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S E M I N A I R E S U R

L'ANALYSE DIOPHANTIENNE ET SES APPLICATIONS

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MEROMORPHIC SOLUTIONS OF TWO-DIMENSIONAL EQUATIONS WITH ALGEBRAIC LAWS OF CONSERVATION

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ABSTRACT

Complete description of meromorphic solutions of several two-dimensional equations with algebraic laws of conservation is obtained. Among them are Zakharov-Shabat systems and, e.g., Kadomtsev-Petiashvili equation.

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We shall consider two-dimensional equations and their meromorphic solutions, especially elliptic solutions (i.e. solutions expressed in terms of elliptic functions). We shall investigate the behaviour of theses solutions using the picture of poles in complex plane, so transferring our problem to many-particles one in the spirit of [2]. Using methods from [2], we will describe completely meromorphic (in particular, rational) solutions of some two-dimensional systems.

§ 0.

Two-dimensional Lax' (or Zakharov-Shabat) equations for the vector function $\bar{u}(x, y, t)$ have the form [5]:

$$\left[L_1 - \frac{\partial}{\partial t}, L_2 - \frac{\partial}{\partial y}\right] = 0 \quad .$$

This is the condition of commutation for two operators acting on functions of x :

(2 dim) $$\frac{\partial L_2}{\partial t} - \frac{\partial L_1}{\partial y} = [L_1, L_2] \quad .$$

This equation is the system of equations on coefficients of operators L_1, L_2 [5]. For such kind of equations, Zakharov-Shabat have constructed "algebraic inverse scattering method".

The first non-trivial example of two-dimensional "inverse scattering integrable" equation is the two-dimensional KdV or Kadomtsev-Petiashvili equation [4], [5] :

(2 KdV) $$\frac{\partial}{\partial x}(u_t + 6uu_x + u_{xxx}) = -3\alpha \frac{\partial^2}{\partial y^2} u$$

for $u = u(x,y,t)$ and $\alpha = \pm 1$. For this equation some solutions were written down by Zakharov-Shabat (constructed from exponentials but without decrease in (x,y) at infinity) ; and some (not the same !) by Krichiver using θ-functions of plane curves [5], [9].

What is the method of solving (2 dim) algebraically ? We consider some auxiliary stationary problem

(S) $$[L_1, Q] = 0$$

or $\frac{\delta}{\delta u} I_p = 0$ for some functional $I_p = \int p\, dx$, such that the stationary ($\frac{\partial \bar{u}}{\partial t} = \frac{\partial \bar{u}}{\partial y} = 0$) manifold (S) is invariant for the system (2 dim). All the solutions of (S) can be found algebraically, when the orders of L_1 and Q are relatively prime.

Then for any solution $\bar{u}(x) = \bar{u}(x,0,0)$ of (S) we find an evolution in y and t according to (2 dim), because for invariant (S), $u(x,y_o,t_o)$ is a solution of (S) for any y_o, t_o.

Thus for (2 KdV) we start e.g. with any finite-lacunary potential u(x) [here L_1 is Schrödinger, $Q = \Sigma\, c_i\, A_{2i+1}$] and obtain through θ-functions (if necessary) solution u(x,y,t) of (2 KdV) such that u(x,0,0) = u(x).

We had already mentioned in [2] that this method can give only meromorphic solutions. Thus it is much more natural to examine all the meromorphic solutions $\bar{u}(x,t,y)$, considering as in [1], [2] the motion of poles $a_i = a_i(t,y)$ of $\bar{u}(x,t,y)$ in all complex x-plane.

§ 1. MEROMORPHIC SOLUTIONS OF TWO-DIMENSIONAL EQUATIONS AND THEIR POLES.

We know by [1], [2], [3] that the evolution of poles of several one-dimensional equations (e.g. KdV, Boussinesq,...) is connected with the Hamiltonian

$$H_\wp = \frac{1}{2} \sum_{i \in I} y_i^2 + G \sum_{i \neq j} \wp(x_i - x_j)$$

for the Weierstrass elliptic function $\wp(X)$ and with the corresponding first integrals $J_n = \frac{1}{n} \operatorname{tr}(L^n)$: n = 1,2,... of $H_\wp$ described in [1] and [2].

These one-dimensional systems have Lax' form

$$\frac{dL_1}{dt} = [L_1, L_2] \quad ,$$

and so are included in more general two-dimensional equation (2 dim). In this paper we shall consider some natural conjectures about meromorphic solutions of (2 dim) and obtain new solutions of (2 KdV) and similar equations in terms of elliptic functions.

The chain (C) and [2] of non-linear equations with constraints $u_n \equiv 0$ gives some particular system of non-linear equations of evolution possessing infinitely many algebraic laws of conservation. As we had already mentioned in [2], the first system in this chain is Boussinesq equation, i.e., corresponds to the Lax pair $\frac{dA}{dt} = [L, A]$, where L is Schrodinger and A of degree 3. For this equation, the motion of the poles $x_i = x_i(t)$ corresponds to the motion according to the Hamiltonian $H = J_2$ with the restrictions $\text{grad } J_3 \equiv 0$ [3]. On the other hand, the KdV equation has the Lax representation $\frac{dL}{dt} = [A, L]$ and the poles $x_i = x_i(t)$ move according to the Hamiltonian J_3 with the restrictions $\text{grad } H = \text{grad } J_2 = 0$ (see [1] and [2]).

So it is natural to put out the following

Conjecture: The pole evolution of the system having the Lax form $\frac{dL}{dt} = [A, L]$ for L order n and A of order m, $n > 1$, $m > 1$, is connected with the system with the Hamiltonian J_m with the restriction $\text{grad } J_n = 0$. By analogy, if we have the system $\frac{dL}{dt} - \frac{dA}{dy} = [A, L]$, then the poles $x_i = x_i(y, t)$ move in y according to J_n and in t according to J_m.

This conjecture was proved in [1] for the case $m = 2$, $n = 3$ of $m = 3$, $n = 2$ (see [1], §10, p. 350). Krichiver [9] tries to say that he obtained these results "very recently" (after [1]). We prove here the conjecture for $n = 2$, $m = 4$ and show close relations with system (C

Part A - Meromorphic, rational and elliptic solutions of the two-dimensional Korteweg-de Vries equation.

In this part, general meromorphic solutions of (2 KdV) are considered and new elliptic solutions are constructed.

A1. For the two-dimensional Korteweg-de Vries equation

(2 KdV) $$\frac{\partial}{\partial x}(u_t + 6uu_x + u_{xxx}) = -3\alpha \frac{\partial^2}{\partial y^2} u ,$$

we shall consider the meromorphic solutions, written in the general form

(1) $$u(x,y,t) = \sum_{i\in I} (-2)(x-a_i)^{-2} \quad , \quad a_i = a_i(y,t)$$

or

(2) $$u(x,y,t) = \sum_{i\in I} (-2)\wp(x-a_i) \; , \; a_i = a_i(y,t) \quad .$$

It is easy to show that, if $u(x,y,t)$ is a meromorphic solution in x of (2 KdV) for $(y,t)\in[0,y_0]\times[0,t_0]$, then the poles in $a_i(y,t)$ are of second order with residues -2. Thus the form (1) or (2) is the general form of the meromorphic solutions of (2 KdV).

In the paper [1] (§ 10, p. 350) it was shown that the motion of the poles $a_i(y,t)$ is in y according to $J_2 \equiv H$ and in t according to J_3. This is possible because J_2 and J_3 commutes. More precisely we have

<u>Proposition 1</u> : The function

$$u(x,y,t) = -2 \sum_{i\in I} \wp(x-a_i) \; , \; a_i = a_i(y,t)$$

is meromorphic solution of (2 KdV) if and only if

(3) $$\begin{cases} \alpha a_{iyy} = 4 \sum_{j\neq i} \wp'(a_i - a_j) \\ a_{it} = 3\alpha a_{iy}^2 - 12 \sum_{j\neq i} \wp(a_i - a_j) \; : \; i \in I \end{cases} \quad .$$

This can be obtained by substituing $u(x,y,t)$ from (2) into (2 KdV) and using the law of addition for $\wp(x)$.

If we consider the commuting flows

$$\overline{J_2} = \sum_{i\in I} \frac{b_i^2}{2} - \frac{4}{\alpha} \sum_{i\neq j} \wp(a_i - a_j)$$

and

$$\overline{J_3} = \alpha \sum_{i\in I} b_i^3 - \sum_{i\neq j} (b_i + b_j)\, \wp(a_i - a_j) \quad ,$$

i.e. $G = 4/\alpha$ and $\overline{J_2} = H = J_2$ and $\overline{J_3} = 3\alpha\, J_3$, then the system (3) is obviously equivalent to

$$
(4) \qquad \left\{ \begin{array}{l} a_{iy} = \dfrac{\partial \overline{J_2}}{\partial b_i} \;, \quad b_{iy} = -\dfrac{\partial \overline{J_2}}{\partial a_i} \;; \\[2em] a_{it} = \dfrac{\partial \overline{J_3}}{\partial b_i} \;, \quad b_{it} = -\dfrac{\partial \overline{J_2}}{\partial a_i} \;, \; i \in I \;. \end{array} \right.
$$

Because $\overline{J_2}$ and $\overline{J_3}$ are commuting we obtain solutions $a_i(y,t)$, $b_i(y,t)$ of the system (4), or of the system (3).

Thus starting from any initial data at $y = y_o$, $t = t_o$

$$
a_i^o = a_i(y_o, t_o) \quad , \quad b_i^o = a_{iy} = b_i(y_o, t_o) \quad , \quad i \in I
$$

we can explicitely find (at least for finite I) -a solution of (3)- $a_i(y,t)$ such that

$$
a_i(y_o, t_o) = a_i^o \quad , \quad a_{iy}(y_o, t_o) = b_i^o \;.
$$

For example, for rational solutions $u(x,y,t)$ of (2 KdV) of degree 2n, this gives solutions depending on 2n arbitrary parameters.

<u>A2</u>. How to obtain e.g. rational solutions of (2 KdV), e.g. those when $I = n$, $\wp(x) = x^{-2}$? According to the theory of commuting Hamiltonians with $H_{x^{-2}}$ described in [1], [2], we have the following

<u>Rule 2</u> : If we have initial conditions at $y = y_o$, $t = t_o$:

$$
a_i^o(y_o, t_o) = a_i^o \quad , \quad b_i^o(y_o, t_o) = b_i^o \quad ,
$$

then for two flows having Hamiltonians

$$
H_1 = \mathrm{tr}.f(L) \quad ; \quad H_2 = \mathrm{tr}.g(L) \quad ,
$$

for which

$$
(5) \qquad \left\{ \begin{array}{l} a_{iy} = \dfrac{\partial H_1}{\partial b_i} \;; \quad b_{iy} = -\dfrac{\partial H_1}{\partial a_i} \\[1.5em] a_{it} = \dfrac{\partial H_2}{\partial b_i} \;; \quad b_{it} = -\dfrac{\partial H_2}{\partial a_i} \;, \\[1.5em] a_i(y_o, t_o) = a_i^o \;, \quad b_i(y_o, t_o) = b_i^o \;, \quad i = 1, \ldots, n \;. \end{array} \right.
$$

the solutions are defined as eigenvalues of the following matrix

$$\mathcal{U}_{H_1,H_2} = \mathrm{diag}(a_1^o,\dots,a_n^o) + (y-y_o)f'(L)(y_o,t_o) + (t-t_o)g'(L)(y_o,t_o) \ .$$

Now it is clear how to obtain $u(x,y,t)$.

<u>Rule 3</u> : If

$$u(x,y,t) = -2\sum_{i=1}^{n}(x-a_i)^{-2} \ , \quad a_i = a_i(y,t) \ , \quad i=1,\dots,n$$

is such that a_i moves in y according to $H_1 = \mathrm{tr}.f(L)$ and in t according to $H_2 = \mathrm{tr}.g(L)$, then

$$u(x,y,t) = 2\frac{d^2}{dx^2}\ln\chi(x,\mathcal{U}_{H_1,H_2}) \ ,$$

where $\chi(x,\mathcal{U}_{H_1,H_2})$ is a characteristic polynomial of the matrix $\mathcal{U}_{H_1,H_2}$ defined before.

Thus for two-dimensional KdV (2 KdV),

(6) $$u(x,y,t) = 2\frac{d^2}{dx^2}\ln\chi(x,\mathcal{U})$$

for $\mathcal{U} = \mathcal{U}_{J_2,3\alpha J_3}$,

(7) $$\mathcal{U} = \mathrm{diag}[a_i^o] + (y-y_o)L(t_o,y_o) + (t-t_o)3\alpha L^2(t_o,y_o) \ ,$$

where as before

(8) $$L_{ij} = (1-\delta_{ij})\sqrt{-6}(a_i-a_j)^{-1} + \delta_{ij}\, b_j \ .$$

So all rational solutions of (2 KdV) are easily described.

Now there exists a paper of Manakov, Zakharov, Bordag, Its and Matveev [7] where they have written similar formulae for rational solutions with $\alpha = -1$. They deduced these solutions from solutions of Zakharov-Shabat type with degenerate kernel taking in $e^{x_i y + \eta_i t}$ the limit $x_i \to 0$, $\eta_i \to 0$, i.e. considering degenerate case of exponential functions.

They even made some kind of speculation, claiming that their rational solutions are solitons. To do this, they consider the 2n-parametric system of $u(x,y,t)$ of degree 4n (we know that there is a 4n-parametric system of solutions) with poles $a_1,\dots,a_n,\overline{a_1},\dots,\overline{a_n}$ and $\mathrm{Im}(a_i)>0$. Of course such $u(x,y,t)$ are non-singular.

For example, let us write down the simplest non-singular rational solution. We also must mention that in Krichiver's review [9] this solution was written wrongly.
It is. :

(9) $$u(x,y,t) = 2\cdot\frac{\frac{4}{\nu}+2\nu y^2-2(x-3\nu^2t-x_o)}{((x-3\nu^2t-x_o)^2+\nu^2y^2+\frac{2}{\nu})^2}\ .$$

Here $\alpha=-1$ and $v_x = 3\nu^2$, velocity in the x-direction, $v_y=-6\nu$ in the y-direction of "soliton". However, this is an incorrect name for such solutions because they are not in any sense general solutions of (2 KdV) even with some particular kind of initial conditions.

Now we shall proceed to exhibit very interesting elliptic solutions of (2 KdV). The simplest such elliptic solution is of the following form

(10) $$u(x,y,t) = -2\wp(x+By+(3V-3\alpha B)t+x_o)+V\ .$$

We shall now consider the elliptic solution of (2 KdV) with two poles in each part of the lattice. So we take

(11) $$u_2(x,y,t) = -2\wp(x-a_1)-2\wp(x-a_2)\ , \quad \text{with}$$

$a_i = a_i(x,t)$, i=1,2. Then accoridng to the equation of evolution, after some changes,we obtain the following formulae :

$$a_1 = R_oy+C_1t+C_2+\frac{\eta(y+6\alpha R_ot)}{2}$$

$$a_2 = R_oy+C_1t+C_2-\frac{\eta(y+6\alpha R_ot)}{2}$$

where the function $\eta(y+G\alpha R_ot)$ satisfies

$$3\alpha\ \eta_y^2/4 = C_1-3\alpha R_o^2+12\wp(\eta)\ .$$

<u>A3.</u> We have already reduced the problem of finding elliptic solutions of 2-dimensional equations, especially of (2 KdV) to the solution of

ordinary differential equations involving $\wp(z)$, such that

$$\frac{\partial a_i}{\partial y} = \frac{\partial H}{\partial b_i} \quad , \quad \frac{\partial b_i}{\partial y} = - \frac{\partial H}{\partial a_i}$$

for $H = H_\wp$ [1], [2].

We know that this system for finite $|I|$ is completely integrable. But we do not know what is the exact form of these solutions when $|I| > 2$. Here we have the following questions :

1) Whether the general trajectory $a_i(y)$, $b_i(y)$ can be expressed using only elliptic functions (namely using : $|I| - 1$ additional ones ; what are the relations between invariants of these functions and $\wp(x)$, if elliptic representation is possible ?

2) What is topologically the variety of solutions of $H_\wp$?

In the case of degenerate $\wp(x)$, e.g. $\wp(x) = x^{-2}$, $\sin^{-2}x$, $sh^{-2}x$, we have simple formulae for exact solutions (cf. [2]). We have also the result for $|I| = 2$.

Let

$$a_{1yy} = -G\wp'(a_1 - a_2) \quad , \quad a_{2yy} = -G\wp'(a_2 - a_1) \quad .$$

Then $a_1 + a_2 = A_0 y + A_1$ and for $a = a_1 - a_2$,

$$a_y^2 = (-4G)\wp(a) - 4GC \quad ,$$

C is a constant. If we put

$$a(y) = a\left(\frac{y_1}{2\sqrt{-G}}\right) ,$$

and set

$$v = \wp(a) \quad ,$$

then for

$$\xi = -(7C^3 - \frac{g_2}{4} C - \frac{g_3}{4})(\wp(a) + C)^{-1} + 3C^2 - \frac{g_2}{12} \quad ,$$

we have the elliptic representation for ξ, ξ satisfying

$$\xi_{y_1}^2 = 4\xi^2 - G_2\xi - G_3 \quad ,$$

where

$$G_2 = -3g_3C + \frac{g_2^2}{12} + g_2C^2 \quad ;$$

$$G_3 = g_3C^3 - \frac{g_2^2}{6}C^2 + \frac{g_2g_3}{4}C - \frac{g_3^2}{4} + \frac{g_2^2}{6^3} \quad .$$

If $\wp_1(z)$ is an elliptic function, satisfying

$$\wp_{1z}^2 = 4\wp_1^3 - G_2\wp_1 - G_3 \quad ,$$

then

$$\xi = \wp_1(2\sqrt{-G}\, y + y_o) \quad .$$

It is very important to mention that for a modular invariant $j = j(\wp)$,

$$j = \frac{g_2^3}{g_2^3 - 27g_3^2} \quad , \quad \text{if } \Delta \neq 0 \quad ,$$

the modular invariant $J = J(\wp_1)$ can be choosen (with variation of C) arbitrarily. Thus the complete solution of $H_\wp$ involves 2 functions the $\wp(z)$ and a new $\wp_1(z)$ with arbitrary invariant.

A4. We turn back to the solution (11) of (2 KdV).
Now we use the solution of $H_\wp$ for $|I| = 2$. We put :

$$\xi(y + 6\alpha R_o t) = -(7C^3 - \frac{g_2}{4}C - \frac{g_3}{4})\frac{1}{\wp(\eta(y + 6\alpha R_o t)) + C} + 3C^2 - \frac{g_2}{12} \quad ,$$

where

$$C = \frac{C_1}{12} - \frac{R_o^2\alpha}{4} \quad .$$

Then we have :

$$\xi(y + 6\alpha R_o t) = \wp_1(\frac{4}{\sqrt{\alpha}}(y + 6\alpha R_o t) + C_3) \quad .$$

Here the function $\wp_1$ satisfy $\wp_1'^2 = 4\wp_1^3 - G_2\wp_1 - G_3$ and G_2, G_3 were defined

This gives exact formulae for $u_2(x, y, t)$ in (11), depending on $g_2, g_3, C_1, C_2, R_0, C_3$. As we already mentioned before this leads to non-simple abelian variety

$$(\wp(x), \wp'(x), \wp_1(x), \wp_1'(x))$$

of dimension 2 and any non-simple abelian variety of dimension 2 can appear as a manifold for solutions of (2 KdV).

The solutions $u_2(x, y, t)$ are non-singular and especially interesting if $\wp(z)$ is with complex multiplication on $\sqrt{-1}$, i.e., [8] : $g_3 = 0$. In this case the a_i can be chosen such that $a_1 = \sqrt{-1}\, a$, $a_2 = -\sqrt{-1}\, a$ and $u_2(x, y, t)$ is bounded in the (x, y)-plane.

Part B - Meromorphic solutions of other two-dimensional equations.

The very general idea of pole expansion can be used in order to obtain information even about algebraic properties of equation. We shall use the pole expansion in this part for different two-dimensional systems of the form (2 dim) with L_1, L_2 of order 2 and 4. In fact, in this case, it is necessary to change the form of equation (2 dim) to obtain our conjecture for elliptic and meromorphic solutions.

B1. Elliptic solutions for (2 dim) in the case $n = 2, m = 4$.

The (2 KdV) corresponds to the case $\frac{dL_2}{dt} - \frac{dL_3}{dy} = [L_2, L_3]$ and we get a good description in terms of Hamiltonians J_2, J_3. The next case (a very non-trivial one) is the system

$$\frac{dL_2}{dy} - \frac{dL_4}{dt} = [L_2, L_4]$$

giving equations for the coefficients of the operators of degree $2 : L_2$ and $4 : L_4$. If we consider them as usual

$$L_2 = \frac{d^2}{dx^2} + u \quad , \quad L_4 = \frac{d^4}{dx^4} + u_2\frac{d^2}{dx^2} + u_1 \frac{d}{dx} + u_0 \quad ,$$

then we have the following system for u_0, u_1, u_2 :

$$u = u_2/2 \quad \text{and}$$

$$(L) \quad \begin{aligned} u_{2t} &= 2u_{2xx} - 2u_{1x} \quad ; \\ u_{1t} &= -u_{1xx} + 2u_{2xxx} + u_2u_{2x} - 2u_{0x} \quad ; \\ u_{0t} - \frac{u_{2y}}{2} &= 1/2\ u_{2xxxx} + 1/2\ u_2u_{2xx} - u_{0xx} + 1/2\ u_{2x}u_1 \end{aligned}$$

We can transform this system into a more convenient form.

If we put

$$u_1 = u_{2x} + \frac{\hat{u}_1}{2} \ ;$$

$$u_0 = 1/3\ u_{2xx} + \frac{\hat{u}_{1x}}{4} + \frac{\hat{u}_0}{4} \ ;$$

then the system (L) takes the form

$$(\hat{L}) \quad \begin{aligned} u_{2t} &= -\hat{u}_{1x} \quad ; \\ \hat{u}_{1t} &= 2/3\ u_{2xxx} + 2u_2u_{2x} - \hat{u}_{0x} \quad ; \\ \hat{u}_{0t} - 2u_{2y} &= 1/3\ \hat{u}_{1xxx} + \hat{u}_1u_{2x} \quad . \end{aligned}$$

Now we consider general purely Weirstrass elliptic solution of $(\hat{L})$. If functions $\hat{u}_0, \hat{u}_1, u_2$ are purely Weirstrass elliptic solutions of $(\hat{L})$, they can have the form

(12) $$u_2 = -4\Sigma_{i \in I}\wp(x - a_i) \quad ;$$

(13) $$\hat{u}_1 = -4\Sigma_{i \in I}\, a_{it}\wp(x - a_i) \quad ;$$

(14) $$\hat{u}_0 = \Sigma_{i \in I}(-4a_{it}^2 + 32\Sigma_{j \neq i}\wp(a_i - a_j))\wp(x-a_i).$$

We must mention that (12)-(14) is not the most general form of meromorphic (and even elliptic) solutions of (L), but it is the general form for purely Weirstrass elliptic solutions.

Solutions (12)-(14) are (a little) similar to those in chain system (C) from [2]. Really, we have in order (12)-(14) to satisfy $(\hat{L})$ the following condition:

(15) $a_{itt} = 8\Sigma_{j\neq i} \wp'(a_i - a_j): i \in I$.

However, it can easily be shown from $(\hat{L})$ that (12)-(14) satisfy $(\hat{L})$,

<u>if</u> $a_{it} = a_{jt}$ for all $i, j, \in I$;

Thus, for all the elliptic solutions of $(\hat{L})$ the velocity of poles in t-direction is the same for all the poles.

The conditions for satisfaction of $(\hat{L})$ for (12)-(14) the following:

$$\Sigma_{j\neq i} \wp'(a_i - a_j) = 0: i \in I \;;$$
$$a_{it} = V: i \in I \;.$$

This is the condition grad $(J_2 - VJ_1) = 0$. In the y-direction, a_i moves according to J_4 but linearly in t .

Even for y-independent (L) this gives N-soliton solutions which have the same velocity not connected with initial conditions. The N-soliton remains thus always solutions <u>connected</u> with a strange interaction of the 1-solitons inside the N-soliton. The simplest 1-soliton of (L) , y-independent, is the following obtained firstly by Manin [10] :

$$\mu_1 = 2a^2 \frac{\text{ch } 2ax - \cos 2ax - 2\sin 4a^2t}{\text{ch } 2ax + \cos 2ax + 2\cos 4a^2t} ;$$

$$\mu_2 = -2a \frac{\text{sh } 2ax - \sin 2ax}{\text{ch } 2ax + \cos 2ax + 2\cos 4a^2t} ;$$

$$u_2 = -4\mu_{2x} , \quad u_1 = -6\mu_{2xx} - 4\mu_{1x} + 4\mu_2\mu_{2x} ;$$

$$u_0 = -4\mu_{2xxx} - 6\mu_{1xx} + 8(\mu_{2x})^2 + 4(\mu_1\mu_2)_x + 6\mu_2\mu_{2xx} - 4\mu_{2x}\mu_2^2 ,$$

or we can change x by $x + c$.

These solutions have the period $\frac{\pi}{2a^2}$, but when $\cos 4a^2t = -1$, we have "explosion" at $x = 0$.

B2. New systems of equations connected with integrals J_2 and J_4

It is clear that because of the strange behavior of elliptic Weierstrass solutions of (L) , even for y-independent case, it is necessary to correct (L) .

We correct (L) using the system of equations (C_0) - (C_2) from [2] in such a way that it has good elliptic (as well as meromorphic) solutions with poles moving according to J_4 in y direction and according to J_2 in t .

Here is the new system:

$$(16) \quad \begin{cases} u_{2t} = - \hat{u}_{1x} ; \\ \hat{u}_{1t} = -\hat{u}_{0x} - \frac{G}{12} u_{2xxx} - \frac{G}{4} u_2 u_{2x} ; \\ \hat{u}_{0t} - u_{2y} = - \frac{G}{6} \hat{u}_{1xxx} - \frac{G}{4} u_{2x}\hat{u}_1 - \frac{G}{4} u_2\hat{u}_{1x} ; \end{cases}$$

The meromorphic solutions of this system are of the same type:

$$\hat{u}_2 = -4 \sum_{i\in I} \wp(x - a_i) \quad ;$$

$$\hat{u}_1 = -4 \sum_{i\in I} a_{it} \wp(x - a_i) \quad ;$$

$$\hat{u}_0 = -4 \sum_{i\in I} (a_{it}^2 + G \sum_{j\neq i} \wp(a_i - a_j)) \wp(x - a_i) \quad ,$$

but the a_i moves in y according to J_4 and in t according to J_2, for any G.

Now we will show that the systems $(\hat{L})$ and its correct form (16) are in fact equivalent. So it is possible to obtain an analogue of elliptic solutions even for $(\hat{L})$.

To do this, we will first investigate the general form of meromorphic solutions for $(\hat{L})$, not necessarily pure Weirstrass elliptic.

B3. <u>General meromorphic solutions of</u> $(\hat{L})$ <u>are connected with</u> J_2 <u>and</u> J_4.

The general meromorphic solutions of $(\hat{L})$ are really connected with Hamiltonians J_2 and J_4, but not with $G = -8$ (as for pure Weierstrass elliptic solutions) and with $G = -4$. However, in this case, u_0 contains the poles of first order.

Now let's find <u>all</u> meromorphic solutions of $(\hat{L})$; they have rather a simple nature, connected with J_2 and J_4. We must have, for an arbitrary meromorphic solution $\hat{u}_0, \hat{u}_1, u_2$ of $(\hat{L})$, the following representation:

(17) $$u_2 = -4 \sum_{i\in I} (x - a_i)^{-2} \quad ;$$

(18) $$u_1 = -4 \sum_{i\in I} a_{it} (x - a_i)^{-2}$$

$$(19)\quad \hat{u}_0 = -4 \sum_{i\in I} (a^2_{it} - 8 \sum_{j\neq i} (a_i - a_j)^{-2}) \times (x - a_i)^{-2}$$
$$-4 \sum_{i\in I} (a_{itt} + 16 \sum_{j\neq i} (a_i - a_j)^{-3}) \times (x - a_i)^{-1} .$$

The general functions u_2, $\hat{u}_1$, $\hat{u}_0$, (17)-(19), as functions from x, t satisfy $(\hat{L})$ if

$$(20)\quad a_{itt} = -8 \sum_{j\neq 1} (a_i - a_j)^{-3} : \quad i \in I .$$

i.e., a_i as functions of t satisfy H_{x-2} for $G = -4$. Similarly $a_i(y, t)$ as a function of y is the solution corresponding to J_4 , also for $(x) = x^{-2}$ and $G = -4$.

Thus, rational solutions of (L) really satisfy our conjecture.

Now we will write $(\hat{L})$ in a more traditional form (cf. (C) in [2] and expressions for $\hat{u}_k(x; t)$ in [2]). To do this, we remember that (17)-(19)is the solution of $(\hat{L})$ if (20) is satisfied:

$$a_{itt} = -8 \sum_{j\neq i} (a_i - a_j)^{-3} .$$

Thus

$$\hat{u}_0 = -4 \sum_{i\in I} (a^2_{it} - 8 \sum_{j\neq i} (a_i - a_j)^{-2}) \times (x - a_i)^{-2}$$
$$-4 \sum_{i\in I} (8 \sum_{j\neq i} (a_i - a_j)^{-3}) (x - a_i)^{-1} .$$

We put

$$(21)\quad \hat{\hat{u}}_0 = -4 \sum_{i\in I} (a^2_{it} - 4 \sum_{j\neq i} (a_i - a_j)^{-2}) (x - a_i)^{-2} .$$

From (21)and (17)-(19) it follows

$$\hat{u}_0 = \hat{\hat{u}}_0 - 4 \sum_{i\in I} \{-4 \sum_{j\neq i} (a_i - a_j)^{-2}\} (x - a_i)^{-2}$$
$$-4 \sum_{i\in I} \{8 \sum_{j\neq i} (a_i - a_j)^{-3}\} (x - a_i)^{-1} ;$$

Now for $\wp(x) = x^{-2}$ we have:

$$(\hat{u}_0 - \hat{\hat{u}}_0)_x = -4 \sum_{i\in I} \{-4 \sum_{j\neq i} \wp(a_i - a_j)\} \wp'(x - a_i) +$$

$$+ 4 \sum_{i\in I} \{-4 \sum_{j\neq i} \wp'(a_i - a_j)\} \wp(x - a_i) \quad .$$

By the law of addition for $\wp(x)$ we have

$$(\hat{u}_0 - \hat{\hat{u}}_0)_x = u_2 u_{2x} - 16 \sum_{i\in I} \wp(x - a_i)\, \wp'(x - a_i) =$$

$$= u_2 u_{2x} + \frac{1}{3} u_{2xxx} \quad ;$$

Thus,

$$\hat{u}_{0x} = \hat{\hat{u}}_{0x} + u_2 u_{2x} + \frac{1}{3} u_{2xxx} \quad ;$$

Now we can rewrite $\widehat{(L)}$ in the following form:

$$u_{2t} = -\hat{u}_{1x}$$

$$\hat{u}_{1t} = \frac{1}{3} u_{2xxx} + u_2 u_{2x} - \hat{\hat{u}}_{0x} \quad ;$$

$$\hat{\hat{u}}_{0xt} - 2u_{2xy} = \frac{1}{3} \hat{u}_{1xxxx} + \hat{u}_{1x} u_{2x} + \hat{u}_1 u_{2xx} - u_{2t} u_{2x} -$$

$$- u_2 u_{2xt} - \frac{1}{3} u_{2txxx} =$$

$$= \frac{2}{3} \hat{u}_{1xxxx} + 2\hat{u}_{1x} u_{2x} + \hat{u}_1 u_{2xx} + \hat{u}_{1xx} u_2 =$$

$$= (\frac{2}{3} \hat{u}_{1xxx} + (\hat{u}_1 u_2)_x)_x \quad ;$$

Now we define $\overline{u}_0(x, t)$ by

$$\overline{u}_0 = \hat{u}_0 - \frac{1}{2} u_2^2 - \frac{1}{3} u_{2xx} \ .$$

Then from $(\hat{L})$ we obtain

$$(\overline{L}) \quad \begin{cases} u_{2t} = -\hat{u}_{1x} \ ; \\ \hat{u}_{1t} = -\overline{u}_{0x} + \frac{1}{3} u_{2xxx} + u_2 u_{2x} \ ; \\ \overline{u}_{0t} - 2u_{2y} = \frac{2}{3} \hat{u}_{1xxx} + \hat{u}_{1x} u_2 + \hat{u}_1 u_{2x} \ . \end{cases}$$

Then, in view of the given supra, this system is equivalent to $(\hat{L})$.

General meromorphic solutions of $(\overline{L})$ have the form (17), (18), (21), with $\hat{\hat{u}}_0 = \overline{u}_0$, where (20) is satisfied and $a_i(y, t)$ moves according to J_2 on t and according to J_4 on y .

But the system $(\overline{L})$ is equivalent also to (16) by B2, if we change y by $2y$.

Thus, after some transformations, we find that (L) has elliptic solutions and in this case also, the conjecture is true.

Conclusion: In fact system (C) from the first part [2] of the paper gives us the possibility to verify the conjecture for min{n, m} = 2, for min {n, m} > 2 besides motion corresponding to J_n and J_m (as we suppose) there can appear some new many-particle systems.

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