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S E M I N A I R E   S U R

L'ANALYSE DIOPHANTIENNE ET SES APPLICATIONS

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MANY-PARTICLE COMPLETELY INTEGRABLE SYSTEMS AND POLES OF MEROMORPHIC SOLUTIONS OF NON-LINEAR EVOLUTION EQUATIONS

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§ 1. FINITE DIMENSIONAL MANY-PARTICLE COMPLETELY INTEGRABLE SYSTEMS.

Lax' procedure gives the first examples of the many particle completely integrable systems. The most known example in the Toda lattice [9] :

$$\ddot{x}_n = e^{x_{n+1}-x_n} - e^{x_n-x_{n-1}} \quad . \tag{1}$$

This system is equivalent to the Lax representation $\frac{dL}{dt} = [A,L]$, where the matrices L and A are the following

$$L_{mn} = i\sqrt{C_n}\,\delta_{u,m+1} - i\sqrt{C_m}\,\delta_{n+1,m} + v_n\,\delta_{mn} \quad ,$$

$$A_{mn} = \frac{i}{2}\left(\sqrt{C_n}\,\delta_{n,m+1} + \sqrt{C}\,\delta_{n+1,m}\right) \quad ,$$

where $C_n = e^{x_n - x_{n-1}}$ and $v_n = \dot{x}_n$.

In fact, the infinite Toda lattice (1) can be considered just in the same way as the KdV equation and in fact, from the topological point of view, it is the same kind of equation.

The system (1) as well as the KdV or non-linear Schrödinger equations arises from the moduli of hyperelliptic curves and there exists a natural analogue of finite band solutions and these solutions can be represented using the same θ-functions on the Jacobian varieties of hyperelliptic curves. Naturally, with the system (1) are also associated higher analogues of the Toda lattice.

In fact these systems arise in some physical problems and, for example, are related to the well-known Fermi-Pasta-Ulam chains [12]. These chains are obtained from (1) by taking only 3 terms in the expansion for the exponent.

There is however mechanical systems of complete different character having no topological relations to the Burchnall-Chaundy-Lax [6] procedure. We mean the system of finitely (or infinitely) many particles interacting via the potential $G\wp(x)$ (where $\wp(x)$ is a Weierstrass elliptic function [10]). In the degenerate case we obtain a system of particles $x_i = x_i(t)$ interacting via the Jacobi potential x^{-2}. Thus there occurs an Hamiltonian of the form

(2) $$H_{\wp} = \frac{1}{2} \sum_{i\in I} \dot{x}_i^2 + G \sum_{i\neq j} \wp(x_i - x_j)$$

or

(3) $$H = \frac{1}{2} \sum_{j=1}^{n} y_j^2 + \sum_{1\leq j\leq k\leq n} (x_j - x_k)^{-2} \quad .$$

It was a surprising result obtained by Moser [2] for finite n in (3) and by Calogero [3] for finite I in (2) that the corresponding systems possess Lax representation

(u) $$\frac{dL}{dt} = [A,L]$$

for finite matrices A and L and, so (as n eigenvalues of L are conserved) possess n first integrals. The form of A and L is very easy for (3) :

$$L_{ij} = (1-\delta_{ij})\sqrt{-1}(x_i - x_j)^{-1} + \delta_{ij}\, y_i \quad ;$$

$$A_{ij} = (1-\delta_{ij})\sqrt{-1}(x_i - x_j)^{-2} - \delta_{ij} \sum_{j\neq j} (x_k - x_j)^{-2}$$

For the case if (2) the matrix $L = (L_{ij})$, $i,j \in I$ have the form

$$L_{ij} = (1-\delta_{ij})\sqrt{-G}\,\alpha(x_i - x_j) + \delta_{ij}\,\dot{x}_i \quad ,$$

where $\alpha^2(x) = \wp(x)$.

Then, of course, the quantities $J_n = \frac{1}{n}\,tr(L^n)$, $n \geq 1$ are the first integrals of $H_{\wp}$. Moreover it is proved that the J_n are involutive and that they are sums of polynomials in $\dot{x}_i$, $\wp(x_i - x_j)$, G with rational coefficients. The form of the first terms of J_n is the following

$$J_n = \frac{1}{n} \sum_{i\in I} \dot{x}_i^2 + G \sum_{i\neq j} (\dot{x}_i^{n-2} + \dot{x}_i^{n-3}\, x_j + \ldots + \dot{x}_j^{n-2})\wp(x_i - x_j) + \ldots \quad .$$

For the Hamiltonians (2) and (3) the exact formulae for solutions can be given for finite $|I| = n$. For $\wp(x) = x^{-2}$ all the solutions $x_i = x_i(t)$ for the Hamiltonian H and for the Hamiltonian J_n are algebraic functions. In fact the x_i are the roots of some polynomial $P(x,t)$ having degree n on x. These solutions can be easily obtained using the matrix L for $\wp(x) = x^{-2}$. For the matrix [1] :

$$M(t) = \mathrm{diag}(x_1(t_o),\ldots,x_n(t_o)) + L(t_o)(t-t_o)$$

the eigenvalues $x_i(t)$ are solutions with given initial values $x_i(t_o)$, $\dot{x}_i(t_o)$, corresponding to the Hamiltonian $H = 1/2\,\mathrm{tr}\,L^2$. For the Hamiltonian $J_n = 1/n\,\mathrm{tr}\,L^n$, in $M(t)$, L must be replaced by L^{n-1}.

Because of this rational character it is unclear how the system $H_{\wp}$ is connected with the usual one. But the connection with the elliptic curve remains : in fact the existence of Lax representation for (2) is simply equivalent to the functional equation defining the $\wp(x)$, namely to the law of addition for $\wp(x)$ [1]. Nevertheless it was very interesting to show that in fact there is a close relation between many particle systems (2)-(3) and the solutions of known completely integrable equations. This connection lies in the so-called pole interpretation. The idea of such pole interpretations were first proposed by Kruskal [4], but at that time they were not taken into consideration.

The idea of this method is the following : to consider solutions $u(x,t)$ of non-linear partial differential equations as meromorphic functions in the complex x-plane and to investigate the motion of the poles $x_i = x_i(t)$ as particles with self-consistent potential. In fact for all the classical non-linear completely integrable systems the pole interpretation (or zero-interpretation in the sense that we consider entire functions, entire solutions) leads either to systems (2), (3) or to Hamiltonians J_n. In this situation there can arise systems of differential equations with some constraints.

The procedure of pole (zero) interpretation is of particular interest as the process of finding all the rational solutions [5].

§ 2. POLE INTERPRETATION OF COMPLETELY INTEGRABLE EQUATIONS OF EVOLUTION

The most intriguing feature of exact solutions, of non-linear differential equations, that were found during these years (1922-1931 [6] and 1967-1977 [4], [13]) is the meromorphic character of the solutions. Moreover the meromorphic functions that can appear as solutions are of a particular nature, because almost all exact formulae are obtained from multidimensional (or even infinitedimensional) θ-functions.

So it is natural to start the investigation of solutions from meromorphic ones. While studying meromorphic solutions $u(x,t)$ or

$u(x,y,t)$ as functions of x we suppose that at least for some non-trivial interval $[0,t_o]$ or $[0,y_o]\times[0,t_o]$, the function $u(x,y)$ or $u(x,y,t)$ is meromorphic as function of x for $t\in[0,t_o]$ or $(y,t)\in[0,y_o]\times[0,t_o]$.

It is necessary to mention that the meromorphy or even rationality of the initial condition $u(x,0)$ or $u(x,y,0)$ does not imply the meromorphy of $u(x,t)$ $(u(x,y,t))$ for $t\gtrless 0$. Even for the KdV equation if the solution $u(x,t)$ is meromorphic for $t\in[0,t_o]$, $t_o>0$, then all the poles $x_i=x_i(t_1)$ of $u(x,t)$ for any given $t_1\in[0,t_o]$ are of second order and of particular type [i.e. $\sum_{j\neq i}(x_i-x_j)^{-3}=0$ for all $x_i=x_i(t_1)$ and any $t_1\in[0,t_o]$].

In particular this shows why the "algebraic inverse scattering method" cannot give complete solution of completely integrable equations : it is necessary to consider also non-meromorphic $u(x,t)$. The main result of pole interpretation for the KdV equation $u_t=12cuu_x-cu_{xxx}$ is the following :

Theorem 1 : If $u(x,t)$ is a solution of the KdV equation $u_t=12cuu_x-cu_{xxx}$, which is meromorphic on x in a complex plane for all t (or for $t\in[t_o,t_1]$, $t_1>t_o$), then

$$u(x,t)=\sum_{i\in I}(x-a_i)^{-2}\quad,\quad a_i=a_i(t)\quad;\quad i\in I\ . \tag{5}$$

Meromorphic functions having the form (5) satisfy $u_t=12cuu_x-cu_{xxx}$ if and only if

$$\dot a_i=-12c\sum_{j\neq i,\,j\in I}(a_i-a_j)^{-2}\quad;\quad i\in I\ ,$$

$$\sum_{j\neq i,\,j\in I}(a_i-a_j)^{-3}=0\quad;\quad i\in I\ .$$

Analogous results are valid for the function $\wp(x)$.

Proposition 2 : For $u(x,t)=\sum_{i\in I}\wp(x-a_i)$, $a_i=a_i(t)$, $i\in I$, the fulfilment of the KdV is equivalent to the fulfilment of the system

$$\dot a_i=-12c\sum_{j\neq i,\,j\in I}\wp(a_i-a_j)\quad,\quad i\in I\ , \tag{6}$$

with restrictions

$$\sum_{j\neq i,\,j\in I}\wp'(a_i-a_j)=0\quad,\quad i\in I\ . \tag{7}$$

It is very important to note that for $G = -12c$ the system (6) describes the Hamiltonian flow induced by J_3 with the restrictions corresponding to the set grad $J_2 \equiv$ grad $H_\wp = 0$. Thus the manifold (7) is invariant for (6) as all J_n are commuting.

Also the pole interpretation for the KdV is connected not only with the Hamiltonian systems corresponding to the potential x^{-2}, but is also connected with Maxwell potential x^{-4} :

<u>Proposition 3</u> : If $u(x,t) = \sum_{i\in I} \wp(x - a_i)$ is a solution of the KdV $u_t = 12cuu_x - cu_{xxx}$, then the $a_i(t)$ are the trajectories of the Hamiltonian system with

$$H_o = \frac{1}{2} \sum_{i\in I} \dot{x}_i^2 + 72c^2 \sum_{i\neq j} \wp^2(x_i - x_j) \quad .$$

The complete description of the meromorphic solutions of KdV having the form $u(x,t) = \sum_{i\in I} \wp(x - a_i)$ for finite I can be given for the finite set I

$$M = \{(a_i) : \text{grad } J_2 = 0\}$$

is non-void, if and only if $|I| = \frac{n(n+1)}{2}$. In this situation the corresponding M has the dimension n and is an invariant manifold for all J_m. In fact to this clase corresponds exactly a n-lacunary potential $n(n+1)\wp(x)$, which arises in the theory of the Lamé equation. Really, all the potentials of the form $u(x) = \sum_{i\in I} \wp(x - a_i)$ for such I and $(a_i) \in M$ are periodical n-band potential and, conversely, all the doubly periodical n-band potentials have the form $\sum_{i\in I} \wp(x - a_i)$ for some $(a_i) \in M$. The last result belongs to Moser and McKean [5].

Moreover the answer to the question posed in [5], as to whether the pole interpretation for higher order KdV equations [13], [14], is connected with J_{2h+1}, can be given. In fact, the evolution of system of poles of solutions of the n-th order KdV equation [13], [14] is governed by J_{2n+1} with constraints (7) : grad $H_\wp = 0$. For example, for the second KdV equation

$$u_t + 10uu_{xxx} + 20u_x u_{xx} + 30u^2 u_x + u_{xxxxx} = 0 \quad ,$$

and $u(x,t) = -2 \sum_{i\in I} \wp(x-a_i)$ we obtain the many-body problem related with J_5 :

$$\dot{a}_i = 120\{(\sum_{j\neq i} \wp(a_i - a_j))^2 + \sum_{j\neq i} \wp^2(a_i - a_j)\} \quad , \quad i \in I$$

with restrictions (7).

In this context instead of a clever and rather old conjecture of Moser [5], Khrichive [7] had made a wrong conjecture about the coincidence of the action J_i on M with the motion of poles of the i-th KdV which corresponds to J_{2i+1}.

Generally speaking, in this review [7], interesting to some extent became of the reproduction of the results on the algebra of differential equations, obtained as we know now in general form 50 years ago by Burchnall and Chaundry [6], Krichiver makes some claim for two-dimensional KdV equation. But for the 2-dimension KdV equation as well as for 1-dimensional the motion of the poles was completely described in § 10 of the paper [1]

Another interesting class of equations with pole interpretation related to (4) is the class of higher-order Burgers-Hopf (BH) equations $u_t = BH_n[u]$ (for $n = 2$ -the ordinary BH equation [11]). These equations are generated as follows :

$$BH_n[u] = \frac{d}{dx} C_n[u]$$

and

$$C_{n+1}[u] = \frac{d}{dx} C_n[u] + u \cdot C_n[u] \quad , \quad C_o[u] = 1 \quad .$$

Then $C_n[u]$ and so $BH_n[u]$ are the polynomials in $u, u_x, \dots$. The first terms are the following :

$$BH_1[u] = u_x \quad ; \quad BH_2[u] = 2uu_x + u_{xx} \quad ,$$

$$BH_3[u] = 3u^2u_x + 3uu_{xx} + 3u_x^2 + u_{xxx} \quad , \quad \dots$$

In the pole interpretation, corresponding systems are without any constraints and we have the general result :

<u>Theorem 4</u> : 1) For meromorphic functions $u(x,t) = \sum_{i\in I} (x - a_i)^{-1}$ the fulfilment of the equation

$$u_t = BH_n[u]$$

is equivalent to the fulfilment of the system

$$(8) \qquad \dot{a}_i = n! \sum_{\{j_1,\dots,j_{n-1}\}\not\ni i} (a_i - a_{j_1})^{-1} \dots (a_i - a_{j_{n-1}})^{-1} \quad , \quad i \in I \quad .$$

2) Any system (8) is embedded into the system with Hamiltonian J_n (4). Moreover for finite I, the trajectories of J_n on which all the integrals J_n vanish, $J_m = 0$, are precisely the solutions of the system (8).

The pole interpretation for the modified KdV equation, nonlinear Schrödinger and their higher analogues are also connected with the Hamiltonians (2)-(3).

2. In general the idea of the pole interpretation and the establishment of a connection with the Hamiltonian $H_{\wp}$ can be described along the lines of the following general scheme [8].

We consider the following special class of meromorphic functions, residues and poles of which are expressed in terms of the variables $(x_i, \dot{x}_i)$ of $H_{\wp}$:

$$(9) \qquad u_k(z,t) = \sum_{i\in I} \frac{\partial \mathfrak{J}_{j+1}}{\partial \dot{x}_i} \wp(z - x_i) \quad , \quad k = 0,1,2,\dots \quad .$$

There exists a special sequence of differential equations connected with (9) :

<u>Theorem 5</u> : Let $u_{k,\underbrace{x,\dots,x}_{m}}$ have the weight k+m+2. The there exist polynomials $\Omega_k(u_o,\dots,u_{k-1})$ in $u_o, u_{o,x},\dots,u_1,u_{1x},\dots,u_{k-1},\dots,u_{k-1,xx\dots},\dots$ of degree two and having all the monomials of weight k+3 such that the system of equations

$$(C)_k \qquad u_{k,t} + u_{k+1,x} + \frac{d}{dx}\Omega_k(u_o,\dots,u_{k-1}) = 0 \quad , \quad k = 0,1,\dots$$

satisfies the following properties :

1) the functions u_k satisfy (C) if and only if $x_i = x_i(t)$ move according to $H_\wp$;

2) if $u_k(x,t)$ satisfy (C) and are meromorphic functions with poles of order 2, then $u_k = u_k$.

Here are the first few Ω_k :

$$\Omega_o = 0 \quad , \quad \Omega_1 = -\frac{G}{2} u_o^2 + \frac{G}{12} u_{oxx} \quad , \quad \Omega_2 = -Gu_o u_1 + \frac{G}{6} u_{1xx} \quad ,$$

$$\Omega_3 = -\frac{G}{2} u_1^2 - Gu_o u_2 - \frac{G^2}{8} u_{ox}^2 - \frac{G^2}{12} u_o u_{oxx} + \frac{G^2}{120} u_{oxxxx} + \frac{G}{4} u_{2xx} \, , \, \ldots$$

The system (C) possesses an important property : if we put $u_n \equiv 0$, then all the u_m, $m>n$, can be obtained from (C) as polynomials in $u_o, u_{ox}, \ldots, u_1, u_{1x}, \ldots, u_{n-1}, \ldots, u_{n-1,xxx\ldots}, \ldots$. Because all the quantities

$$\int_{-\infty}^{\infty} u_m \, dx$$

are the first integrals of (C), this shows

Corollary 6 : If in the system (C) we put $u_n \equiv 0$, then the system $(C)_1 - (C)_n$ possesses infinitely many polynomial conservation laws.

The first such non-trivial system coincides with the Boussinesq equation [16] $u_{tt} + (u^2)_{xx} + u_{xxxx} = 0$. In general the system (C) describes one of the scheme of approximations of two-dimensional shallow water equation [15].

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Erratum "Many-particle completely integrable systems and poles of meromorphic solutions of non-linear equations", D.V. Choodnovsky, 21.1.78 (1

Page	Ligne	Lire
1	9	$A_{mn} = \frac{i}{2} (\sqrt{C_n}\ \delta_{n,m+1} + \sqrt{C_m}\ \delta_{n+1,m})$
2	3	$H = \frac{1}{2} \sum_{j=1}^{n} y_j^2 + \sum_{1 \leq j < k \leq n} (x_j - x_k)^{-2}$.
2	7	Supprimer (4).
2	12	$A_{ij} = (1 - \delta_{ij})\sqrt{-1}(x_i - x_j)^{-2} - \delta_{ij} \sum_{j \neq k} (x_k - x_j)^{-2} \sqrt{-1}$.
2	21	(4) $J_n = \frac{1}{n} \sum_{i \in I} \dot{x}_i^n + G \sum_{i \neq j} (\dot{x}_i^{n-2} + \dot{x}_i^{n-3}\ \dot{x}_j + \ldots + \dot{x}_j^{n-2})$ $\wp(x_i - x_j) + \ldots$.
7	5	$-\dot{a}_i = n! \sum_{\{j_1, \cdots, j_{n-1}\} \not\ni i} (a_i - a_{j_1})^{1} \ldots (a_i - a_{j_{n-1}})^{-1}$, i
7	19	$\widehat{u_k}(z,t) = \sum_{i \in I} \frac{\partial J_{k+1}}{\partial \dot{x}_i} \wp(z - x_i)$, $k = 0,1,2,\ldots$.
8	15	$(C)_1 - (C)_{n-1}$ ...
9	15	G.B. Whitham.
