 0$ for $\min\{i,j\} > 1$, $U_{11} = 0$.

Then the multicomponent non-linear Schrödinger equation (MCNS) [2]

(2)
$$-\frac{\partial u_i}{\partial t} + \frac{\partial^2 u_i}{\partial x^2} + u_i(\Sigma_{j=1}^n u_j v_j) = 0 : i = 1,\dots,n \quad ;$$
$$\frac{\partial v_i}{\partial t} + \frac{\partial^2 v_i}{\partial x^2} + v_i(\Sigma_{j=1}^n u_j v_j) = 0 : i = 1,\dots,n \quad ,$$

is equivalent (cf. computation in [5]) to Lax's equation

$$\frac{dL}{dt} = [L,M]$$

for matrix differential operator M of order 2 . Substituting in (2), $u_i = e^{\lambda_i t} y_{1i}$, $v_i = e^{-\lambda_i t} y_{2i}$ $y_{\chi i} = y_{\chi i}(x,t)$ we obtain Lax's system

(3)
$$(-1)^{\chi}\frac{\partial y_{\chi i}}{\partial t} + \frac{\partial^2 y_{\chi i}}{\partial x^2} = \lambda_i y_{\chi i} - y_{\chi i}(\Sigma_{j=1}^n y_{1j} y_{2j}) : i = 1,\dots,n,\ \chi = 1,2 \ .$$

The stationary subsystem of (3) : $\partial y_{\chi i}/\partial t = 0 : i = 1,\dots,n$, $y_{1i} = y_{2i} : i = 1,\dots,n$, is simply (1) . By [5], [6] this is also Lax's system and is, by Liouville's theorem completely integrable.

Below we'll give n independent and involutive first integrals of (1).

2. - The system (1) is also closely connected with stationary solutions of higher Korteweg-de Vries (KdV) equations [1], [7]. In order to find them we consider the resolvent $R(x,\lambda)$ of Sturm-Liouville problem $-\varphi'' + [u(x) + \lambda]\varphi = 0$, sasitfying the Ricatti equation

(4)
$$-2RR'' + (R')^2 + 4(u + \lambda)R^2 = 0 \quad .$$

Then we have the asymptotic expansion

(5)
$$R(x,\lambda) = \Sigma_{i=0}^{\infty} R_i[u]\lambda^{-i-1/2}$$

and all $R_i[u]$ are polynomials from u and its derivatives $u',u'',u''',\dots$. For $R_i[u]$ there exists a lot of recursive formulae [1], [7], [8], most of which can be deduced from (4) and (5). We'll list only

$$R_0 = 1/2 \ , \ R_1 = -u/4 \ , \ R_2 = 1/16(3u^2 - u''),\dots$$

and

(6) $$\frac{\delta}{\delta u} R_\ell[u] = -(\ell-1/2)R_{\ell-1}[u] \quad .$$

The equations

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x} \Sigma_{i=1}^{n} c_i \frac{\delta}{\delta u} I_i[u]$$

for $I_i[u] = \int R_i[u]dx$, are called higher KdV. By (6) they have the form

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x} \Sigma_{i=1}^{n+1} d_i R_i[u] \quad .$$

Their stationary subsystems

(7) $$\Sigma_{k=0}^{n+1} c_k R_k[u] = 0 \quad , c_{n+1} = 1 \quad ,$$

are called stationary n-th order KdV equations (or Lax-Novikov equations) [7], [9]. The system (7) with given $c_0,\ldots,c_n$ is completely integrable Hamiltonian system with n degrees of freedom [7], [8] . All the solutions are expressed in terms of θ-functions on Jacobians of hyperelliptic curves of genus n[7]-[9] . The equations (7) have also the spectral sense ([7]-[9]) : periodic (or quasi-periodic) potential $u(x)$ have n-lacunary spectrum - i.e. having finetely many forbidden intervals for $-\varphi'' + [u+\lambda]\varphi = 0$ - if and only if $u(x)$ is a solution of some of the equation (7) with fixed $c_0,\ldots,c_n$.

Now we can prove the main result concerning the solutions of (1) :

<u>Theorem 2</u> - <u>If</u> $x_1,\ldots,x_n$ <u>are solutions of (1), then the potential</u>

$$u = -\Sigma_{i=1}^{n} x_i^2 + C \quad ,$$

<u>is a solution of stationary</u> n-th <u>order KdV equation</u> (7) :

$$\Sigma_{k=0}^{n+1} d_k R_k[u] = 0 \quad .$$

For the proof cf. [4]. In fact the proof follows immediately from the following lemma, easely to verify directly, using (4) and (5) :

Lemma 3 - Let $u = -\Sigma_{i=1}^{n} x_i^2 - 8C_o$. Then there exist such constants of motion $C_1,\dots,C_\ell,\dots$ (first integrals of (1)), that for polynomials $P_\ell(x) : \ell = 0,1,2,\dots$ defined by

(8) $P_o(x) = 1/4 \ , \ P_{\ell+1}(x) = P_\ell(x).x + 8C_o P_\ell(x) + C_\ell \ : \ \ell = 0,1,2,\dots$

and moments

(9) $M_\ell = \Sigma_{i=1}^{n} P_\ell(\lambda_i) x_i^2 + 2C_\ell : \ell = 0,1,2,\dots$

we have

(10) $R_{\ell+1}[u] = M_\ell$ for all $\ell = 0,1,2,\dots$.

By the theory of n-lacunary (or n-band, or n-gap) solutions [7]-[9] we obtain

Corollary 4 - For any solutions $x_1,\dots,x_n$ of (1) and the potential $u = -\Sigma_{i=1}^{n} x_i^2$, $\lambda_1,\dots,\lambda_n$ are the eigenvalues and $x_1,\dots,x_n$ are the eigenfunctions corresponding to the ends of lacunae (in the sense of [8]) .

Using Borg's uniquiness theorem we obtain

Theorem 5 - For any periodic potential u , which is n-lacunary, there exist such $x_1,\dots,x_n,\lambda_1,\dots,\lambda_n$ and C that

$$u = -\Sigma_{i=1}^{n} x_i^2 + C$$

and

$$x_i \ : \ i = 1,\dots,n$$

are the solutions of (1). Moreover, $\lambda_1,\dots,\lambda_n$ are the eigenvalues corresponding to solutions of periodic or anti-periodic Sturm-Liouvile problem and $x_1,\dots,x_n$ are corresponding eigenfunctions (periodic or antiperiodic).

In theorem 5 $\lambda_1,\dots,\lambda_n$ can be choosen as arbitrary n from 2n+1 ends of lacunae. Thus any n + 1 square $\Psi_i^2 : i = 1,\dots,n+1$ of Bloch's eigenfunctions, corresponding to periodic or antiperiodic eigenvalue problem for n-lacunary potential are linearly dependent (cf. [1]).

Corollary 6 - If $\Psi_1,\dots,\Psi_n$ are Bloch's eigenfunctions of periodic or anti-periodic problem for n-lacunary potential that are linearly independent, then for some $k_i : i = 1,\dots,n$ the potential $v = -\sum k_i^2\Psi_i^2$ is also n-lacunary.

The theorem 5 can be generalized for arbitrary n-lacunary potential (i.e. quasi-periodic, not neccessary periodic) :

Theorem 7 - Any n-lacunary potential u (i.e. the solution of (7)) can be presented in the form

$$u = -\Sigma_{i=1}^{n} x_i^2 + C \quad ,$$

where $\{x_1,\dots,x_n\}$ is some solution of (1) with $\lambda_1,\dots,\lambda_n$ defined by (7) and hyperelliptic curve correspond to u .

3. - As the systems (1) and (7) are equivalent, we can obtain the expressions of the first integrals of (7) from those of (1). It is very important because it's very difficult problem to write down first integrals of (7) for $n > 3$. We'll remember from [4] the second (different from H) integral of (1) (use lemma 5) :

$$\begin{aligned} J = (\sum_{i=1}^{n} p_i^2)(\sum_{i=1}^{n} q_i^2) + (\sum_{i=1}^{n} q_i^2)(\sum_{i=1}^{n} \lambda_i q_i^2) - \\ -(\sum_{i=1}^{n} p_i q_i)^2 + 2(\sum_{i=1}^{n} \lambda_i p_i^2 - \Sigma_{i=1}^{n} \lambda_i^2 q_i^2) \end{aligned} \quad . \tag{11}$$

It is very simple to make small changes in (11) in order to obtain another integrals of (1). In order to do this it's enough to consider such $\pi_1,\dots,\pi_n$ and $\xi_1,\dots,\xi_n$, that

$$\frac{\pi_i - \pi_j}{\lambda_i - \lambda_j} = \xi_i \xi_j \qquad (\text{for } \lambda_i \neq \lambda_j \text{ and } \pi_i = \pi_j \text{ for } \lambda_i = \lambda_j) \quad . \tag{12}$$

e.g. $\pi_i = \lambda_i$, $\xi_i = 1$ $(i = 1,\dots,n)$, $\pi_i = (\lambda_i + \zeta)^{-1}$, $\xi_i = \sqrt{-1}(\lambda_i + \zeta)^{-1}$ $(i = 1,\dots,n)$. Then it is easy to verify that

$$J' = (\sum_{i=1}^{n} \xi_i q_i^2)(\sum_{i=1}^{n} \xi_i p_i^2) + (\sum_{i=1}^{n} q_i^2)(\sum_{i=1}^{n} \pi_i q_i^2) -$$

(13)

$$- (\sum_{i=1}^{n} \xi_i q_i p_i)^2 + 2(\sum_{i=1}^{n} \pi_i p_i^2 - \sum_{i=1}^{n} \lambda_i \pi_i q_i^2) \quad .$$

E.g. for $\xi_i = 0$, $\pi_i = 1$ $(i = 1,\ldots,n)$, $J' = 2H$. It is clear that integrals J' for $\pi_i = (\lambda_i + \zeta)^{-1}$, $\xi_i = \sqrt{-1} \times (\lambda_i + \zeta)^{-1}$ after expansion by powers of ζ give n independent integrals in involution for (1).

From (13) it can be deduced various forms of first integrals of (13) (one of such forms was obtained by H. Grosse recently using (11)). It is natural to use (11), (13) to write down the system of algebraic equations defining Jacobians of hyperelliptic curves of genus n . For $n = 2$ these equations are as follows :

$$(p_1 q_2 - q_1 p_2)^2 + 2(\lambda_1 - \lambda_2)p_1^2 - 2\lambda_1(\lambda_1 - \lambda_2)q_1^2 + (\lambda_1 - \lambda_2)q_1^2(q_1^2 + q_2^2) = C_1 \ ;$$

$$2(p_1^2 + p_2^2) - (\lambda_1 q_1^2 + \lambda_2 q_2^2) + (q_1^2 + q_2^2)^2 = C_2 \quad .$$

Here $\lambda_1, \lambda_2, C_1, C_2$ are arbitrary constant. It is interesting that Jacobian $J(F)$ of the curve $F : y^2 = P_{2n+2}(x)$ can be defined by n equations in p_i, q_i $(i = 1,\ldots,n)$ of degree 2 with $2n$ constants $C_1,\ldots,C_n$, $\lambda_1,\ldots,\lambda_n$.

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Page	Ligne	Lire
6	14	p_i, q_i $(i = 1, \ldots, n)$ of degree 4 ....
