



SEMINAIRE

**Equations aux
Dérivées
Partielles**

2009-2010

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Séminaire É. D. P. (2009-2010), Exposé n° XX, 15 p.

<http://sedp.cedram.org/item?id=SEDP_2009-2010____A20_0>

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*Exposé mis en ligne dans le cadre du
Centre de diffusion des revues académiques de mathématiques*
<http://www.cedram.org/>

NON ZERO FLUX SOLUTIONS OF KINETIC EQUATIONS

MIGUEL ESCOBEDO

1. INTRODUCTION

We are interested in a particular type of solutions of kinetic equations that do not preserve one of the natural invariant quantities related to the physical model. We shall describe in some details two examples coming from different problems and with different properties: the Uehling Uhlenbeck equation for Bose particles and the Smoluchowski's coagulation equation. A motivation of our work is that we believe that the results are related with two interesting phenomena, the Bose Einstein condensation and the gelation.

1.1. Uehling Uhlenbeck equation. A spatially homogeneous diluted weakly interacting gaz of Bose particles is described by the following Boltzmann type equation:

$$(1.1) \quad \frac{\partial f}{\partial t}(t, p) = Q(f)(t, p), \quad t > 0, p \in \mathbb{R}^3.$$

$$(1.2) \quad Q(f)(t, p) = \int \int \int_{\mathbb{R}^9} W(p, p_2, p_3, p_4) q(f) dp_2 dp_3 dp_4$$

$$(1.3) \quad q(f) = f_3 f_4 (1 + f)(1 + f_2) - f f_2 (1 + f_3)(1 + f_4)$$

$$(1.4) \quad W(p, p_2, p_3, p_4) = \delta(p + p_2 - p_3 - p_4) \delta(|p|^2 + |p_2|^2 - |p_3|^2 - |p_4|^2).$$

where the time has been scaled in such a way that the constant in front of the collision integral is made equal to one, the mass m of the particles and the Planck constant k_B have been taken equal to one. The function $f(t, p)$ is the density of particles that at time $t \geq 0$ have momentum $p \in \mathbb{R}^3$. The gas is assumed to be spatially homogeneous and therefore the density of particles is independent of the spatial variable.

This equation was first derived by L. W. Nordheim in 1928 (cf. [21]), then by E. A. Uehling and G. E. Uhlenbeck in 1933 (cf. [25]). For more recent derivations see E. Zaremba, T. Nikuni, A. Griffin ([26]) and R. Baier, T. Stockkamp ([1]).

If we assume moreover that the density of particles is radially symmetric, i.e. $f(t, p) = f(t, |p|^2)$ with some minor abuse of notations, and after integration of the angular variables, the equation reduces to:

$$(1.5) \quad \left\{ \begin{array}{l} \frac{\partial f}{\partial t}(t, x_1) = \int \int_{D(x_1)} w(x_1, x_3, x_4) q(f) dx_3 dx_4 \equiv Q_r(f) \\ q(f) = f_3 f_4 (1 + f_1)(1 + f_2) - f_1 f_2 (1 + f_3)(1 + f_4) \\ x = |p|^2, \quad x_2 = x_3 + x_4 - x_1 \\ D(x_1) \equiv \{(x_3, x_4) : x_3 > 0, x_4 > 0, x_3 + x_4 \geq x_1 > 0\} \\ w(x_1, x_3, x_4) = \frac{\min(\sqrt{x_1}, \sqrt{x_2}, \sqrt{x_3}, \sqrt{x_4})}{\sqrt{x_1}}. \end{array} \right.$$

Due to the symmetries of the measure W it is easy to obtain formally the following conservation laws:

$$(1.6) \quad \frac{d}{dt} \int_{\mathbb{R}^3} f(t, p) dp = 0, \quad \text{conservation of the total number of particles } N(f)$$

$$(1.7) \quad \frac{d}{dt} \int_{\mathbb{R}^3} pf(t, p) dp = 0, \quad \text{conservation of the momentum } N(f)$$

$$(1.8) \quad \frac{d}{dt} \int_{\mathbb{R}^3} |p|^2 f(t, p) dp = 0 \quad \text{energy conservation } E(f).$$

The term formally means the following. If the function f satisfies (1.1)-(1.4) and all the calculations that are needed are allowed, then multiplying (1.1) by a test function $\varphi(p)$, integrating with respect to $p \in \mathbb{R}^3$ using the symmetries of W and $q(f)$ and using Fubini's theorem, we obtain

$$(1.9) \quad \begin{aligned} \frac{d}{dt} \int_{\mathbb{R}^3} \varphi(p) f(t, p) dp &= \int_{\mathbb{R}^{12}} W(p_1, p_2, p_3, p_4) f_3 f_4 (1+f)(1+f_2) \times \\ &\times [\varphi(p) + \varphi(p_2) - \varphi(p_3) - \varphi(p_4)] dp_1 dp_2 dp_3 dp_4 \end{aligned}$$

from where (1.6)-(1.8) follow by choosing $\varphi = 1$, $\varphi(p) = p$ and $\varphi(p) = |p|^2$ respectively.

An important property of the equation is the existence of equilibria called Bose-Einstein distributions :

$$F(p) = \frac{1}{e^{\beta|p-p_0|^2 - \mu} - 1}, \quad \beta > 0, \quad p_0 \in \mathbb{R}^3, \quad \mu \leq 0,$$

$$F(p) = \frac{1}{e^{\beta|p-p_0|^2} - 1} + \rho \delta_{p_0}, \quad \rho > 0.$$

These equilibria satisfy:

$$\begin{aligned} \forall p_i \in \mathbb{R}^3; \quad p_1 + p_2 = p_3 + p_4, \quad &\& \quad |p_1|^2 + |p_2|^2 = |p_3|^2 + |p_4|^2 : \\ F(p_3) F(p_4) (1 + F(p_1)) (1 + F(p_2)) &\equiv F(p_1) F(p_2) (1 + F(p_3)) (1 + F(p_4)) \end{aligned}$$

from where $Q(F) \equiv 0$.

The equilibria describe a gas at equilibrium: the distribution of velocities is at equilibrium. In particular, for any $p_0 \in \mathbb{R}^3$ and $\rho > 0$:

$$(1.10) \quad \frac{d}{dt} \int_{|p-p_0| < \rho} f(t, p) dp = \int_{|p-p_0| < \rho} Q(f)(t, p) dp$$

$$(1.11) \quad = 0.$$

The quantity in the left hand side of (1.10) can be seen as the flux of particles through the sphere $S_\rho(p_0)$ of center p_0 and radius ρ . The identity (1.10) is then telling us that the flux of particles through any sphere $S_\rho(p_0)$ is zero whatever are the point p_0 and the radius ρ .

1.2. The coagulation equation. In 1936, Carothers [3] first suggested that gelation is due to the formation of macroscopic branched polymers. In 1941, however, Flory [13, 14] put this idea into quantitative terms and formulated a theory for the polycondensation of bi-functional monomers with tri- or tetra-functional branch points. In this equilibrium model, bonds are randomly formed with probability p between adjacent nodes on an infinite Cayley tree, or Bethe lattice. Stockmayer [24] subsequently refined Flory's theory to include the case of arbitrary branch point functionality f and also showed that the Bethe lattice approach is equivalent to a kinetic rate equation formulation with the

unscreened reaction kernel $W(i, j) = ij$. We consider in our second example the continuous version of this kinetic rate equation with more general reaction kernels $W(x, y)$. Let us then consider particles of mass denoted by x . When two particles, of masses x_1 and x_2 , collide they aggregate and form a new particle of mass $x_1 + x_2$. If we denote by $f(t, x)$ the density of particles of mass x at time t the coagulation equation reads:

$$(1.12) \quad \frac{\partial f}{\partial t}(t, x) = C(f)(t, x)$$

$$(1.13) \quad C(f)(t, x) = \frac{1}{2} \int_0^x W(x-y, y) f(t, x-y) f(t, y) dy - f(t, x) \int_0^\infty W(x, y) f(t, y) dy$$

where W is a homogeneous function of degree λ satisfying:

$$W(x, y) = W(y, x).$$

Here again since the set of particles is spatially homogeneous the density function is independent of the spatial variable. Moreover since all the particles have the same momentum, the density function is independent of that momentum. This situation may be obtained for example if all the particles move following the same brownian motion assuming that they are contained in a dilute gas of particles that are themselves undergoing elastic collisions. The kernel $W(x, y)$ is in that case $W(x, y) = (x^{1/3} + y^{1/3})(x^{-1/3} + y^{-1/3})$. This example was the first considered by Smoluchowski in 1916 (cf. [22]) where his goal was to describe diffusion-limited aggregation and the particles were then following a brownian motion. Other types of physical situations give rise to different functions W .

An argument similar to that used for the Uehling Uhlenbeck equation shows again formally that if f is a solution of (1.12), (1.13):

$$(1.14) \quad \frac{d}{dt} \int_0^\infty x f(t, x) dx = 0 \quad \text{conservation of the total mass.}$$

The only equilibrium in that case is $f \equiv 0$, that obviously satisfies

$$\int_{x_0-\rho}^{x_0+\rho} f(t, x) dx = 0$$

for all $x_0 > 0$ and $\rho > 0$.

2. NON ZERO FLUX SOLUTIONS.

It turns out that the coagulation equation has other stationary solutions. It may be easily checked that if $W(x, y) = x^\alpha y^\beta + y^\alpha x^\beta$ with $\alpha + \beta = \lambda$, then the function

$$(2.1) \quad G(x) = x^{-(3+\lambda)/2}$$

satisfies:

$$(2.2) \quad C(G)(x) = \int_0^{x/2} [W(x-y, y)G(x-y) - W(x, y)G(x)] G(y) dy - G(x) \int_{x/2}^\infty W(x, y)G(y) dy = 0.$$

Moreover, for every $R > 0$:

$$(2.3) \quad \int_0^R x C(G)(x) dx = -2\pi$$

Identity (2.3) may be seen as describing a flow of particles going out of the (mass) interval $(0, R)$ at a constant and finite rate. This was remarked by P.G.J. van Dongen in [5]

The Uehling Uhlenbeck equation (1.1)-(1.4) does not possess such type of solutions due to the fact that the term $q(f)$ is not homogeneous with respect to f . Nevertheless this equation is sometimes approximated by the following:

$$(2.4) \quad \frac{\partial f}{\partial t}(t, p) = Q_m(f)(t, p) \equiv \int \int \int W[p, p_2, p_3, p_4] q_m(f) dp_2 dp_3 dp_4$$

$$(2.5) \quad q_m(f) = f_3 f_4 (f_1 + f_2) - f_1 f_2 (f_3 + f_4).$$

that we shall denote mU-U. This modified equation still has a family of equilibria:

$$(2.6) \quad \tilde{F} = 1; \quad \tilde{F} = \frac{1}{|p|^2 - \mu}, \quad \mu \leq 0$$

such that

$$W[\dots] q_m(f) \equiv 0.$$

Moreover it has been observed by V. E. Zakharov (cf. [2]) that the function

$$(2.7) \quad G(p) = |p|^{-7/3}$$

satisfies:

$$(2.8) \quad \forall R > 0 : \quad \int_{B(0,R)} Q_m(G)(p) dp = -C_*$$

The particle density $G(p)$ defines a stationary distribution of particles with non zero flux at the origin for the operator Q_m .

3. OUR MAIN RESULT.

For none of the two particular solutions that we have considered in the previous Section, $G(x) = x^{-(3+\lambda)/2}$ nor $G(p) = |p|^{-7/3}$, are the natural quantities of the corresponding equation well defined. For the mU-U equation the total number of particles contained in a ball of radius $R > 0$ since

$$\int_{B(0,R)} G(p) dp = C \int_0^R G(p) |p|^2 d|p|$$

diverges as $R \rightarrow +\infty$) while for the coagulation equation it is the total mass contained in an interval $(0, R)$:

$$\int_\delta^\infty x G(x) dx$$

that diverges as $\delta \rightarrow 0$. We then address the following question: is it possible to find solutions for which the natural quantities are well defined and not preserved for $t > 0$?

The answer in both cases is yes. In both cases, the results may be roughly described as follows. Consider an initial data such that

$$\begin{aligned} f_0(p) &\sim |p|^{-7/3} & |p| \rightarrow 0, & \text{ for the Uehling Uhlenbeck equation} \\ f_0(x) &\sim x^{-(3+\lambda)/2} & x \rightarrow +\infty, & \text{ for the coagulation equation} \end{aligned}$$

and for which the total number of particles or the total mass respectively, are well defined. Then there exists local (in time) solutions of the equations for $t \in (0, T)$ for some finite $T > 0$, such that:

$$\begin{aligned} f(t, p) &\sim \lambda_1(t) |p|^{-7/3} & |p| \rightarrow 0, & \text{ for the Uehling Uhlenbeck equation} \\ f(t, x) &\sim \lambda_2(t) x^{-(3+\lambda)/2} & x \rightarrow +\infty, & \text{ for the coagulation equation} \end{aligned}$$

where λ_1 and λ_2 are continuous functions bounded from above and from below by two positive constants on the time interval $t \in [0, T]$, and such that, for all $t \in [0, T]$, the total number of particles or the total mass respectively are well defined. Since the precise result in both cases is very similar we only write in detail the Theorem for the Uehling Uhlenbeck equation. Moreover, since we only consider radially symmetric solutions and the solutions that we obtain have the same property we state our result in terms of the Uehling Uhlenbeck equation in radial variables (1.5). We recover the unified presentation of our two examples in the description of the proofs in Section 4.

Theorem 3.1. (*U-U equation in radial coordinates.*)

$$\begin{aligned} \text{Suppose that:} \quad & \left| f_0(x) - A x^{-7/6} \right| \leq \frac{B}{x^{7/6-\delta}}, \quad 0 \leq x \leq 1, \\ & \left| f_0'(x) + \frac{7}{6} A x^{-13/6} \right| \leq \frac{B}{x^{13/6-\delta}}, \quad 0 \leq x \leq 1, \\ & f_0 \in \mathbf{C}(\mathbb{R}^+) \quad y \quad f_0(x) \leq B \frac{e^{-Dx}}{x^{7/6}}, \quad x \geq 1 \end{aligned}$$

where A, B, C, D, δ are positive constants Then, there exists a solution $f(t, x)$ of (1.5), $f \in \mathbf{C}^{1,0}((0, T) \times (0, +\infty))$, a function $\lambda(t) \in \mathbf{C}[0, T] \cap \mathbf{C}^1(0, T)$, and constants $L > 0$, $T > 0$ such that:

$$\begin{aligned} (i) \quad & 0 \leq f(x, t) \leq L \frac{e^{-Dx}}{x^{7/6}}, \quad \text{if } x > 0, t \in (0, T), \\ (ii) \quad & |f(x, t) - \lambda(t) x^{-7/6}| \leq L x^{-7/6+\delta/2}, \quad x \leq 1, t \in (0, T) \\ (iii) \quad & |\lambda(t)| \leq L, \quad \text{for all } t \in (0, T). \end{aligned}$$

Remark 3.2. If f is the solution described in Theorem 3.1 then for all $t \in (0, T)$ the total number of particles is well defined:

$$\int_0^\infty x f(t, x) \sqrt{x} dx < +\infty$$

and, due to the behavior of the solution f as $x \rightarrow 0$, we have

$$\int_{|p|^2 \geq R} Q_r(f(t, p)) dp = -C\lambda^3(t) + \mathcal{O}(R^{1/10}),$$

as $R \rightarrow 0$.

Remark 3.3. (*The question of uniqueness.*) It has been proved by X. Lu in [18] that for all non negative measure h_0 on \mathbb{R}^+ satisfying:

$$(3.1) \quad \int_0^\infty d \left((x^{1/2} + x^{3/2}) h_0(x) \right) < \infty.$$

there exists a solution $h(t, x)$ of the equation (1.5) such that, for all time $t > 0$ $h(t)$ is a non negative measure, $h(t)$ takes h_0 as initial value in some suitable weak sense and such that, for all $t > 0$:

$$\int_0^\infty d \left((x^{1/2} + x^{3/2})h(t, x) \right) < \infty$$

This solution h also satisfies an entropy inequality and, what is most important to us here, the conservation of the total number of particles:

$$\frac{d}{dt} \int_0^\infty d \left(x^{1/2}h(t, x) \right) = 0.$$

Notice that the initial data in Theorem 3.1 satisfy condition (3.1) and are then perfectly admissible data for the Theorem of Lu. It is an open question to determine if the solution obtained by Lu's Theorem and ours are related in some way.

4. METHOD AND MAIN IDEAS OF THE PROOF.

The proof is based on the linearisation of the equation around the initial data f_0 as follows. We look for solutions of the form:

$$(4.1) \quad f(t, x) = \lambda(t) f_0(x) + g(t, x)$$

where λ and g have to be determined. We then write $q(f)$ as follows:

$$q(f) = q(\lambda(t) f_0 + g) = \underbrace{\ell_2(\lambda(t) f_0, g)}_{\text{quadratic in } \lambda(t) f_0} + \underbrace{\ell_1(\lambda(t) f_0, g)}_{\text{linear in } \lambda(t) f_0} + q(\lambda(t) f_0) + R(f_0, g)$$

linear in g

where $R(f_0, g)$ is quadratic in g . Using now that $f_0(x) \sim x^{-7/6}$ as $x \rightarrow 0$ we use:

$$\ell_2(\lambda(t) f_0, g) = \lambda(t)^2 \ell_2(f_0, g) \approx \lambda(t)^2 \ell_2(|p|^{-7/3}, g)$$

and that introduces a new term in the remainder $R(f_0, g)$. Rescaling the time variable, the resulting equation may then be written as:

$$(4.2) \quad \frac{\partial g}{\partial \tau} = \mathcal{L}(g) + \mathcal{R}(f_0, g, \tau)$$

$$(4.3) \quad \mathcal{L}(g) = \int \int \int_{D(x_1)} w(x_1, x_2, x_3, x_4) \ell_2(x^{-7/6}, g) dx_3 dx_4$$

In both problems, the Uheling Uhlenbeck equation and the coagulation equation, one arrives at the same type of equation (4.2) for $t > 0$ and $x > 0$, where L is a linear integral operator. The equation (4.2) can be solved in a very classical way. In a first step we study the linear semigroup generated by the operator L (4.3). Then, in a second step, we prove the existence of a local solution g of (4.2) by means of a fixed point argument. Moreover the study of the linear semigroup generated by L , i.e. the study of the solutions:

$$(4.4) \quad \frac{\partial g}{\partial t} = L(g), \quad t > 0, x > 0$$

$$(4.5) \quad g(0, x) = \delta_{x_0}$$

for any fixed $x_0 > 0$ may be performed with essentially the same methods and may then be presented in a unified way. The second step - study of the nonlinear problem - is more technical and is rather different in both cases due to the different properties of the

respective linear semigroups. I will then mainly describe here our study of the linear semigroups and will only add a brief remark on the non linear problems.

Remark 4.1. *The proof of existence of a solution g to (4.2) is based on a fixed point and the semigroup generated by L . One is led, as usual, to prove that the nonlinear term $\mathcal{R}(f_0, g, \tau)$ is small in some suitable norm for t small. The operator L is then a good linearisation at small times of the Uehling Uhlenbeck equation around f_0 under the assumptions of Theorem 3.1. Notice that L is not the linearised of the Uehling Uhlenbeck equation around $x^{-7/6}$, but that of the modified equation (2.4), (2.5).*

4.1. The linear equation. We first perform a change of variables $X = \log x$, $w(x_1, \dots, x_4) = \omega(X_1, \dots, X_4)$ and $G(t, X) = g(t, x)$, where now $X \in \mathbb{R}$, and take Fourier transform in X defining:

$$\widehat{G}(t, \xi) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-iX\xi} G(t, X) dX.$$

After a lengthly but explicit calculation the equation (4.4) is transformed in:

$$(4.6) \quad \partial_t \widehat{G}(t, \xi) = \widehat{G}(t, \xi + ih) \Phi(\xi + ih)$$

$$(4.7) \quad \widehat{G}(0, \xi) = 1$$

for some $h \in \mathbb{R}$ and some meromorphic function Φ from \mathbb{C} to \mathbb{C} , all of them explicit and determined by the kernel ω . In our two examples:

- $h = \frac{\lambda - 1}{2}$; $\Phi(\xi) = -\frac{2\sqrt{\pi} \Gamma(i\xi + 1 + \frac{\lambda}{2})}{\Gamma(i\xi + \frac{\lambda+1}{2})}$ (Coagulation equation)

- $\Phi(\xi) = -a + \sum_{j=0}^{\infty} \frac{A_1(j)}{(1 - 6i\xi + 12j)} + \sum_{j=0}^{\infty} \frac{A_2(j)}{(1 - 3i\xi + 3j)} + \sum_{j=0}^{\infty} \frac{A_3(j)}{(3 + 2i\xi + 2j)} + \sum_{j=0}^{\infty} \frac{A_4(j)}{(10 + 3i\xi + 6j)}$; $h = -\frac{1}{3}$ (Uehling Uhlenbeck equation)

Moreover, the behavior of the kernel ω as $X \rightarrow \pm\infty$ imposes some decay conditions on $G(t, X)$ as $X \rightarrow \pm\infty$ and therefore the condition that $\widehat{G}(t, \cdot)$ must be analytic in some strip

$$(4.8) \quad S = \{\xi; \xi = u + iv, A_1 < v < A_2, u \in \mathbb{R}\}$$

for two explicitly known real constants $A_1 < A_2$.

We are then lead to look for a solution \widehat{G} of (4.7) analytic in the domain S defined in (4.8). This roblem is solved in two steps by using the Laplace transform in time.

Lemma 4.2. *Suppose that \mathcal{V} is an analytic function in the strip S satisfying:*

$$(4.9) \quad \int_{Im(y)=\beta_0} \left| \frac{1}{\mathcal{V}(y)} \right| e^{-\frac{\pi}{2h}|y|} \sqrt{1 + |y|} |dy| < \infty$$

for some $\beta_0 \in (A_1, A_2)$ and

$$(4.10) \quad \mathcal{V}(\xi) = -\mathcal{V}(\xi + ih) \Phi(\xi + ih)$$

for $Im\xi \in (A_1, A_2)$. Then, the function

$$(4.11) \quad \widehat{G}(t, \xi) = \frac{i}{\sqrt{2\pi h}} \int_{Imy=\beta_0} \frac{\mathcal{V}(\xi)}{\mathcal{V}(y)} t^{-\frac{y-\xi}{h} i} \Gamma\left(\frac{y-\xi}{h} i\right) dy$$

$$(4.12) \quad A_1 < \beta_0 < A_2$$

for $\text{Im}\xi > \beta_0$ may be extended analytically to S and solves (4.6), (4.7) for $\text{Im}\xi \in (A_1, A_2)$.

Proof. The proof follows by direct complex variable computation. \square

A heuristic explanation for the formula (4.11) can be given using Laplace transform. Suppose that we define the Laplace transform of $\widehat{G}(t, \xi)$ in t as:

$$\widetilde{G}(z, \xi) = \int_0^\infty \widehat{G}(t, \xi) e^{-zt} dt$$

Then, (4.6), (4.7) become:

$$(4.13) \quad z\widetilde{G}(z, \xi) = \widetilde{G}(z, \xi + hi) \Phi(\xi + hi) + \frac{1}{\sqrt{2\pi}}$$

The solution of this equation can be formally reduced to (4.10) by means of the transformation:

$$(4.14) \quad \widetilde{G}(z, \xi) = \exp\left(-\frac{i}{h} \log(-z)\xi\right) \mathcal{V}(\xi) H(z, \xi).$$

If \mathcal{V} satisfies (4.10), such transformation brings (4.13) to:

$$(4.15) \quad H(z, \xi) - H(z, \xi + hi) = \frac{1}{\sqrt{2\pi}} \frac{e^{\frac{i}{h} \log(-z)\xi}}{z\mathcal{V}(\xi)}$$

Equation (4.15) can be transformed into a Riemann-Hilbert problem by means of the following conformal mapping:

$$(4.16) \quad H(z, \xi) = h(z, \zeta) \quad , \quad \zeta = e^{\frac{2\pi}{h}(\xi - \beta_0 i)}$$

where, for the sake of simplicity we will write, with some slight abuse of notation

$$\mathcal{V}(\xi) = \mathcal{V}(\zeta).$$

Then (4.15) becomes:

$$(4.17) \quad h(z, \zeta + i0) - h(z, \zeta - i0) = \frac{e^{\frac{2\pi}{h}\beta_0\alpha(z)}}{\sqrt{2\pi}} \frac{\zeta^{\alpha(z)}}{z\mathcal{V}(\zeta)} \quad , \quad \zeta \in \mathbb{R}^+$$

with h analytic in $\mathbb{C} \setminus \mathbb{R}^+$ and:

$$\alpha(z) = \frac{1}{2\pi i} \arg(-z)$$

It is well known that the solution of Riemann-Hilbert problems can be obtained using Wiener Hopf methods (cf. [19], [20]). However, in this particular case, assuming that $\frac{\zeta^{\alpha(z)}}{\mathcal{V}(\zeta)}$ satisfies suitable boundedness estimates for small and large ζ , we can solve (4.17) just using Cauchy's formula to obtain:

$$h(z, \zeta) = \frac{1}{2\pi i} \frac{1}{\sqrt{2\pi}} \frac{e^{\frac{2\pi}{h}\beta_0\alpha(z)}}{z} \int_0^\infty \frac{s^{\alpha(z)}}{\mathcal{V}(s)} \frac{ds}{(s - \zeta)}$$

and, using (4.16):

$$(4.18) \quad H(z, \xi) = \frac{1}{2\pi i} \frac{1}{\sqrt{2\pi}} \frac{1}{z} \int_{-\infty}^\infty \frac{e^{\frac{2\pi\alpha(z)}{h}y}}{\mathcal{V}(y)} \frac{dy}{1 - e^{\frac{2\pi}{h}(\xi - y)}}$$

It then follows from (4.14) that:

$$(4.19) \quad \tilde{G}(z, \xi) = \frac{1}{2\pi i} \frac{1}{\sqrt{2\pi}} \frac{1}{z} \mathcal{V}(\xi) \int_{-\infty}^{\infty} \frac{e^{\frac{2\pi\alpha(z)}{h}(y-\xi)}}{\mathcal{V}(y)} \frac{dy}{1 - e^{\frac{2\pi}{h}(\xi-y)}}$$

and inverting the Laplace transform we finally obtain (4.11).

On the other hand, equation (4.10) has infinitely many solutions. Notice indeed that given a solution \mathcal{V}_{part} another is obtained as:

$$\mathcal{V}(\xi) = \mathcal{V}_{part}(\xi) p(\xi)$$

where $p(\xi) = p(\xi + h)$. Examples of such function p are $e^{\frac{2\pi\ell}{h}}$ with $\ell \in \mathbb{N}$ and any linear combination of them.

Given such a non uniqueness a natural and essential question is then how to chose one of them. We may state several sufficient conditions that would ensure that \hat{G} is the Fourier transform of a tempered distribution. First we want the function \hat{G} to be defined. This is guaranteed by the condition (4.9) above. However, this condition is not sufficient to prove that $\hat{G}(t, \xi)$ is globally bounded with respect to ξ . The difficulty comes from the fact that, if the behaviors of $\mathcal{V}(\xi)$ are too disparate as $Re(\xi)$ tends to plus or minus infinity, the quotient $\frac{\mathcal{V}(\xi)}{\mathcal{V}(\xi+Y)}$ may be strongly increasing in some regions of the integral in (4.11). A sufficient condition to avoid this difficulty is to have:

$$(4.20) \quad |\mathcal{V}(\xi)| \approx e^{B_{\pm}|\xi|}, \quad |B_{\pm}| \leq \frac{2\pi}{h}$$

as $Re(\xi) \rightarrow \pm\infty$. The decay rate of the Gamma function in (4.11) may then control the possible growth of the quotient $\frac{\mathcal{V}(\xi)}{\mathcal{V}(\xi+Y)}$ uniformly on ξ .

A particular solution of (4.10) can be easily obtained using Cauchy's formula. To this end we take the logarithm of both sides of (4.10) to obtain:

$$(4.21) \quad \log(\mathcal{V}(\xi)) = \log(\mathcal{V}(\xi + hi)) + \log(-\Phi(\xi + hi))$$

or equivalently

$$(4.22) \quad \log(\mathcal{V}(\xi - hi)) = \log(\mathcal{V}(\xi)) + \log(-\Phi(\xi)).$$

Let us take any β such that $\Phi(\xi)$ has no zeros nor poles along the line $Im(\xi) = \beta$. We define:

$$\psi(\zeta) = \log(\mathcal{V}(\xi)) \quad , \quad \zeta = e^{\frac{2\pi}{h}(\xi - \beta i)} \quad , \quad Q(\zeta) = \log(-\Phi(\xi))$$

Equation (4.22) then becomes

$$(4.23) \quad \psi(\zeta + i0) = \psi(\zeta - i0) + Q(\zeta - i0) \quad , \quad \zeta \in \mathbb{R}^+$$

with ψ analytic in $\mathbb{C} \setminus \mathbb{R}^+$. Taking into account that $|Q(\zeta)| \leq C(1 + |\log(\zeta)|)$ we can obtain a particular solution of (4.23) as:

$$\psi(\zeta) = \frac{1}{2\pi i} \int_{\mathbb{R}^+} Q(s) \left[\frac{1}{s - \zeta} - \frac{1}{s + 1} \right] ds$$

where the term $1/(s + 1)$ has been added to the classical Cauchy integral in order to ensure the convergence of the integral. Then, returning to the variable ξ we obtain:

$$(4.24) \quad V_{part}(\xi) = \exp \left[-\frac{i}{h} \int_{Imy=\beta} \ln(\Phi(y)) \left(\frac{1}{1 - e^{\frac{2\pi}{h}(\xi-y)}} - \frac{1}{1 - A e^{-\frac{2\pi}{h}y}} \right) dy \right]$$

where β is chosen in such a way that $\Phi(\cdot)$ has no zeros nor pole on $Im\xi = \beta$ and $A \in \mathbb{C}$ is an auxiliary constant chosen such that $1 - A e^{-\frac{2\pi}{h}y} \neq 0$ on $Imy = \beta$. This formula only

defines a unique function if the value of β and the argument of the function $\ln(-\Phi(\eta))$ are prescribed. These choices depend in an essential way on the particular function Φ and very particularly on the location of its zeros and poles. Once that choice has been made the formulas (4.11) and (4.20) yield an explicit solution $\tilde{G}(t, \xi)$ of (4.6), (4.7). Inverting the Fourier transform, we obtain a solution $g(t, x, x_0)$ of (4.4), (4.5) that has in both cases the self similar form:

$$g(t, x, x_0) = \frac{1}{x_0} g\left(tx_0^h, \frac{x}{x_0}, 1\right).$$

It is then possible to obtain the asymptotic behaviors of the function $g(t, x, 1)$ in the different domains of the variables x and t . We state the Theorems corresponding to our two different examples. First for the Uehling Uhlenbeck equation:

Theorem 4.3. *(The Uehling Uhlenbeck equation [9]) Let g be the solution of the equation (4.4) corresponding to the Uehling Uhlenbeck equation that has been previously obtained. For $x \in (0, 2)$ the function $g(t, x, 1)$ can be written, for all $t > 0$ as:*

$$(4.25) \quad g(t, x, 1) = e^{-at} \delta(x-1) + \sigma(t) x^{-7/6} + \mathcal{R}_1(t, x) + \mathcal{R}_2(t, x),$$

where $\sigma \in \mathbf{C}[0, +\infty)$ satisfies:

$$\sigma(t) = \begin{cases} At^4 + \mathcal{O}(t^{4+\varepsilon}) & \text{as } t \rightarrow 0^+, \\ \mathcal{O}(t^{-(3v_0-5/2)}) & \text{as } t \rightarrow +\infty \end{cases}$$

\mathcal{R}_1 satisfies,

$$\begin{aligned} \mathcal{R}_1(t, x) &\equiv 0 \quad \text{for } |x-1| \geq \frac{1}{2}, \\ |\mathcal{R}_1(t, x)| &\leq C \frac{e^{-(a-\varepsilon)t}}{|x-1|^{5/6}} \quad \text{for } |x-1| \leq \frac{1}{2}, \end{aligned}$$

and \mathcal{R}_2 satisfies:

$$\mathcal{R}_2(t, x) \leq \begin{cases} \frac{C}{t^{5/2+\varepsilon}} \left(\frac{t^3}{x}\right)^{\tilde{b}} & \text{for } 0 \leq t \leq 1 \\ \frac{C}{t^{3v_0-\varepsilon}} \left(\frac{t^3}{x}\right)^{\tilde{b}} & \text{for } t > 1. \end{cases}$$

On the other hand, for $x > 2$,

$$F(t, x, 1) \leq \begin{cases} \frac{C}{t^{\frac{9}{2}+\varepsilon}} \left(\frac{t^3}{x}\right)^{\frac{11}{6}} & \text{for } 0 \leq t \leq 1 \\ \frac{C}{t^{1+3v_0-\varepsilon}} \left(\frac{t^3}{x}\right)^{\frac{11}{6}} & \text{for } t > 1, \end{cases}$$

In these formulae, A is an explicit numerical constant, $\varepsilon > 0$ is an arbitrarily small number, \tilde{b} is an arbitrary number in the interval $(1, 7/6)$, and $v_0 \approx 1.84$. The constant C depends on ε and \tilde{b} but is independent on t .

And for the coagulation equation.

Theorem 4.4. *(The coagulation equation [11]) Let g be the solution of the equation (4.4) corresponding to the coagulation equation that has been previously obtained. Then, there exists positive constants δ and ε_1 only depending on λ , such that for any $0 < \varepsilon < \varepsilon_1$*

the following statements hold.

For all $t \geq 1$:

$$g(t, x, 1) = t^{\frac{2}{\lambda-1}} \varphi_1(\sigma) + \varphi_2(t, \sigma)$$

where σ is the self similar variable:

$$\sigma = t^{\frac{2}{\lambda-1}} x,$$

and the functions φ_1 and φ_2 satisfy the following estimates:

$$\varphi_1(\sigma) = \begin{cases} a_1 \sigma^{-\frac{3}{2}} + \mathcal{O}_\varepsilon \left(\sigma^{-\frac{4-\lambda}{2} + \varepsilon} \right) & \text{for } 0 \leq \sigma < 1 \\ a_2 \sigma^{-\frac{3+\lambda}{2}} + \mathcal{O}_\varepsilon \left(\sigma^{-(1+\lambda+\varepsilon)} \right) & \text{for } \sigma > 1 \end{cases}$$

where a_1 and a_2 are two explicit constants,

$$\varphi_2(t, \sigma) = \begin{cases} b_1(t) \sigma^{-\frac{3}{2}} + \mathcal{O} \left(t^{\frac{2}{\lambda-1} - \delta} \sigma^{-\frac{3}{2} + \delta} \right) & \text{for } 0 \leq \sigma < 1 \\ b_2(t) \sigma^{-\frac{3+\lambda}{2}} + \mathcal{O} \left(t^{\frac{2}{\lambda-1} - \delta} \sigma^{-\frac{3+\lambda}{2} - \delta} \right) & \text{for } \sigma > 1 \end{cases}$$

where b_1 and b_2 are two continuous functions such that $|b_1(t)| + |b_2(t)| \leq C t^{\frac{2}{\lambda-1} - \delta}$.

For all $0 < t < 1$:

$$g(t, x, 1) = \begin{cases} t x^{-\frac{3}{2}} + b_3(t) x^{-\frac{3}{2}} + \mathcal{O} \left(t x^{-\frac{3}{2} + \delta} \right) & \text{for } 0 \leq x \leq \frac{1}{2} \\ a_3 t x^{-\frac{3+\lambda}{2}} + b_4(t) x^{-\frac{3+\lambda}{2}} + \mathcal{O} \left(t x^{-\frac{3+\lambda}{2} - \delta} \right) & \text{for } x \geq \frac{3}{2}, \\ \mathcal{O}_\varepsilon \left(\frac{t^{1-2\varepsilon}}{|x-1|^{\frac{3}{2}-\varepsilon}} \right) & \text{for } t^2 < |x-1| < \frac{1}{2} \end{cases}$$

where a_3 is an explicit numerical constant and b_3 and b_4 are continuous functions such that $|b_3(t)| + |b_4(t)| \leq C t^{1+\delta}$.

Finally:

$$(4.26) \quad \lim_{t \rightarrow 0} t^2 g(t, 1 + t^2 \chi, 1) = \Psi(\chi) \quad \text{uniformly on compact subsets of } \mathbb{R}$$

where the function Ψ is given by:

$$(4.27) \quad \Psi(\chi) = \begin{cases} \frac{\pi}{\chi^{3/2}} e^{-\frac{\pi}{\chi}}, & \text{for all } \chi \geq 0, \\ 0 & \text{for all } \chi < 0. \end{cases}$$

Remark 4.5. The two linear equations behave very differently with respect to the initial Dirac measure. As it is shown by (4.25), in the Uehling Uhlenbeck equation the Dirac measure is present for all time although it is exponentially damped. On the contrary, the coagulation equation regularizes instantaneously the Dirac measure by means of a self similar process described in (4.26), (4.27).

4.2. The non linear equation. With the two Theorems 4.3 and 4.4 we can build the linear semigroup G generated by the linear operator L . We may then look for mild solutions of the nonlinear equation (4.2), (4.3), i.e. solutions of the integral equation:

$$(4.28) \quad g(\tau, x) = G(\tau)g_0(x) + \int_0^\tau G(\tau - s)\mathcal{R}(f_0, g(s); s) ds$$

This nonlinear equations is now solved using a fixed point argument. As it is well known this relies very much on the precise properties of the linear semigroup G generated by the operator L . As we have seen, these properties are different for the Uehling Uhlenbeck equation and the coagulation equation, the details of the fixed point arguments have then to be different.

The case of the Uehling Uhlenbeck equation turns out to be simpler. It is possible in that case to prove that the remainder term $\int_0^\tau \tau G(\tau - s)\mathcal{R}(f_0, g, s)ds$ is small in some weighted L^∞ norm when τ is sufficiently small.

The problem is a little more involved for the coagulation equation and the reason is the following. In order to solve the non linear equation we want to look for solutions $f(t, x)$ that are small perturbations of the initial data f_0 . That leads to an operator \mathcal{L} that is the linearization of the non linear operator (Q or C), around f_0 . The study that we must perform of the linear operator \mathcal{L} is delicate because the estimates that we need will only appear if we are able to take into account the cancelations that take place in the integral defining the operator. To this end the operator \mathcal{L} is treated as a perturbation of the operator L that has been introduced and studied above. An added difficulty in the case of the coagulation operator is that the operator L as regularizing effects that are only present “at infinity” in the operator \mathcal{L} (the region where $f_0(x)$ behaves like $x_0^{-(3+\lambda)/2}$). The passage from L to \mathcal{L} may then be seen as a singular perturbation problem. The fixed point argument requires in that case a more involved type of norms measuring the regularising effect of the operator \mathcal{L} . Although these norms may seem rather involved it may be useful to give them here.

$$N_{2,\sigma}(f; t_0, R) = \left(R^{\frac{\lambda-1}{2}+2\sigma-1} \int_{t_0}^{\min(t_0+R^{-(\lambda-1)/2}, T)} \|D_x^\sigma f(t)\|_{L^2(R/2, 2R)}^2 dt \right)^{1/2}, \quad \sigma \geq 0$$

$$M_{2,\sigma}(f; R) = \left(R^{2\sigma-1} \int_0^T \|D_x^\sigma f(t)\|_{L^2(R/2, 2R)}^2 dt \right)^{1/2}, \quad \sigma \geq 0$$

$$N_\infty(f; t_0, R) = \left(R^{\frac{\lambda-1}{2}} \int_{t_0}^{\min(t_0+R^{-(\lambda-1)/2}, T)} \|f(t)\|_{L^\infty(R/2, 2R)}^2 dt \right)^{1/2}$$

$$M_\infty(f; R) = \left(\int_0^T \|f(t)\|_{L^\infty(R/2, 2R)}^2 dt \right)^{1/2}$$

Then, for any $\sigma > 0$ we define the following norms:

$$\begin{aligned} \|f\|_{Y_{q,p}^\sigma(T)} &= \sup_{0 < R \leq 1} R^q M_{2,0}(f; R) + \sup_{0 < R \leq 1} R^q M_{2,\sigma}(f; R) + \\ &\quad + \sup_{0 \leq t_0 \leq T} \sup_{R \geq 1} R^p N_{2,0}(f; t_0, R) + \sup_{0 \leq t_0 \leq T} \sup_{R \geq 1} R^p N_{2,\sigma}(f; t_0, R) \\ \|f\|_{X_{q,p}(T)} &= \sup_{0 < R \leq 1} R^q M_\infty(f; R) + \sup_{0 \leq t_0 \leq T} \sup_{R \geq 1} R^p N_\infty(f; t_0, R) \end{aligned}$$

$$\begin{aligned} |||f|||_{q,p} &= \sup_{0 \leq x \leq 1} \{x^q |f(x)|\} + \sup_{x > 1} \{x^p |f(x)|\} \\ |||f|||_{\sigma} &= \sup_{0 \leq t \leq T} |||f|||_{\frac{3}{2}, \frac{3+\lambda}{2}} + \|f\|_{Y_{\frac{3}{2}, \frac{3+\lambda}{2}}^{\sigma}(T)} \end{aligned}$$

and the following spaces:

$$\begin{aligned} Y_{q,p}^{\sigma}(T) &= \left\{ f : \|f\|_{Y_{q,p}^{\sigma}(T)} < \infty \right\}, \\ X_{q,p}^{\sigma}(T) &= \left\{ f : \|f\|_{X_{q,p}^{\sigma}(T)} < \infty \right\}, \\ \mathcal{E}_{T;\sigma} &= \{f : |||f|||_{\sigma} < \infty\} \end{aligned}$$

The suitable choice of σ for our purpose is $\sigma \in (\frac{1}{2}, 1)$. The Sobolev embeddings implies then:

$$Y_{q,p}^{\sigma}(T) \subset X_{q,p}(T).$$

We may then solve the problem (1.1)-(1.3) using a functional space that measures the regularizing effects of the coagulation equation as $x \rightarrow \infty$. Let $\eta \in C^{\infty}(\mathbb{R}^+)$ a cutoff function satisfying $\eta(x) = 1$ for $x \in (\frac{1}{4}, 3)$, $\eta(x) = 0$ for $x \notin (\frac{1}{8}, 4)$. Given $f \in C(\mathbb{R}^+)$, $t_0 \in [0, T]$, $R \geq 1$ we define:

$$F_{R,t_0}(\theta, X) = \eta(RX) f\left(t_0 + \theta R^{-(\lambda-1)/2}, RX\right)$$

and:

$$[f]_p^{\sigma; \frac{1}{2}} = \sup_{R \geq 1} \sup_{0 \leq t_0 \leq T} R^p \left(\int_{t_0}^{\min(t_0 + R^{-(\lambda-1)/2}, T)} \int_{\mathbb{R}} |\widehat{F}_{R,t_0}(\theta, k)|^2 (1 + |k|^{2\sigma}) (1 + \min\{|k|, R\}) dk d\theta \right)^{1/2}$$

$$\begin{aligned} \|f\|_{\mathcal{Z}_p^{\sigma; \frac{1}{2}}(T)} &= \|f\|_{L^2((0,T); H_x^{\sigma}(0,2))} + [f]_p^{\sigma; \frac{1}{2}} + \sup_{0 \leq t \leq T} |||f|||_{\frac{3}{2}, p} + \|f\|_{Y_{\frac{3}{2}, p}^{\sigma}(T)} \\ \mathcal{Z}_p^{\sigma; \frac{1}{2}}(T) &= \left\{ f : \|f\|_{\mathcal{Z}_p^{\sigma; \frac{1}{2}}(T)} < \infty \right\} \end{aligned}$$

These spaces are well adapted to the study of the non linear coagulation operator C as show the following result whose proof may be found in [12]:

Lemma 4.6. *Let C be the coagulation operator defined in (1.13). Then, for any $\sigma \in (\frac{1}{2}, 1)$, and any $\bar{\delta} > 0$ there exists a positive constant $c = c(\sigma, \bar{\delta})$ such that for any $h \in \mathcal{Z}_{\left(\frac{3+\lambda}{2} + \bar{\delta}\right)}^{\sigma; \frac{1}{2}}(T)$:*

$$\|C[h]\|_{Y_{\frac{3}{2}, (2+\bar{\delta})}^{\sigma}(T)} \leq c \|h\|_{\mathcal{Z}_{\left(\frac{3+\lambda}{2} + \bar{\delta}\right)}^{\sigma; \frac{1}{2}}(T)}^2.$$

5. THE PHENOMENA OF B-E CONDENSATION AND GELATION.

The solutions with non zero fluxes are related with two interesting phenomena. As for the first one, it is shown by the experiments that below a critical temperature a gas of bosons undergoes the Bose Einstein (B-E) condensation: at finite time, a positive fraction of the density of particles concentrate at the lowest energy level giving rise to what is called a Bose Einstein condensate. At the same time takes place a loss of the total

number of particles in the gas. The presence of a nonzero flux of particles towards the particles of zero momentum makes tempting to think that the solutions constructed in this paper could provide some information about the dynamic growth of Bose-Einstein condensates. However, this does not seem to be the case since the zero momentum particles would not interact at all with the particles outside the condensate and, a more careful analysis rather yields models (cf. [1, 26]) where the condensate interacts with the particles that are not in the condensate. A recent and detailed description of the mathematical questions related with this phenomena and several enlightening results may be found in [23].

Our second example considers reactive polymers in liquid sol, whose density function is described by the coagulation equation. If the reaction rate is strong enough, the gelation phenomena takes place: macroscopic branched chain molecules appear containing part of the mass previously in the polymer. Simultaneously, the total mass of polymers still in the sol phase decreases. The sudden transformation of the reaction mixture from a viscous liquid to an elastic material of infinite viscosity, which transformation characterizes the gel point, has been attributed to the formation of infinite network structures, i.e., molecular structures which assume macroscopic size and which extend throughout the medium.

This phenomena has been related with the following property of the coagulation equation (1.12), (1.13) with kernel $W(x, y) = x^\alpha y^\beta + x^\beta y^\alpha$ that was first proved by I. Jeon in [15] and later by different methods in [8]: if $\lambda = \alpha + \beta \in (1, 2]$ then for all solution $f(t)$ with non negative initial data f_{in} of finite mass there exists $T > 0$ and $C > 0$ such that:

$$\int_0^\infty x f(t, x) dx = \int_0^\infty x f_{in}(x) dx \quad \forall 0 < t < T$$

$$\int_0^\infty x f(t, x) dx \leq \frac{C}{(1+t)^{1/\lambda}}, \quad \forall t > 0.$$

These two facts show that the total mass of the particles whose density function is f decreases after some finite time T^* . The non zero flux solutions whose existence we have presented above may then be seen as possible descriptions of the density function of polymers after the gelation phenomena has happened at $t = T^*$.

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