SEMINAIRE

Equations aux Dérivées Partielles

2008-2009

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Séminaire É. D. P. (2008-2009), Exposé n° VII, 19 p.

<http://sedp.cedram.org/item?id=SEDP_2008-2009____A7_0>

Séminaire *Équations aux dérivées partielles*
Centre de mathématiques Laurent Schwartz, 2008-2009
Exposé n^{o} VII, 1-19

DERIVATION AND MATHEMATICAL ANALYSIS OF A NONLOCAL MODEL FOR LARGE AMPLITUDE INTERNAL WAVES

by

David Lannes

Abstract. — This note is devoted to the study of a bi-fluid generalization of the nonlinear shallow-water equations. It describes the evolution of the interface between two fluids of different densities. In the case of a two-dimensional interface, this systems contains unexpected nonlocal terms (that are of course not present in the usual one-fluid shallow water equations). We show here how to derive this systems from the two-fluid Euler equations and then show that it is locally well-posed.

1. Introduction

This note is devoted to the study of the equations describing the interface between two layers of immiscible fluids of different densities. The focus is here in a particular regime called shallow-water/shallow water (more briefly SW/SW) because both fluids layers are in a shallow-water regime (i.e. their height is small compared to the wavelength of the interfacial waves under consideration). In this particular regime, we show how to derive an asymptotic model from the two-fluids Euler equation, and then analyze this model.

The idealized system that will be the focus of the discussion here, when it is at rest, consists of a homogeneous fluid of depth d_1 and density ρ_1 lying over another homogeneous fluid of depth d_2 and density $\rho_2 > \rho_1$. The bottom on which both fluids rest is presumed to be horizontal and featureless while the top of fluid 1 is restricted by the rigid lid assumption, which is to say, the top is viewed as an impenetrable, bounding surface. We also assume that the deviation of the interface is a graph over the flat bottom (see Figure 1 for a definition sketch).

In [**3**], a rigorous and systematic derivation of a plethora of asymptotic models for this system has been presented. The fact that many different asymptotic models can

Ce travail a bnfici d'une aide de l'Agence Nationale de la Recherche portant la rfrence ANR-08-BLAN-0301-01.

be derived comes from the large number of physical parameters playing a role on the dynamics: heights of both fluids, wavelength, amplitude...

In Section 2, we describe the strategy of [**3**] in the particular case of the SW/SW regime (see (14)) and proceed to the derivation of the so-called SW/SW model, which is a generalization of the 2D nonlinear shallow water equations to the two fluids system. The originality of this generalization is that it contains some quite unexpected nonlocal terms.

In Section 3 we then show that this model is locally well-posed in Sobolev spaces.

2. Derivation if the SW-SW model

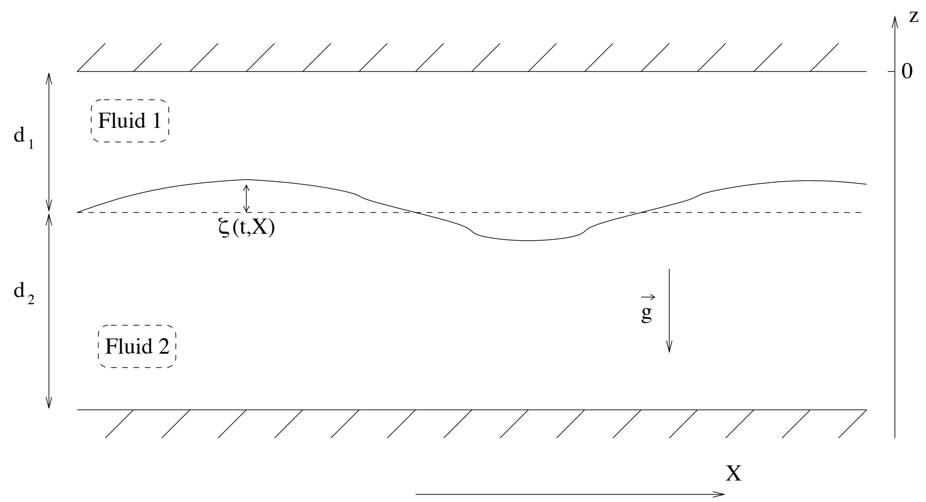
2.1. The two layers Euler equations. — As in Figure 1, the origin of the vertical coordinate z is taken at the rigid top of the two-fluid system. Assuming each fluid is incompressible and each flow irrotational, there exists velocity potentials Φ_i $(i = 1, 2)$ associated to both the upper and lower fluid layers which satisfy

$$\Delta_{X,z}\Phi_i = 0 \quad \text{in } \Omega_t^i \tag{1}$$

for all time t, where Ω_t^i denotes the region occupied by fluid i at time t, $i = 1, 2$. As above, fluid 1 refers to the upper fluid layer whilst fluid 2 is the lower layer (see again Figure 1). Assuming that the densities ρ_i, $i = 1, 2$, of both fluids are constant, we also have two Bernoulli equations, namely,

$$\partial_t\Phi_i + \frac{1}{2}|\nabla_{X,z}\Phi_i|^2 = -\frac{P}{\rho_i} - gz \quad \text{in } \Omega_t^i, \tag{2}$$

where g denotes the acceleration of gravity and P the pressure inside the fluid. These equations are complemented by two boundary conditions stating that the velocity must be horizontal at the two rigid surfaces $\Gamma_1 := \{z = 0\}$ and $\Gamma_2 := \{z = -d_1 - d_2\}$,



which is to say

$$\partial_z \Phi_i = 0 \quad \text{on} \quad \Gamma_i, \qquad (i = 1, 2). \tag{3}$$

Finally, as mentioned earlier, it is presumed that the interface is given as the graph of a function $\zeta(t, X)$ which expresses the deviation of the interface from its rest position $(X, -d_1)$ at the spatial coordinate X at time t. The interface $\Gamma_t := \{z = -d_1 + \zeta(t, X)\}$ between the fluids is taken to be a bounding surface, or equivalently it is assumed that no fluid particle crosses the interface. This condition, written for fluid i, is classically expressed by the relation $\partial_t \zeta = \sqrt{1 + |\nabla\zeta|^2} v_n^i$, where v_n^i denotes the upwards normal derivative of the velocity of fluid i at the surface. Since this equation must of course be independent of which fluid is being contemplated, it follows that the normal component of the velocity is continuous at the interface. The two equations

$$\partial_t \zeta = \sqrt{1 + |\nabla\zeta|^2}\partial_n \Phi_1 \quad \text{on} \quad \Gamma_t, \tag{4}$$

and

$$\partial_n \Phi_1 = \partial_n \Phi_2 \quad \text{on} \quad \Gamma_t, \tag{5}$$

with

$$\partial_n := \mathbf{n} \cdot \nabla_{X,z} \quad \text{and} \quad \mathbf{n} := \frac{1}{\sqrt{1 + |\nabla\zeta|^2}}(-\nabla\zeta, 1)^T$$

follow as a consequence. A final condition is needed on the pressure to close this set of equations, namely,

$$P \text{ is continuous at the interface}, \tag{6}$$

if we neglect surface tension effects (see Remark 8 for a comment on this point).

2.2. Transformation of the Equations. — In this subsection, a new set of equations is deduced from the internal-wave equations (1)-(6). Introduce the trace of the potentials Φ_1 and Φ_2 at the interface,

$$\psi_i(t, X) := \Phi_i(t, X, -d_1 + \zeta(t, X)), \qquad (i = 1, 2).$$

One can evaluate Eq. (2) at the interface and use (4) and (5) to obtain a set of equations coupling ζ to ψ_i $(i = 1, 2)$, namely

$$\partial_t \zeta - \sqrt{1 + |\nabla\zeta|^2}\partial_n \Phi_i \quad = 0, \tag{7}$$

$$\rho_i\Big(\partial_t \psi_i + g\zeta + \frac{1}{2}|\nabla\psi_i|^2 - \frac{(\sqrt{1 + |\nabla\zeta|^2}(\partial_n \Phi_i) + \nabla\zeta \cdot \nabla\psi_i)^2}{2(1 + |\nabla\zeta|^2)}\Big) \quad = -P, \tag{8}$$

where in (7) and (8), $(\partial_n \Phi_i)$ and P are both evaluated at the interface $z = -d_1 + \zeta(t, X)$. Notice that $\partial_n \Phi_1$ is fully determined by ψ_1 since Φ_1 is uniquely given as the solution of Laplace's equation (1) in the upper fluid domain, the Neumann condition (3) on Γ_1 and the Dirichlet condition $\Phi_1 = \psi_1$ at the interface. Following the formalism introduced for the study of surface water waves in [**5, 6, 15**], we can therefore define the Dirichlet-Neumann operator $G[\zeta]\cdot$ by

$$G[\zeta]\psi_1 = \sqrt{1 + |\nabla\zeta|^2}(\partial_n \Phi_1)_{|z=-d_1+\zeta}. \tag{9}$$

Similarly, one remarks that ψ_2 is determined up to a constant by ψ_1 since Φ_2 is given (up to a constant) by the resolution of the Laplace equation (1) in the lower fluid domain, with Neumann boundary conditions (3) on Γ_2 and $\partial_n\Phi_2 = \partial_n\Phi_1$ at the interface (this latter being provided by (5)). It follows that ψ_1 fully determines $\nabla\psi_2$ and we may thus define the operator $\mathbf{H}[\zeta]\cdot$ by

$$\mathbf{H}[\zeta]\psi_1 = \nabla\psi_2. \tag{10}$$

Using the continuity of the pressure at the interface expressed in (6), we may equate the left-hand sides of $(8)_1$ and $(8)_2$ using the operators $G[\zeta]$ and $\mathbf{H}[\zeta]$ just defined. This yields the equation

$$\partial_t(\psi_2 - \gamma\psi_1) + g(1-\gamma)\zeta + \frac{1}{2}\big(|\mathbf{H}[\zeta]\psi_1|^2 - \gamma|\nabla\psi_1|^2\big) + \mathcal{N}(\zeta,\psi_1) = 0$$

where $\gamma = \rho_1/\rho_2$ and

$$\mathcal{N}(\zeta,\psi_1) := \frac{\gamma\big(G[\zeta]\psi_1 + \nabla\zeta\cdot\nabla\psi_1\big)^2 - \big(G[\zeta]\psi_1 + \nabla\zeta\cdot\mathbf{H}[\zeta]\psi_1\big)^2}{2(1+|\nabla\zeta|^2)}.$$

Taking the gradient of this equation and using (7) then gives the system of equations

$$\begin{cases} \partial_t\zeta - G[\zeta]\psi_1 = 0, \\ \partial_t(\mathbf{H}[\zeta]\psi_1 - \gamma\nabla\psi_1) + g(1-\gamma)\nabla\zeta \\ \qquad + \frac{1}{2}\nabla\big(|\mathbf{H}[\zeta]\psi_1|^2 - \gamma|\nabla\psi_1|^2\big) + \nabla\mathcal{N}(\zeta,\psi_1) = 0, \end{cases} \tag{11}$$

for ζ and ψ_1. This is the system of equations that will be used in the next sections to derive asymptotic models.

Remark 1. — Setting $\rho_1 = 0$, and thus $\gamma = 0$, in the above equations, one recovers the usual surface water-wave equations written in terms of ζ and ψ as in [**5, 6, 15**].

2.3. Non-Dimensionalization of the Equations. — The asymptotic behavior of (11) is more transparent when these equations are written in dimensionless variables. Denoting by a a typical amplitude of the deformation of the interface in question, and by λ a typical wavelength, the following dimensionless independent variables

$$\widetilde{X} := \frac{X}{\lambda}, \quad \widetilde{z} := \frac{z}{d_1}, \quad \widetilde{t} := \frac{t}{\lambda/\sqrt{gd_1},},$$

are introduced. Likewise, we define the dimensionless unknowns

$$\widetilde{\zeta} := \frac{\zeta}{a}, \quad \widetilde{\psi_1} := \frac{\psi_1}{a\lambda\sqrt{g/d_1}},$$

as well as the dimensionless parameters

$$\gamma := \frac{\rho_1}{\rho_2}, \quad \delta := \frac{d_1}{d_2}, \quad \varepsilon := \frac{a}{d_1}, \quad \mu := \frac{d_1^2}{\lambda^2}.$$

Though they are redundant, it is also notationally convenient to introduce two other parameter's ε_2 and μ_2 defined as

$$\varepsilon_2 = \frac{a}{d_2} = \varepsilon\delta, \qquad \mu_2 = \frac{d_2^2}{\lambda^2} = \frac{\mu}{\delta^2}.$$

The equations (11) can then be written in dimensionless variables as

$$\left\{\begin{array}{lcl} \partial_{\widetilde{t}}\widetilde{\zeta} - \dfrac{1}{\mu}G^{\mu}[\varepsilon\widetilde{\zeta}]\widetilde{\psi}_1 & = & 0, \\ \partial_{\widetilde{t}}\big(\mathbf{H}^{\mu,\delta}[\varepsilon\widetilde{\zeta}]\widetilde{\psi}_1 - \gamma\nabla\widetilde{\psi}_1\big) + (1-\gamma)\nabla\widetilde{\zeta} & & \\ \quad + \dfrac{\varepsilon}{2}\nabla\big(|\mathbf{H}^{\mu,\delta}[\varepsilon\widetilde{\zeta}]\widetilde{\psi}_1|^2 - \gamma|\nabla\widetilde{\psi}_1|^2\big) + \varepsilon\nabla\mathcal{N}^{\mu,\delta}(\varepsilon\widetilde{\zeta},\widetilde{\psi}_1) & = & 0, \end{array}\right. \tag{12}$$

where $\mathcal{N}^{\mu,\delta}$ is defined for all pairs (ζ,ψ) smooth enough by the formula

$$\mathcal{N}^{\mu,\delta}(\zeta,\psi) := \mu\frac{\gamma\big(\frac{1}{\mu}G^{\mu}[\zeta]\psi + \nabla\zeta\cdot\nabla\psi\big)^2 - \big(\frac{1}{\mu}G^{\mu}[\zeta]\psi + \nabla\zeta\cdot\mathbf{H}^{\mu,\delta}[\zeta]\psi\big)^2}{2(1+\mu|\nabla\zeta|^2)},$$

and where the operators G^{μ} and $\mathbf{H}^{\mu,\delta}$ are the nondimensionalized versions of the Dirichlet-Neumann and interface operators defined in (9) and (10) (see §2.5 and §2.6 for precise definitions).

Notation 1. — The tildes which indicate the non-dimensional quantities will be systematically dropped henceforth.

Remark 2. — Linearizing the equations (12) around the rest state, one finds the linearized dispersion relation

$$\omega^2 = (1-\gamma)\frac{|\mathbf{k}|}{\sqrt{\mu}}\frac{\tanh(\sqrt{\mu}|\mathbf{k}|)\tanh(\frac{\sqrt{\mu}}{\delta}|\mathbf{k}|)}{\tanh(\sqrt{\mu}|\mathbf{k}|)+\gamma\tanh(\frac{\sqrt{\mu}}{\delta}|\mathbf{k}|)}; \tag{13}$$

corresponding to plane-wave solutions $e^{i\mathbf{k}\cdot X - i\omega t}$. In particular, the expected instability is found when $\gamma > 1$, corresponding to the case wherein the heavier fluid lies over the lighter one.

2.4. The Shallow-Water/Shallow-Water (SW/SW) regime. — In [**3**], many asymptotics of the non-dimensionalized equations (12) are studied, in various physical regimes corresponding to different relationships among the dimensionless parameters ε, μ and δ. Here, we mainly focus on the *Shallow-Water/Shallow-Water* regime characterized by

$$\text{Shallow Water/Shallow Water (SW/SW) regime:} \qquad \mu \sim \mu_2 \ll 1. \tag{14}$$

Since no assumption is made on the amplitude of the interfacial waves, this regime allows large interfacial amplitudes ($\varepsilon \sim \varepsilon_2 = O(1)$).

2.5. Asymptotic expansion of the Dirichlet-Neumann operator. — Let us first define the nondimensionalized Dirichlet-Neumann operator $G^{\mu}[\varepsilon\zeta]\cdot$ that appears in (12). Denoting the non-dimensionalized upper fluid domain by

$$\Omega_1 = \{(X,z)\in\mathbb{R}^{d+1}, -1+\varepsilon\zeta(X) < z < 0\}$$

and assuming that the height of this domain never vanishes,

$$\exists H_1 > 0, \qquad 1 - \varepsilon\zeta \geq H_1 \quad \text{on} \quad \mathbb{R}^d, \tag{15}$$

we can state the following definition:

Definition 1. — Let $\zeta \in W^{2,\infty}(\mathbb{R}^d)$ be such that (15) is satisfied and let $\psi_1 \in H^{3/2}(\mathbb{R}^d)$. If Φ_1 is the unique solution in $H^2(\Omega_1)$ of the boundary-value problem

$$\left\{ \begin{array}{ll} \mu\Delta\Phi_1 + \partial_z^2\Phi_1 = 0 & \text{in } \Omega_1, \\ \partial_z\Phi_{1\,|_{z=0}} = 0, & \Phi_{1\,|_{z=-1+\varepsilon\zeta(X)}} = \psi_1, \end{array} \right. \tag{16}$$

then $G^\mu[\varepsilon\zeta]\psi_1 \in H^{1/2}(\mathbb{R}^d)$ is defined by

$$G^\mu[\varepsilon\zeta]\psi_1 = -\mu\varepsilon\nabla\zeta\cdot\nabla\Phi_{1\,|_{z=-1+\varepsilon\zeta}} + \partial_z\Phi_{1\,|_{z=-1+\varepsilon\zeta}}.$$

Remark 3. — Another way to approach G^μ is to define

$$G^\mu[\varepsilon\zeta]\psi_1 = \sqrt{1+\varepsilon^2|\nabla\zeta|^2}\partial_n\Phi_{1\,|_{z=-1+\varepsilon\zeta}}$$

where $\partial_n\Phi_{1\,|_{z=-1+\varepsilon\zeta}}$ stands for the upper conormal derivative associated to the elliptic operator $\mu\Delta\Phi_1 + \partial_z^2\Phi_1$.

The following lemma connects ζ with the vertically integrated horizontal velocity via the Dirichlet-Neumann operator $G^\mu[\varepsilon\zeta]\cdot$ (the proof is a consequence of Green's identity).

Lemma 1. — *Let $\zeta \in W^{2,\infty}(\mathbb{R}^d)$ be such that (15) is satisfied and let $\psi \in H^{3/2}(\mathbb{R}^d)$ and Φ_1 be the solution of (16) with $\psi_1 = \psi$. If V^μ is defined by*

$$V^\mu[\varepsilon\zeta]\psi := \int_{-1+\varepsilon\zeta}^0 (\sqrt{\mu}\nabla\Phi_1)dz,$$

then one has

$$G^\mu[\varepsilon\zeta]\psi = \sqrt{\mu}\nabla\cdot(V^\mu[\varepsilon\zeta]\psi).$$

If $\mu \ll 1$ (shallow water regime for the upper fluid), it is possible to obtain an expansion of $V^\mu[\varepsilon\zeta]\psi$ with respect to μ which is uniform with respect to $\varepsilon \in [0,1]$ (for the asymptotic regimes where μ is not small, other techniques must be used [**3**]).

Proposition 1. — *Let $s > d/2$ and $\zeta \in H^{s+3/2}(\mathbb{R}^d)$. Then for all $\mu \in (0,1)$ and ψ such that $\nabla\psi \in H^{s+5/2}(\mathbb{R}^d)$, one has*

$$\big|\sqrt{\mu}V^\mu[\varepsilon\zeta]\psi - \mu(1-\varepsilon\zeta)\nabla\psi\big|_{H^s} \leq \mu^2 C(|\zeta|_{H^{s+3/2}}, |\nabla\psi|_{H^{s+5/2}}),$$

uniformly with respect to $\varepsilon \in [0,1]$), where $V^\mu[\varepsilon\zeta]\psi$ is as defined in Lemma 1 (so that $G^\mu[\varepsilon\zeta]\psi = \sqrt{\mu}\nabla\cdot V^\mu[\varepsilon\zeta]\psi$).

Proof. — This follows from well known results on the Dirichlet-Neumann operator in the case of one single fluid layer (e.g. Proposition 3.8 of [**1**]). □

2.6. Asymptotic expansion of the interface operator. — We first define here the dimensionless operator $\mathbf{H}^{\mu,\delta}[\varepsilon\zeta]\cdot$ that appears in (12). Denoting the non-dimensionalized lower fluid domain by

$$\Omega_2 = \{(X,z) \in \mathbb{R}^{d+1}, -1-1/\delta < z < -1+\varepsilon\zeta(X)\},$$

and assuming that the height of this domain never vanishes,

$$\exists H_2 > 0, \qquad 1+\varepsilon\delta\zeta \geq H_2 \quad \text{on} \quad \mathbb{R}^d, \tag{17}$$

we can state the following definition:

Definition 2. — Let $\zeta \in W^{2,\infty}(\mathbb{R}^d)$ be such that (15) and (17) are satisfied, and suppose that $\psi_1 \in H^{3/2}(\mathbb{R}^d)$ is given. If the function Φ_2 is the unique solution (up to a constant) of the boundary-value problem

$$\left\{ \begin{array}{ll} \mu\Delta\Phi_2 + \partial_z^2\Phi_2 = 0 & \text{in } \Omega_2, \\ \partial_z\Phi_2\,_{|z=-1-1/\delta} = 0, & \partial_n\Phi_2\,_{|z=-1+\varepsilon\zeta(X)} = \frac{1}{(1+\varepsilon^2|\nabla\zeta|^2)^{1/2}} G^\mu[\varepsilon\zeta]\psi_1, \end{array}\right. \tag{18}$$

then the operator $\mathbf{H}^{\mu,\delta}[\varepsilon\zeta]\cdot$ is defined on ψ_1 by

$$\mathbf{H}^{\mu,\delta}[\varepsilon\zeta]\psi_1 = \nabla(\Phi_2\,_{|z=-1+\varepsilon\zeta}) \in H^{1/2}(\mathbb{R}^d).$$

Remark 4. — In the statement above, $\partial_n\Phi_2\,_{|z=-1+\varepsilon\zeta}$ stands here for the upwards conormal derivative associated to the elliptic operator $\mu\Delta\Phi_2 + \partial_z^2\Phi_2$,

$$\sqrt{1+\varepsilon^2|\nabla\zeta|^2}\partial_n\Phi_2\,_{|z=-1+\varepsilon\zeta} = -\mu\varepsilon\nabla\zeta\cdot\nabla\Phi_2\,_{|z=-1+\varepsilon\zeta} + \partial_z\Phi_2\,_{|z=-1+\varepsilon\zeta}.$$

The Neumann boundary condition of (18) at the interface can also be stated as $\partial_n\Phi_2\,_{|z=-1+\varepsilon\zeta} = \partial_n\Phi_1\,_{|z=-1+\varepsilon\zeta}$.

Remark 5. — Of course, the solvability of (18) requires the condition $\int_\Gamma \partial_n\Phi_2 d\Gamma = 0$ (where $d\Gamma = \sqrt{1+\varepsilon^2|\nabla\zeta|^2}dX$ is the Lebesgue measure on the surface $\Gamma = \{z = -1+\varepsilon\zeta\}$). This is automatically satisfied thanks to the definition of $G^\mu[\varepsilon\zeta]\psi_1$. Indeed, applying Green's identity to (16), one obtains

$$\int_\Gamma \partial_n\Phi_2 d\Gamma = \int_\Gamma \partial_n\Phi_1 d\Gamma = -\int_{\Omega_1}(\mu\Delta\Phi_1 + \partial_z^2\Phi_1) = 0.$$

The boundary-value problem (18) plays a key role in the analysis of the operator $\mathbf{H}^{\mu,\delta}[\varepsilon\zeta]\cdot$. The analysis of this problem is easier if we first transform it into a variable-coefficient, boundary-value problem on the flat strip $\mathcal{S} := \mathbb{R}^d \times (-1,0)$ using the diffeomorphism

$$\sigma : \begin{array}{ccc} \mathcal{S} & \to & \Omega_2 \\ (X,z) & \mapsto & \sigma(X,z) := (X, (1+\varepsilon\delta)\frac{z}{\delta} + (-1+\varepsilon\zeta)). \end{array}$$

As shown in Proposition 2.7 of [**11**] (see also §2.2 of [**1**]), Φ_2 solves (18) if and only if $\underline{\Phi}_2 := \Phi_2 \circ \sigma$ solves

$$\left\{ \begin{array}{ll} \nabla^{\mu_2}_{X,z}\cdot Q^{\mu_2}[\varepsilon_2\zeta]\nabla^{\mu_2}_{X,z}\underline{\Phi}_2 = 0 & \text{in } \mathcal{S}, \\ \partial_n\underline{\Phi}_2\,_{|z=0} = \frac{1}{\delta}G^\mu[\varepsilon\zeta]\psi_1, & \partial_n\underline{\Phi}_2\,_{|z=-1} = 0, \end{array}\right. \tag{19}$$

with

$$Q^{\mu_2}[\varepsilon_2\zeta] = \begin{pmatrix} (1+\varepsilon_2\zeta)I_{d\times d} & -\sqrt{\mu_2}\varepsilon_2(z+1)\nabla\zeta \\ -\sqrt{\mu_2}\varepsilon_2(z+1)\nabla\zeta^T & \frac{1+\mu_2\varepsilon_2^2(z+1)^2|\nabla\zeta|^2}{1+\varepsilon_2\zeta} \end{pmatrix},$$

and where, as before, $\varepsilon_2 = \varepsilon\delta$, $\mu_2 = \frac{\mu}{\delta^2}$, and $\nabla^{\mu_2}_{X,z} = (\sqrt{\mu_2}\nabla, \partial_z)^T$.

Remark 6. — As always in the present exposition, $\partial_n\underline{\Phi}_2$ stands for the *upward* conormal derivative associated to the elliptic operator involved in the boundary-value problem,

$$\partial_n\underline{\Phi}_2\,_{|z=0 \text{ or } z=-1} = \mathbf{e_z}\cdot Q^{\mu_2}[\varepsilon_2\zeta]\nabla^{\mu_2}_{X,z}\underline{\Phi}_2\,_{|z=0 \text{ or } z=-1},$$

where $\mathbf{e_z}$ is the upward-pointing unit vector along the vertical axis.

An asymptotic expansion of

$$\mathbf{H}^{\mu,\delta}[\varepsilon\zeta]\psi_1 = \nabla(\underline{\Phi}_{2\,|z=0}), \tag{20}$$

is obtained by finding an approximation $\underline{\Phi}_{app}$ to the solution of (19) and then using the formal relationship $\mathbf{H}^{\mu,\delta}[\varepsilon\zeta]\psi_1 \sim \nabla(\underline{\Phi}_{app\,|z=0})$. This procedure is justified in the following proposition. To state the result, it is useful to have in place the spaces

$$H^{s,k}(\mathcal{S}) = \{f \in \mathcal{D}'(\overline{\mathcal{S}}) : \|f\|_{H^{s,k}} < \infty\}$$

for $s \in \mathbb{R}$ and $k \in \mathbb{N}$, where $\|f\|_{H^{s,k}} = \sum_{j=0}^{k} \|\Lambda^{s-j}\partial_z^j f\|$.

Proposition 2. — *Let $s_0 > d/2$, $s \geq s_0 + 1/2$, and $\zeta \in H^{s+3/2}(\mathbb{R}^d)$ be such that (15) and (17) are satisfied (the interface does not touch the horizontal boundaries). If $\mathbf{h} \in H^{s+1/2,1}(\mathcal{S})^{d+1}$ and $V \in H^{s+1}(\mathbb{R}^d)^d$ are given, then the boundary-value problem*

$$\begin{cases} \nabla^{\mu_2}_{X,z} \cdot Q^{\mu_2}[\varepsilon_2\zeta]\nabla^{\mu_2}_{X,z} u = \nabla^{\mu_2}_{X,z} \cdot \mathbf{h} & \text{in } \mathcal{S}, \\ \partial_n u_{|z=0} = \sqrt{\mu_2}\nabla \cdot V + \mathbf{e_z} \cdot \mathbf{h}_{|z=0}, & \partial_n u_{|z=-1} = \mathbf{e_z} \cdot \mathbf{h}_{|z=-1} \end{cases} \tag{21}$$

admits (up to a constant) a unique solution u. Moreover, the solution u obeys the inequality

$$\big|\nabla u_{|z=0}\big|_{H^s} \leq \frac{1}{\sqrt{\mu_2}} C(\frac{1}{H_2}, \varepsilon_2^{max}, \mu_2^{max}, |\zeta|_{H^{s+3/2}})\big(\|\mathbf{h}\|_{H^{s+1/2,1}} + |V|_{H^{s+1}}\big),$$

uniformly with respect to $\varepsilon_2 \in [0, \varepsilon_2^{max}]$ and $\mu_2 \in (0, \mu_2^{max})$.

Remark 7. — Suppose we take $\mathbf{h} = 0$ and $V = V^{\mu}[\varepsilon\zeta]\psi$ in Proposition 3. By Lemma 1, one has $\nabla u_{|z=0} = \mathbf{H}^{\mu,\delta}[\varepsilon\zeta]\psi$. and the Proposition thus provides an estimate of the operator norm of $\mathbf{H}^{\mu,\delta}[\varepsilon\zeta]$.

Proof. — The main lines of the proof are:

1. Check the coercivity of $Q^{\mu_2}[\varepsilon_2\zeta]$
2. Derive estimates on $\nabla^{\mu_2}_{X,z} u$ in $H^{r,1}$ ($r \geq 0$) by elliptic estimates
3. Use the trace theorem to control $|\nabla u_{|z=0}|_{H^s} \lesssim \|u\|_{H^{s+1/2,1}} \lesssim \frac{1}{\mu_2}\|\nabla^{\mu_2}_{X,z} u\|_{H^{s+1/2,1}}$ and use Step 2

□

The remaining task is therefore to find an approximation $\underline{\Phi}_{app}$ to the solution of (19). As for the expansion of the Dirichlet-Neumann operator, various techniques must be used depending on the regime under consideration (see [**3**]). We focus here in the SW/SW regime (14).

In this regime, large amplitude waves are allowed for the upper fluid ($\varepsilon = O(1)$) and for the lower fluid ($\varepsilon_2 = O(1)$). Assuming that $\mu \ll 1$ and $\mu_2 \ll 1$ raises the prospect of making asymptotic expansions of shallow-water type, in terms of μ and μ_2. As before, the plan is to formally construct an approximate solution $\underline{\Phi}_{app}$ to (19) having the form

$$\underline{\Phi}_{app} = \Phi^{(0)} + \mu_2\Phi^{(1)}.$$

(such a form exploits the assumption that μ_2 is small). From the expression for $Q^{\mu_2}[\varepsilon_2\zeta]$, we may write

$$\nabla^{\mu_2}_{X,z} \cdot Q^{\mu_2}[\varepsilon_2\zeta]\nabla^{\mu_2}_{X,z} = \frac{1}{h_2}\partial_z^2 + \mu_2 \nabla_{X,z} \cdot Q_1 \nabla_{X,z},$$

with $h_2 = 1 + \varepsilon_2\zeta$ and where an explicit formula can be derived for Q_1. At leading order, the elliptic operator of (19) thus reduces to $\frac{1}{h_2}\partial_z^2$, which amounts to discard the horizontal derivatives of the original Laplace operator. Consequently, the nonlocal effects of the Laplace operators disappear in this regime (but new, unexpected enough, nonlocal effects appear, as shown below).
Using Proposition 1 (and thus the assumption that μ is small) to approximate the Neumann condition at the interface of (19), one readily checks that $\Phi^{(0)}$ and $\Phi^{(1)}$ must solve

$$\begin{cases} \partial_z^2 \Phi^{(0)} = 0, \\ \partial_z \Phi^{(0)}{}_{|z=0} = 0, \qquad \partial_z \Phi^{(0)}{}_{|z=-1} = 0, \end{cases}$$

which is obviously solved by any $\Phi^{(0)}(X,z) = \Phi^{(0)}(X)$ independent of z, and

$$\begin{cases} \partial_z^2 \Phi^{(1)} = -h_2^2 \Delta \Phi^{(0)}, \\ \partial_z \Phi^{(1)}{}_{|z=0} = h_2\big(\varepsilon_2 \nabla\zeta \cdot \nabla\Phi^{(0)} + \delta \nabla \cdot (h_1 \nabla \psi_1)\big), \qquad \partial_z \Phi^{(1)}{}_{|z=-1} = 0, \end{cases}$$

where we have used the fact that $\Phi^{(0)}$ does not depend on z. Solving this second order ordinary differential equation in the variable z with the boundary condition at $z = 0$ yields (up to a function independent of z which we take equal to 0 for the sake of simplicity),

$$\Phi^{(1)} = -\frac{z^2}{2} h_2^2 \Delta \Phi^{(0)} + z(\partial_z \Phi_1{}_{|z=0}).$$

Matching the boundary condition at $z = -1$ leads to the restriction

$$\nabla \cdot (h_2 \nabla \Phi^{(0)}) = -\delta \nabla \cdot (h_1 \nabla \psi_1),$$

and we thus deduce the following asymptotic expansion of the interface operator:

$$\mathbf{H}^{\mu,\delta}[\varepsilon\zeta]\psi_1 \sim \nabla(\underline{\Phi}^{(0)}_{|z=0}) \sim -\delta(I + \Pi(\varepsilon_2\zeta\Pi\cdot))^{-1}\Pi(h_1 \nabla\psi_1), \tag{22}$$

where $\Pi = -\frac{\nabla\nabla^T}{|D|^2}$ is the orthogonal projector onto the gradient vector fields of $L^2(\mathbb{R}^d)^d$ defined earlier (and $h_1 = 1 - \varepsilon\zeta$, $h_2 = 1 + \varepsilon\delta\zeta$).

2.7. The SW-SW model. — We derive here the SW/SW model and establish that the internal-wave equations (12) are *consistent* with this model in the following precise sense.

Definition 3. — The internal wave equations (12) are *consistent* with a system S of $d+1$ equations for ζ and $\mathbf{v}$ if for all sufficiently smooth solutions (ζ, ψ_1) of (12) such that (15) and (17) are satisfied, the pair $(\zeta, \mathbf{v})$, with

$$\mathbf{v} = \mathbf{H}^{\mu,\delta}[\varepsilon\zeta]\psi_1 - \gamma\nabla\psi_1), \tag{23}$$

solves S up to a small residual called the *precision* of the asymptotic model.

Remark 8. — It is worth emphasis that above definition does not require the well-posedness of the internal wave equations (12). In fact, the two-layer water-wave system is known to be well-posed in Sobolev spaces in the presence of surface tension but ill-posed without surface tension due to Kelvin-Helmholtz instabilities (see for instance [**10**]). In consequence, one could simply add a small amount of surface tension at the interface between the two homogeneous layers to put oneself in a well-posed situation. The resulting analysis would be exactly the same and would, in fact, lead to the same asymptotic models. (Such an approach is used in [**14**] for the Benjamin-Ono equation). As the resulting model systems do not change, such a regularization has been eschewed here. We refer to [**13**] for a detailed analysis of the role of surface tension for the control of Kelvin-Helmholtz instabilities.

The following theorem shows that the internal wave equations are consistent in the SW/SW regime (14) with the Shallow water/Shallow water system,

$$\left\{\begin{array}{l} \partial_t\zeta + \nabla\cdot\big(h_1\mathfrak{R}[\varepsilon\zeta]\mathbf{v}\big) = 0, \\ \partial_t\mathbf{v} + (1-\gamma)\nabla\zeta + \dfrac{\varepsilon}{2}\nabla\Big(\big|\mathbf{v}-\gamma\mathfrak{R}[\varepsilon\zeta]\mathbf{v}\big|^2 - \gamma\big|\mathfrak{R}[\varepsilon\zeta]\mathbf{v}\big|^2\Big) = 0, \end{array}\right. \tag{24}$$

where $h_1 = 1-\varepsilon\zeta$, $h_2 = 1+\varepsilon\delta\zeta$, and the operator $\mathfrak{R}$ is defined by (recalling that $\Pi = -\frac{\nabla\nabla^T}{|D|^2}$)

$$\mathfrak{R}[\varepsilon\zeta]\mathbf{v} = \frac{1}{\gamma+\delta}\big(1-\Pi(\frac{1-\gamma}{\gamma+\delta}\varepsilon\delta\zeta\Pi\cdot)\big)^{-1}\Pi(h_2\mathbf{v}).$$

Theorem 1. — *Let* $0 < \delta^{min} < \delta^{max} \leq (1-\delta(1-H_1))^{-1}$. *The internal waves equations (12) are consistent with the SW/SW equations (24) in the sense of Definition 3, with a precision* $O(\mu)$, *and uniformly with respect to* $\varepsilon\in[0,1]$, $\mu\in(0,1)$ *and* $\delta\in[\delta^{min},\delta^{max}]$.

Remark 9. — Taking $\gamma = 0$ and $\delta = 1$ in the SW/SW equations (24) yields the usual shallow water equations for surface water waves.

Remark 10. — In the one-dimensional case $d = 1$, one has

$$\mathfrak{R}[\varepsilon\zeta]\mathbf{v} = \frac{h_2}{\delta h_1+\gamma h_2}\mathbf{v}$$

and the equations (24) take the simpler form

$$\left\{\begin{array}{l} \partial_t\zeta + \partial_x\big(\frac{h_1h_2}{\delta h_1+\gamma h_2}\mathbf{v}\big) = 0, \\ \partial_t\mathbf{v} + (1-\gamma)\partial_x\zeta + \frac{\varepsilon}{2}\partial_x\big(\frac{(\delta h_1)^2-\gamma h_2^2}{(\delta h_1+\gamma h_2)^2}|\mathbf{v}|^2\big) = 0, \end{array}\right. \tag{25}$$

which coincides of course with the system (5.26) of [**4**]. The presence of the nonlocal operator $\mathfrak{R}$, which does not seem to have been noticed before, appears to be a purely two dimensional effect. V. Duchene showed in a recent paper [**7**] that these nonlocal effects are due to the rigid lid assumption (they disappear if the top bottom is replaced by a free surface).

Proof. — First remark that with the range of parameters considered in the theorem, one has $\mu \sim \mu_2$ as $\mu \to 0$ while $\varepsilon \sim \varepsilon_2 = O(1)$.
By the definition (23) of $\mathbf{v}$ and using Proposition 1 and (22), one deduces from (12) that

$$\left\{ \begin{array}{l} \partial_t \zeta - \nabla \cdot ((1-\varepsilon\zeta)\nabla\psi_1) = O(\mu), \\ \partial_t \mathbf{v} + (1-\gamma)\nabla\zeta + \frac{\varepsilon}{2}\nabla(|\mathbf{H}^{\mu,\delta}[\varepsilon\zeta]\psi_1|^2 - \gamma|\nabla\psi_1|^2) = O(\mu). \end{array} \right. \tag{26}$$

Recall now that $\mathbf{H}^{\mu,\delta}[\varepsilon\zeta]\psi_1 = \mathbf{v} + \gamma\nabla\psi_1$; using this relation together with (22), one can get

$$\nabla\psi_1 = -\mathfrak{R}[\varepsilon\zeta]\mathbf{v} + O(\mu)$$

and consequently,

$$\begin{array}{rcl} \mathbf{H}^{\mu,\delta}[\varepsilon\zeta]\psi_1 & = & \mathbf{v} + \gamma\nabla\psi_1 \\ & = & \mathbf{v} - \gamma\mathfrak{R}[\varepsilon\zeta]\mathbf{v} + O(\mu). \end{array}$$

Replacing $\nabla\psi_1$ and $\mathbf{H}^{\mu,\delta}[\varepsilon\zeta]\psi_1$ by these two expressions in (26) yields the result. □

3. Mathematical analysis of the SW/SW system (24)

This section is devoted to the mathematical analysis of the SW/SW system (24). In 1D, this system reduces to (25) which is of quasilinear type. A detailed analysis of this system (with precise blow-up conditions) is given in [**9**]. We focus here on the 2D case because the presence of the nonlocal effects induces the main difficulties.

For the sake of notational simplicity, we take $\varepsilon = 1$ throughout this section.

3.1. Preliminary results on the operator $\mathfrak{R}[\zeta]$. — We choose to state here some properties of the operator $\mathfrak{R}[\zeta]$ that will be used in the analysis of the two-dimensional SW/SW equations (24). The first property deals with the operator norm of the nonlocal operator.

Proposition 3. — *Let $\gamma \in [0,1)$, $\delta > 0$ and $t_0 > 1$. Assume also that $\zeta \in L^\infty(\mathbb{R}^2)$ and satisfies*

$$(1 - |\zeta|_\infty) > 0 \quad \text{and} \quad (1 - \delta|\zeta|_\infty) > 0.$$

1. *The operator $\mathfrak{R}[\zeta] : L^2(\mathbb{R}^2)^2 \to L^2(\mathbb{R}^2)^2$ is well defined (see Definition 1) and (with $h_2 = 1 + \delta\zeta$)*

$$\forall \mathbf{v} \in L^2(\mathbb{R}^2)^2, \qquad |\mathfrak{R}[\zeta]\mathbf{v}|_2 \leq \frac{1}{\gamma + \delta - \delta(1-\gamma)|\zeta|_\infty}|h_2\mathbf{v}|_2.$$

2. *If moreover $\zeta \in H^s \cap H^{t_0+1}(\mathbb{R}^2)$ $(s \geq 0)$ then for all $v \in H^s(\mathbb{R}^2)^2$,*

$$\begin{array}{rcl} |\mathfrak{R}[\zeta]\mathbf{v}|_{H^s} & \leq & C\big(\frac{1}{\gamma + \delta - \delta(1-\gamma)|\zeta|_\infty}, \delta(1-\gamma)|\zeta|_{H^{t_0+1}}\big) \\ & & \times\big(|h_2\mathbf{v}|_{H^s} + \delta(1-\gamma)|\zeta|_{H^s}|\Pi(h_2\mathbf{v})|_{H^{t_0}}\big). \end{array}$$

Proof. — Since $\|\Pi(\frac{\gamma-1}{\gamma+\delta}\delta\zeta\Pi\cdot)\|_{L^2\to L^2} \leq \frac{1-\gamma}{\gamma+\delta}\delta|\zeta|_\infty$, the bilinear form $a(\mathbf{u},\mathbf{v})$ defined as

$$a(\mathbf{u},\mathbf{v}) = \big((1-\Pi(\frac{1-\gamma}{\gamma+\delta}\delta\zeta\Pi\cdot))\mathbf{u},\mathbf{v}\big)$$

is coercive and continuous on $L^2(\mathbb{R}^d)^2$, with coercivity and continuity constants respectively given by

$$k(\zeta) = 1 - \frac{1-\gamma}{\gamma+\delta}\delta|\zeta|_\infty \quad \text{and} \quad M(\zeta) = 1 + \frac{1-\gamma}{\gamma+\delta}\delta|\zeta|_\infty.$$

It follows therefore from Lax-Milgram's theorem that for all $f \in L^2(\mathbb{R}^d)^2$, there exists a unique solution to the equation

$$(1-\Pi(\frac{1-\gamma}{\gamma+\delta}\delta\zeta\Pi\cdot))\mathbf{u} = f,$$

and that $|\mathbf{u}|_2 \leq k(\zeta)^{-1}|f|_2$. The result is thus proved for the particular case $s=0$. The general case requires the control of commutator terms [**9**]. □

In the following proposition, we show how the divergence and partial differentiation operators act on the operator $\mathfrak{R}[\zeta]$. Let us introduce first the following notation:

$$\mathfrak{S}[\zeta]\mathbf{v} = \mathbf{v} + (1-\gamma)\mathfrak{R}[\zeta]\mathbf{v} \tag{27}$$

(so that $\mathfrak{S}[\zeta]\mathbf{v}$ degenerates into $\mathfrak{S}[\zeta]\mathbf{v} = \dfrac{1+\delta}{\delta h_1 + \gamma h_2}\mathbf{v}$ when $d=1$).

Proposition 4. — *Let $\gamma \in [0,1)$, $\delta > 0$ and $t_0 > 1$. Assume also that $\zeta \in H^s(\mathbb{R}^2)$, with $s \geq t_0 + 1$, and satisfies*

$$\inf_{\mathbb{R}}(1-|\zeta|_\infty) > 0 \quad \textit{and} \quad \inf_{\mathbb{R}}(1-\delta|\zeta|_\infty) > 0.$$

Then, for all $\mathbf{v} \in L^2(\mathbb{R}^2)^2$, one has

$$\nabla\cdot\mathfrak{R}[\zeta]\mathbf{v} = \delta\frac{\mathfrak{S}[\zeta]\mathbf{v}}{\delta h_1+\gamma h_2}\cdot\nabla\zeta + \frac{h_2}{\delta h_1+\gamma h_2}\nabla\cdot\mathbf{v}$$

and, for $j = 1,2$,

$$\partial_j\big(\mathfrak{R}[\zeta]\mathbf{v}\big) = \delta\mathfrak{R}[\zeta]\big(\frac{\mathfrak{S}[\zeta]\mathbf{v}}{h_2}\partial_j\zeta\big) + \mathfrak{R}[\zeta]\partial_j\mathbf{v}.$$

Proof. — For the first identity, remark first that from Lemma 3 of [**3**] one has

$$\nabla\cdot\big((\delta h_1+\gamma h_2)\mathfrak{R}[\zeta]\mathbf{v}\big) = \nabla\cdot(h_2\mathbf{v});$$

since the assumptions on ζ imply $\delta h_1 + \gamma h_2 > 0$, it follows that

$$\nabla\cdot\mathfrak{R}[\zeta]\mathbf{v} = \frac{1}{\delta h_1+\gamma h_2}\nabla\cdot(h_2\mathbf{v}) + \delta(1-\gamma)\frac{\nabla\zeta}{\delta h_1+\gamma h_2}\cdot\mathfrak{R}[\zeta]\mathbf{v},$$

from which the result follows easily.
For the second identity, remark that by definition of $\mathfrak{R}[\zeta]\mathbf{v}$, one has

$$(\gamma+\delta)\big(1+\Pi(\frac{\gamma-1}{\gamma+\delta}\delta\zeta\Pi\cdot)\big)\mathfrak{R}[\zeta]\mathbf{v} = \Pi\big(h_2\mathbf{v}\big);$$

differentiating this identity, one gets

$$(\gamma+\delta)\big(1+\Pi(\frac{\gamma-1}{\gamma+\delta}\delta\zeta\Pi\cdot)\big)\partial_j(\mathfrak{R}[\zeta]\mathbf{v}) = \partial_j\Pi(h_2\mathbf{v}) + (1-\gamma)\Pi(\delta\partial_j\zeta\mathfrak{R}[\zeta]\mathbf{v}),$$

and thus

$$\partial_j(\mathfrak{R}[\zeta]\mathbf{v}) = \mathfrak{R}[\zeta]\big(\frac{\partial_j(h_2\mathbf{v})}{h_2}\big) + \delta(1-\gamma)\mathfrak{R}[\zeta]\big(\frac{\partial_j\zeta\mathfrak{R}[\zeta]\mathbf{v}}{h_2}),$$

from which the result follows. □

One has $\mathfrak{R}[\zeta](\frac{\mathbf{v}}{h_2}) = \frac{1}{\delta h_1+\gamma h_2}\mathbf{v}$ in the one-dimensional case $d=1$; when $d=2$, this identity is of course false but the following proposition establishes that when $\mathbf{v}$ is a gradient vector field (i.e. when $\Pi\mathbf{v}=\mathbf{v}$) then this identity is true up to a more regular term.

Proposition 5. — *Let $\gamma\in[0,1)$, $\delta>0$ and $t_0>1$. Let also $\zeta\in H^{t_0+1}(\mathbb{R}^2)$ be such that*

$$\inf_{\mathbb{R}}(1-|\zeta|_\infty)>0 \quad \textit{and} \quad \inf_{\mathbb{R}}(1-\delta|\zeta|_\infty)>0.$$

Then, for all $\mathbf{v}\in L^2(\mathbb{R}^2)^2$,

$$|\mathfrak{R}[\zeta](\frac{\mathbf{v}}{h_2}) - \frac{1}{\delta h_1+\gamma h_2}\Pi\mathbf{v}|_2 \le C\big(\frac{1}{\gamma+\delta-\delta(1-\gamma)|\zeta|_\infty}, \delta(1-\gamma)|\zeta|_{H^{t_0+1}}\big)|\Pi\mathbf{v}|_{H^{-1}}.$$

Proof. — Remark first that one can write

$$\mathfrak{R}[\zeta](\frac{\mathbf{v}}{h_2}) - \frac{1}{\delta h_1+\gamma h_2}\Pi\mathbf{v} = (1-\Pi(\frac{1-\gamma}{\gamma+\delta}\delta\zeta\Pi\cdot))^{-1}\mathbf{w},$$

with

$$\mathbf{w} = \frac{1}{\gamma+\delta}\Pi\mathbf{v} - (1-\Pi(\frac{1-\gamma}{\gamma+\delta}\delta\zeta\Pi\cdot))\big(\frac{1}{\delta h_1+\gamma h_2}\Pi\mathbf{v}\big).$$

With the same notation as in the proof of Proposition 3, we thus have

$$|\mathfrak{R}[\zeta](\frac{\mathbf{v}}{h_2}) - \frac{1}{\delta h_1+\gamma h_2}\Pi\mathbf{v}|_2 \lesssim \frac{1}{k(\zeta)}|\mathbf{w}|_2,$$

and we are thus led to control $|\mathbf{w}|_2$. In order to do so, let f_1 and f_2 be defined as

$$f_1 = \frac{1-\gamma}{\gamma+\delta}\delta\zeta, \qquad f_2 = \frac{1}{\delta h_1+\gamma h_2}.$$

Simple computations show that

$$\mathbf{w} = f_1[\Pi,f_2]\Pi\mathbf{v} + [\Pi,f_1]\Pi(f_2\Pi\mathbf{v}).$$

Using the commutator estimate (which can be deduced from the general commutator estimates for pseudo-differential operators given in Theorem 6 of [**12**])

$$\forall r, -t_0<r\le t_0+1, \quad |[\Pi,g]h|_{H^r} \lesssim |g|_{H^{t_0+1}}|h|_{H^{r-1}} \tag{28}$$

with $r=0$ and the product estimate (valid for $t_0>1=d/2$, see for instance [**8**]),

$$|fg|_{H^{-1}} \lesssim |f|_{H^{t_0}}|g|_{H^{-1}} \tag{29}$$

we deduce that

$$|\mathbf{w}|_2 \lesssim |f_1|_{H^{t_0+1}}|f_2|_{H^{t_0+1}}|\Pi\mathbf{v}|_{H^{-1}},$$

and the result follows easily. □

3.2. An "almost" quasilinear formulation of the equations. — In the one dimensional case $d = 1$, the SW/SW systems (25) is quasilinear; in the two dimensional case ($d = 2$), it is more tricky to put (24) in a quasilinear form because of the presence of the nonlocal term $\mathfrak{R}[\zeta]\mathbf{v}$. The main result of this section is to prove that one can write (24) in the equivalent form

$$\partial_t U + A^j[U]\partial_j U = 0, \qquad U = (\zeta, \mathbf{v})^\top, \tag{30}$$

where

$$A^j[U] = \begin{pmatrix} a^j(U) & \mathbf{b}^j(U)^\top \\ \mathbf{c}^j[U] & D^j[U] \end{pmatrix}, \qquad (j = 1, 2),$$

and

$$a^j(U) = (\mathbf{v} - \gamma\mathfrak{R}[\zeta]\mathbf{v})_j - \gamma(\mathfrak{S}[\zeta]\mathbf{v})_j \frac{h_2}{\delta h_1 + \gamma h_2}, \tag{31}$$

$$\mathbf{b}^j(U) = \frac{h_1 h_2}{\delta h_1 + \gamma h_2}\mathbf{e}^j, \tag{32}$$

$$\mathbf{c}^j[U]\bullet = \mathbf{e}^j - \gamma\Big[\mathbf{e}^j + \delta(\mathfrak{S}[\zeta]\mathbf{v})_j \mathfrak{R}[\zeta]\big(\frac{\mathfrak{S}[\zeta]\mathbf{v}}{h_2}\bullet\big)\Big], \tag{33}$$

$$D^j[U]\bullet = (\mathbf{v} - \gamma\mathfrak{R}[\zeta]\mathbf{v})_j \mathrm{Id}_{2\times 2} - \gamma(\mathfrak{S}[\zeta]\mathbf{v})_j \mathfrak{R}[\zeta]\bullet, \tag{34}$$

the operator $\mathfrak{S}[\zeta]$ being as defined in (27).

***Proposition 6* (The case $d = 2$).** — *Let $T > 0$, $t_0 > 1$ and $s \geq t_0 + 1$. Let also $U = (\zeta, \mathbf{v})^\top \in C([0,T]; H^s(\mathbb{R}^2)^3)$ be such that for all $t \in [0,T]$,*

$$(1 - |\zeta(t,\cdot)|_\infty) > 0 \quad \textit{and} \quad (1 - \delta|\zeta(t,\cdot)|_\infty) > 0 \quad \text{and} \quad \operatorname{curl} \mathbf{v}(t,\cdot) = 0.$$

Then U solves (24) if and only U solves (30).

Remark 11. — The system (30) is not *stricto sensu* a quasilinear system since $\mathbf{c}^j[U]$ (resp. $D^j[U]$) is not an $\mathbb{R}^2$-vector-valued (resp. 2×2-matrix-valued) function but a linear operator defined over the space of $\mathbb{R}^2$-vector-valued (resp. 2×2-matrix-valued) functions. However, these operators are of order zero and, as shown below, (30) can be handled roughly as a quasilinear system.

Proof. — One can use Proposition 4 to express the quantities involved in (24) in the following form:

Lemma 2. — *Let $t_0 > 1$ and $U = (\zeta, \mathbf{v})^\top \in H^{t_0+1}(\mathbb{R}^2)^3$ be such that*

$$\inf_{\mathbb{R}^2}(1 - |\zeta|_\infty) > 0 \quad \textit{and} \quad \inf_{\mathbb{R}^2}(1 - \delta|\zeta|_\infty) > 0.$$

Then one has

$$\nabla \cdot (h_1 \mathfrak{R}[\zeta]\mathbf{v}) = \big(a^j(U)\partial_j\zeta + \mathbf{b}^j(U)\cdot\partial_j\mathbf{v}\big),$$

where $a^j(U)$ and $\mathbf{b}^j(U)$ are given by (31) and (32).
If moreover $\operatorname{curl} \mathbf{v} = 0$ *then*

$$(1-\gamma)\nabla\zeta + \frac{1}{2}\nabla\Big(\big|\mathbf{v} - \gamma\mathfrak{R}[\zeta]\mathbf{v}\big|^2 - \gamma\big|\mathfrak{R}[\zeta]\mathbf{v}\big|^2\Big) = \mathbf{c}^j[U]\partial_j\zeta + D^j[U]\partial_j\mathbf{v},$$

where the operators $\mathbf{c}^j[U]$ and $D^j[U]$ are given by (33) and (34).

Proof. — We refer to [**9**] for a proof. The curl-free assumption is needed to simplify the right-hand-side in the following identity, used in the computations: for all $\mathbf{u} \in H^1(\mathbb{R}^2)^2$, one has

$$\frac{1}{2}\nabla|\mathbf{u}|^2 = (\mathbf{u}\cdot\nabla)\mathbf{u} + (\mathrm{curl}\ \mathbf{u})\mathbf{u}^\perp$$

with $\mathbf{u}^\perp = (u_2, -u_1)^\top$ and we recall that curl $\mathbf{u} = \partial_1 u_2 - \partial_2 u_1$. □

The proposition is then a direct consequence of Lemma 2. □

The next proposition is crucial in order to prove that a solution of (30) which is initially curl free remains curl free and thus yields a solution of (24); its proof relies essentially on a Gronwall-type argument (see [**9**]).

Proposition 7. — *Let $T > 0$, $t_0 > 1$ and $s \geq t_0 + 1$. Let also $U = (\zeta, \mathbf{v})^\top \in C([0,T]; H^s(\mathbb{R}^2)^3)$ be a solution of* (30) *such that* curl $\mathbf{v}(0,.) = 0$. *Then* curl $\mathbf{v}(t,.) = 0$ *for all $t \in [0,T]$.*

3.3. Local well-posedness of (24). — We show here that the two-dimensional Shallow Water/Shallow Water equations (24) are locally well-posed under the following conditions that generalize the hyperbolicity conditions of the one-dimensional case (see [**9**]),

$$\left\{\begin{array}{l} 1 - |\zeta|_\infty > 0, \\ 1 - \delta|\zeta|_\infty > 0, \\ 1 - \gamma - \gamma\delta\dfrac{|\mathfrak{S}[\zeta]\mathbf{v}|_\infty^2}{\gamma + \delta - \delta(1-\gamma)|\zeta|_\infty} > 0, \end{array}\right. \tag{35}$$

with $\mathfrak{S}[\zeta]\mathbf{v}$ as in (27).

Theorem 2. — *Let $\delta > 0$ and $\gamma \in [0,1)$. Let also $t_0 > 1$, $s \geq t_0 + 1$ and $U^0 = (\zeta^0, \mathbf{v}^0)^\top \in H^s(\mathbb{R}^2)^3$ be such that (35) is satisfied and* curl $\mathbf{v}^0 = 0$. *Then there exists $T_{max} > 0$ and a unique maximal solution $U = (\zeta, \mathbf{v})^\top \in C([0, T_{max}); H^s(\mathbb{R}^2)^3)$ to (24) with initial condition U^0. Moreover, if $T_{max} < \infty$ then at least one of the following conditions holds:*

(*i*) $\lim_{t\to T_{max}} |U(t)|_{H^{t_0+1}} = \infty.$

(*ii*) *One of the three conditions of (35) is enforced as $t \to T_{max}$.*

Proof. — Throughout this proof, we denote by $\mathfrak{c}(U)$ any constant of the form

$$\mathfrak{c}(U) = C\Big(\frac{1}{1-|\zeta|_{H^{t_0}}}, \frac{1}{1-\delta|\zeta|_{H^{t_0}}}, \frac{1}{1-\gamma-\gamma\delta\frac{|\mathfrak{S}[\zeta]\mathbf{v}|_\infty^2}{\delta+\gamma-\delta(1-\gamma)|\zeta|_\infty}}, |U|_{H^{t_0+1}}\Big).$$

Step 1. Regularized equations. We construct a regularized system of equations. Denote $\chi_\iota = \chi(\iota|D|)$, with χ a smooth, compactly supported function equal to 1 in a neighborhood of the origin. The regularization of (30) is then

$$\partial_t U^\iota + \chi_\iota(A^j[U^\iota]\chi_\iota(\partial_j U^\iota)) = 0; \tag{36}$$

existence/uniqueness of a maximal solution $U^\iota = (\zeta^\iota, \mathbf{v}^\iota)^\top \in C([0, T^\iota); H^s)$ ($s \geq t_0$, $T^\iota > 0$) with initial condition U^0 to (36) satisfying (35) is thus obtained by classical theorems on ODEs. Moreover, proceeding as for Proposition 7, one has (since $\mathrm{curl}\,\mathbf{v}^0 = 0$)

$$\forall t \in [0, T^\iota), \qquad \mathrm{curl}\,\mathbf{v}^\iota = 0. \tag{37}$$

Step 2. Choice of a symmetrizer. Let us look for $S[U]$ in the form

$$S[U] = \begin{pmatrix} s_1(U) & 0 \\ 0 & S_2[U] \end{pmatrix}, \tag{38}$$

with $s_1(\cdot) : H^s(\mathbb{R}^2)^3 \mapsto H^s(\mathbb{R}^2)$ and $S_2[U]$ a linear operator mapping $L^2(\mathbb{R}^2)^2$ into itself. Defining $C[U]$ as

$$\forall \widetilde{\mathbf{v}} = (\widetilde{\mathbf{v}}_1, \widetilde{\mathbf{v}}_2)^\top \in L^2(\mathbb{R}^2)^2, \qquad C[U]\widetilde{\mathbf{v}} = \mathbf{c}_1[U]\widetilde{\mathbf{v}}_1 + \mathbf{c}_2[U]\widetilde{\mathbf{v}}_2,$$

a straightforward generalization of the one-dimensional case consists in taking $s_1(U) = b(U)^{-1}$ and $S_2[U] = C[U]^{-1}$; unfortunately, such a choice is not correct because the operator $C[U]$ is not self-adjoint. It turns out however that $C[U]$ is self-adjoint (up to a smoothing term) on the restriction of $L^2(\mathbb{R}^2)^2$ to gradient vector fields, as shown in the following lemma. We first need to define the operator $C_1[U]$ as

$$C_1[U] = (1-\gamma)\mathrm{Id} + \frac{1}{2}\delta\gamma \begin{pmatrix} c_1[U] + c_1[U]^* & 0 \\ 0 & c_1[U] + c_1[U]^* \end{pmatrix}, \tag{39}$$

with $c_1[U] : L^2(\mathbb{R}^2) \to L^2(\mathbb{R}^2)$ given by

$$c_1[U] = \frac{1}{\delta h_1 + \gamma h_2}(2\mathfrak{S}_1\mathfrak{S}_2\Pi(\mathbf{e}^2\cdot)_1 + \mathfrak{S}_1^2\Pi(\mathbf{e}^1\cdot)_1 + \mathfrak{S}_2^2\Pi(\mathbf{e}^2\cdot)_2), \tag{40}$$

and where $\mathfrak{S}_j = (\mathfrak{S}[\zeta]\mathbf{v})_j$.

Lemma 3. — *Let $t_0 > 1$ and $U = (\zeta, \mathbf{v})^\top \in H^{t_0+1}(\mathbb{R}^2)^3$ be such that (35) is satisfied. Define also $C_1[U]$ as in (39) and let $C_2[U] = C[U] - C_1[U]$. For all $\widetilde{\zeta} \in L^2(\mathbb{R}^2)$, one has*

$$|C_2[U]\nabla\widetilde{\zeta}|_2 \leq \mathfrak{c}(U)|\widetilde{\zeta}|_2.$$

Proof. — The proof is quite technical (see [**9**]) and relies on heavily on Proposition 5. ☐

In view of this lemma, we now choose the coefficients $s_1[U]$ and $S_2[U]$ of the symmetrizer $S[U]$ given by (38) as follows

$$s_1(U) = b(U)^{-1}, \tag{41}$$

$$S_2[U] = C_1[U]^{-1}; \tag{42}$$

the invertibility of $C_1[U]$ is ensured by the following lemma (see [**9**] for a proof)

Lemma 4. — *Let $t_0 > 1$ and $U = (\zeta, \mathbf{v})^\top \in H^{t_0}(\mathbb{R}^2)^3$ be such that (35) is satisfied.*

1. *The operator $C_1[U]$ is invertible in $\mathcal{L}(L^2(\mathbb{R}^2)^2; L^2(\mathbb{R}^2))$ and*

$$\|C_1[U]^{-1}\|_{L^2\to L^2} \leq \mathfrak{c}(U).$$

2. *The following coercivity property holds, for all $\widetilde{\mathbf{v}} \in L^2(\mathbb{R}^2)^2$,*

$$|\widetilde{\mathbf{v}}|_2^2 \leq \mathfrak{c}(U)(C_1[U]^{-1}\widetilde{\mathbf{v}}, \widetilde{\mathbf{v}}).$$

It follows directly from this lemma that $S[U]$ satisfies: for all $V \in L^2(\mathbb{R}^2)^{1+2}$,

$$|V|_2^2 \leq \mathfrak{c}(U)\,(S[U]V,V) \quad \text{and} \quad (S[U]V,V) \leq \mathfrak{c}(U)|V|_2^2. \tag{43}$$

The operator $S[U]$ would therefore be a symmetrizer if $S[U]A^j[U]$ $(j=1,2)$ were symmetric, which is unfortunately not the case. However, $\Pi S[U]A^j[U]\Pi$, where Π denotes as before the projection onto gradient vector fields is symmetric at leading order. This crucial property will be exploited in Step 3 below and is a consequence of Lemma 3 and of the following lemma:

Lemma 5. — *Let $t_0 > 1$ and $U = (\zeta, \mathbf{v})^\top \in H^{t_0+1}(\mathbb{R}^2)^3$ satisfying (35). One can decompose the operators $D^j[U]$ $(j=1,2)$ given by (34) into*

$$D^j[U] = d_1^j(U)\mathrm{Id} + D_2^j[U]$$

with

$$d_1^j(U) = \big(\mathbf{v} - \gamma\mathfrak{R}[\zeta]\mathbf{v} - \gamma\frac{h_2}{\delta h_1 + \gamma h_2}\mathfrak{S}[\zeta]\mathbf{v}\big)_j.$$

Then, for all $\widetilde{\mathbf{v}} \in L^2(\mathbb{R}^2)^2$ such that $\Pi\widetilde{\mathbf{v}} = \widetilde{\mathbf{v}}$, one has

$$|D_2^j[U]\partial_j\widetilde{\mathbf{v}}|_2 \leq \mathfrak{c}(U)|\widetilde{\mathbf{v}}|_2.$$

Step 3. Energy estimate. One can check that $\widetilde{U} = \Lambda^s U^\iota$ solves

$$\partial_t\widetilde{U} + \chi_\iota\big(A^j[U^\iota]\chi_\iota(\partial_j\widetilde{U})\big) = \chi_\iota\big([A^j[U^\iota],\Lambda^s]\chi_\iota(\partial_j U^\iota)\big), \tag{44}$$

and one obtains

$$\begin{aligned} &\frac{1}{2}\partial_t(S[U^\iota]\widetilde{U},\widetilde{U}) + (S[U^\iota]A^j[U^\iota]\partial_j\chi_\iota\widetilde{U},\chi_\iota\widetilde{U}) = \frac{1}{2}([\partial_t, S[U^\iota]]\widetilde{U},\widetilde{U}) \\ &+ \quad \big(\chi_\iota\big([A^j[U^\iota],\Lambda^s]\chi_\iota(\partial_j U^\iota)\big), S[U^\iota]\widetilde{U}\big) + ([\chi_\iota, S[U^\iota]]A^j[U^\iota]\chi_\iota(\partial_j\widetilde{U}),\widetilde{U}). \end{aligned} \tag{45}$$

We now intend to control all the components of (45).

- *Control of* $(S[U^\iota]A^j[U^\iota]\partial_j\chi_\iota\widetilde{U},\chi_\iota\widetilde{U})$. Using the explicit expression of $A^j[U^\iota]$, we get

$$\begin{aligned} (S[U^\iota]A^j[U^\iota]\partial_j\chi_\iota\widetilde{U},\chi_\iota\widetilde{U}) &= (s_1(U^\iota)a^j(U^\iota)\partial_j\chi_\iota\widetilde{\zeta},\chi_\iota\widetilde{\zeta}) \\ &+ \quad (s_1(U^\iota)b(U^\iota)\nabla\cdot\chi_\iota\widetilde{\mathbf{v}},\chi_\iota\widetilde{\zeta}) + (S_2[U^\iota]C[U^\iota]\nabla\chi_\iota\widetilde{\zeta},\chi_\iota\widetilde{\mathbf{v}}) \\ &+ \quad (S_2[U^\iota]D^j[U^\iota]\partial_j\chi_\iota\widetilde{\mathbf{v}},\chi_\iota\widetilde{\mathbf{v}}), \end{aligned} \tag{46}$$

and we thus have to bound from above the different components of the right-hand side of (46).

- *Estimate on* $(s_1(U^\iota)a^j(U^\iota)\partial_j\chi_\iota\widetilde{\zeta},\chi_\iota\widetilde{\zeta})$. With a simple integration by parts, and using the explicit formulas of $s_1(U^\iota)$ and $a^j(U^\iota)$ provided by (41) and (31), one obtains

$$|(s_1(U^\iota)a^j(U^\iota)\partial_j\chi_\iota\widetilde{\zeta},\chi_\iota\widetilde{\zeta})| \leq \mathfrak{c}(U^\iota)|\widetilde{\zeta}|_2^2. \tag{47}$$

- *Estimate on* $I := (s_1(U^\iota)b(U^\iota)\nabla\cdot\chi_\iota\widetilde{\mathbf{v}},\chi_\iota\widetilde{\zeta}) + (S_2[U^\iota]C[U^\iota]\nabla\chi_\iota\widetilde{\zeta},\chi_\iota\widetilde{\mathbf{v}})$. Replacing $s_1(U^\iota)$ and $S_2[U^\iota]$ by their expressions given by (41) and (42), we get immediately

$$\begin{aligned} I &= -(\nabla\chi_\iota\widetilde{\zeta},\chi_\iota\widetilde{\mathbf{v}}) + (C_1[U^\iota]^{-1}C[U^\iota]\nabla\chi_\iota\widetilde{\zeta},\chi_\iota\widetilde{\mathbf{v}}) \\ &= (C_1[U^\iota]^{-1}C_2[U^\iota]\nabla\chi_\iota\widetilde{\zeta},\chi_\iota\widetilde{\mathbf{v}}), \end{aligned}$$

where we used the decomposition $C[U^\iota]=C_1[U^\iota]+C_2[U^\iota]$ of Lemma 3. Using the bounds on $\|C_1[U^\iota]^{-1}\|_{L^2\to L^2}$ and $|C_2[U^\iota]\nabla\chi_\iota\widetilde{\zeta}|$ provided by Lemmas 4 and 3 respectively, we deduce that

$$|I|\leq \mathfrak{c}(U^\iota)|\widetilde{\zeta}|_2|\widetilde{\mathbf{v}}|_2. \tag{48}$$

– *Estimate on* $J:=(S_2[U^\iota]D^j[U^\iota]\partial_j\chi_\iota\widetilde{\mathbf{v}},\chi_\iota\widetilde{\mathbf{v}})$. With $d_1^j[U^\iota]$ and $D_2^j[U^\iota]$ as in Lemma 5, we can write

$$\begin{aligned} J &= (S_2[U^\iota]d_1^j(U^\iota)\partial_j\chi_\iota\widetilde{\mathbf{v}},\chi_\iota\widetilde{\mathbf{v}})+(S_2[U^\iota]D_2^j[U^\iota]\partial_j\chi_\iota\widetilde{\mathbf{v}},\chi_\iota\widetilde{\mathbf{v}})\\ &:= J_1+J_2. \end{aligned}$$

Let us decompose J_1 into

$$\begin{aligned} 2J_1 &= -(S_2[U^\iota]\partial_j(d_1^j(U^\iota))\chi_\iota\widetilde{\mathbf{v}},\chi_\iota\widetilde{\mathbf{v}})-([\partial_j,S_2[U^\iota]]d_1^j(U^\iota)\chi_\iota\widetilde{\mathbf{v}},\chi_\iota\widetilde{\mathbf{v}})\\ &+ ([S_2[U^\iota],d_1^j(U^\iota)]\partial_j\chi_\iota\widetilde{\mathbf{v}},\chi_\iota\widetilde{\mathbf{v}}). \end{aligned}$$

Recalling that $S_2[U^\iota]=C_1[U^\iota]^{-1}$, the first term on the right-hand side is easily controlled thanks to Lemma 4 and the explicit expression of $d_1^j(U^\iota)$. After controlling some commutator terms, we can thus deduce that

$$|J_1|\leq\mathfrak{c}(U^\iota)|\widetilde{\mathbf{v}}|_2^2.$$

In order to control J_2, first remark that $\Pi(\chi_\iota\widetilde{\mathbf{v}})=\chi_\iota\widetilde{\mathbf{v}}$ (this follows from the identity curl $\mathbf{v}^\iota=0$ stated in (37)). It is thus a direct consequence of Lemmas 5 and 4 that $|J_2|$ has the same upper bound as $|J_1|$, so that

$$|J|\leq\mathfrak{c}(U^\iota)|\widetilde{\mathbf{v}}|_2^2. \tag{49}$$

It is now easy to deduce from (46), (47), (48) and (49) that

$$|(S[U^\iota]A^j[U^\iota]\partial_j\chi_\iota\widetilde{U},\chi_\iota\widetilde{U})|\leq\mathfrak{c}(U^\iota)|\widetilde{U}|_2^2. \tag{50}$$

- *Control of the r.h.s. of (45).* Quite classically, one gets that

$$\text{the right-hand-side of (45) is controlled by}\quad \mathfrak{c}(U^\iota)|\widetilde{U}|_2^2. \tag{51}$$

Thanks to (50)-(51), we can conclude by a Gronwall type argument that there exists $T>0$ independent of ι such that $T^\iota>T$ and

$$|U^\iota|_{L^\infty([0,T];H^s)}\leq M,$$

for some $M>0$ independent of ι.

Step 4. Convergence of U^ι to a solution U of (24) as $\iota\to0$. Very classically, one can check that there exists $T>0$ such that U^ι converges to $U\in C([0,T];H^s(\mathbb{R}^2))^3$ ($s\geq t_0+1$) which solves (30) and that such a solution is unique. The fact that this solution is also the unique solution to (24) requires $\mathbf{v}$ to remain curl-free. This property is ensured by Proposition 7 since we assumed that curl $\mathbf{v}^0=0$.

Step 5. Blow-up condition. The blow-up condition is provided by a completely standard continuation argument. □

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