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SEMINAIRE SUR

LES EQUATIONS NON-LINEAIRES

- I -

MULTISOLITON FORMULA FOR COMPLETELY INTEGRABLE TWO-DIMENSIONAL SYSTEMS.

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0. For general two-dimensional completely integrable systems, we give the exact formulae for multisoliton-type solutions. These formulae are obtained algebraically from solutions of two linear partial differential equations $\varphi_t = \varphi_{\underbrace{x\ldots x}_{n}}$, $\varphi_y = \varphi_{\underbrace{x\ldots x}_{m}}$. The formulae are similar to second logarithmic derivatives of Fregholm determinant formulae occurring frequently in solutions arising from inverse scattering method and Gelfand-Levitan type equations. Our formula contains all previously known starting from the first multisoliton formula of Kay and Moses [1] in 1950, as well as [2], [3], [4]. Our formalism resembles Crum's investigation [5] on removing of bound states for the Schrodinger equation. It should be mentioned that solutions obtained here correspond to the continuous spectrum as well as to the discrete.

1. We consider the most general form of two-dimensional nonlinear integrable partial differential equations arising as a condition of commutativity of two linear operators. These systems are equations for functions $u(x, y, t)$ with two space and one time variable. Such systems of the Zaharov-Shabat [3] or Lax [4] type, have the following operator representation:

$$[L_n - \frac{\partial}{\partial t}, L_m - \frac{\partial}{\partial y}] = 0$$

for linear operators L_n, L_m in $\frac{\partial}{\partial x}$, as a system on non-linear equations in $\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial t}$ for undetermined coefficients of L_n, L_m .

In other words, we define a two-dimensional completely integrable system as

$$(1) \qquad \frac{dL_m}{dt} - \frac{dL_n}{dy} = [L_n, L_m] \quad ,$$

where

$$(2) \qquad L_n = \sum_{i=0}^{n} u_i \frac{d^i}{dx^i} \, , \quad L_m = \sum_{j=0}^{m} v_j \frac{d^j}{dx^j} \quad ,$$

$$u_n = v_m = 1 \quad , \quad u_{n-1} = v_{m-1} = 0 \quad .$$

We give the set of solutions for (1), (2) based on solutions of linear partial differential equations or, equivalently, Burgers-Hopf (BH) equation [6]. The simplest solutions of this kind were already constructed by one of us in [7].

Basic (multisoliton) solutions can be constructed using higher Burgers-Hopf equations $w_t = BH_n[w]$: $n = 1, 2, \ldots,$ that were defined in [6], [7].

These equations can be generated inductively as

$$BH_n[w] = \frac{\partial}{\partial x} C_n[w] \quad \text{and}$$

$$C_{n+1}[w] = u \cdot C_n[w] + \frac{\partial}{\partial x} C_n[w], \quad C_0[w] = 1 \quad .$$

However, the main property of higher Burgers-Hopf equations is that it can be linearized using Hopf-Cole substitution $w = \frac{\partial}{\partial x} \log \varphi$:

<u>Proposition 1</u>: If $w = w(z, x)$ and $w = \frac{\partial}{\partial x} \log \varphi$, then the equation $w_x = BH_\ell[w]$ is equivalent to $\varphi_x = \varphi_{\underbrace{x \ldots x}_{\ell}} + \lambda\varphi$ for some constant λ.

A series of multisoliton solutions for (1), (2) give us the following general result. Here, and later, we consider the Wronskian determinant W.

For arbitrary functions $f_1, \ldots, f_k$ we put

$$W(f_1, \ldots, f_k) = \det\left(\frac{d^{i-1} f_j}{dx^{i-1}}\right) \quad i, j = 1, \ldots, k \quad . \tag{3}$$

Our general result is the following

<u>Theorem 2</u>: For any solutions $\varphi_1, \ldots, \varphi_k$ of two linear systems

$$\varphi_{it} = \varphi_{i\underbrace{x \ldots x}_{n}} \; ; \quad \varphi_{iy} = \varphi_{i\underbrace{x \ldots x}_{m}} \; : \quad i = 1, \ldots, k \tag{4}$$

and function

$$\psi(x, y, t, k) = \frac{W(\varphi_1, \ldots, \varphi_k, e^{kx + k^n t + k^m y})}{W(\varphi_1, \ldots, \varphi_k)} \tag{5}$$

there exist unique operators L_n, L_m of the form (2) satisfying

$$L_m \psi = \frac{\partial \psi}{\partial t} \; , \quad L_n \psi = \frac{\partial \psi}{\partial y} \quad . \tag{6}$$

Then the coefficients of L_n, L_m constitute the solution of system (1).

All the coefficients of L_n, L_m - solutions of (1), (2) corresponding to our solution (5), (6) as well as all eigenfunctions of L_n, L_m in (6) in this case can be found explicitly.

In particular, we can obtain from theorem 2 the following corollary expressed in terms of the Burgers-Hopf equation.

Corollary 3: For any solution $w_1, \ldots, w_k$ of system

(7) $$w_{it} = BH_n[w_i], \quad w_{iy} = BH_m[w_i]: \quad i = 1, \ldots, k$$

we can find the solution u_i, v_j, $i = 1, \ldots, u$, $j = 1, \ldots, m$ of system (1), (2). For example, if we put

$$W = \frac{d}{dx} \log W(\varphi_1, \ldots, \varphi_k)$$

for

$$w_i = \frac{d}{dx} \log \varphi_i: \quad i = 1, \ldots, k$$

then

$$u_{n-2} = nW_x, \; v_{m-2} = mW_x$$

or

$$u_{n-2} = n \frac{d^2}{dx^2} \log W(\varphi_1, \ldots, \varphi_k) \; ;$$

$$v_{m-2} = m \frac{d^2}{dx^2} \log W(\varphi_1, \ldots, \varphi_k) \; .$$

All the eigenfunctions of L_n, L_m corresponding to solutions (4)-(6) (or (7)-(8)) have a very simple form:

$$\Psi_1 = \frac{W(\varphi_1, \ldots, \varphi_k, \varphi_{k+1})}{W(\varphi_1, \ldots, \varphi_k)}$$

is common eigenfunction of L_n, L_m :

$$L_n \Psi_1 = \Psi_{1t} \; , \quad L_n \Psi_1 = \Psi_{1y}$$

for any φ_{k+1} also satisfying (4):

$$\varphi_{k+1t} = \varphi_{\underbrace{x \ldots x}_{n}}, \quad \varphi_{k+1y} = \varphi_{\underbrace{x \ldots x}_{m}} \; .$$

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