Philippe G. LeFloch

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Philippe G. LeFloch*

Abstract

We investigate the initial value problem for the Einstein-Euler equations of general relativity under the assumption of Gowdy symmetry on T^3. Given an arbitrary initial data set, we establish the existence of a globally hyperbolic future development and we provide a global foliation of this spacetime in terms of a geometrically defined time-function coinciding with the area of the orbits of the symmetry group. This allows us to construct matter spacetimes with weak regularity which admit, both, impulsive gravitational waves and shock waves. The cosmic censorhip conjecture is established in the polarized case.

1 Introduction

Spacetimes with Gowdy symmetry on T^3, by definition, admit a two-parameter group of isometries generated by two orthogonally transitive, commuting Killing fields. Under this symmetry assumption, the initial value problem for the Einstein equations has received a lot of attention in recent years, both in the vacuum case and in the matter case when the matter is governed by the Vlasov equation of the kinetic theory of gases.

In the present work, we are interested in self-gravitating perfect fluids and, in the context of Gowdy symmetry, we study the initial value problem and we construct a global foliation for a future development of a given initial data set. The foliation under consideration here is based on a global

*Laboratoire Jacques-Louis Lions & Centre National de la Recherche Scientifique, Université Pierre et Marie Curie (Paris 6), 4 Place Jussieu, 75252 Paris, France. E-mail: pgLeFloch@gmail.com. 2000 *AMS Subject Classification.* 83C, 76L05. *Key Words and Phrases.* General relativity, Gowdy symmetry, Einstein-Euler equations, perfect fluid, impulsive gravitational wave, shock wave.

time function that coincides with the area of the orbits of symmetry. Only an outline of the theory will be provided in this short review and, for further details on Sections 3 and 4 below, the reader is referred to LeFloch and Rendall [19] and LeFloch and Stewart [20, 21], respectively.

2 Background on the Einstein equations

We begin with some terminology from Lorentzian geometry and general relativity. Let (M, g) be a time-oriented $(3+1)$-Lorentzian manifold with signature $(-,+,+,+)$. A vector $X \in T_pM$ is called time-like, null, or space-like if its norm is negative, zero, or positive. A vector is said to be causal if it is time-like or null, and to be achronal if it is spacelike or null. A time-orientation corresponds to a continuous selection of one particular component of the null cone. Time-like curves correspond to trajectories of physical observers, and null curves (or, rather, null geodesics) to the trajectories of light. Spacelike hypersurfaces on which the induced metric is Riemannian play an important role in general relativity.

By definition, a trip from $p \in M$ to $q \in M$ is a future-oriented, time-like, Lipschitz continuous curve connecting p to q. The future domain of dependence of a subset $S \subset M$ is defined as

$$\mathcal{D}^+(S) := \Big\{ p \in M \, / \text{ every past-endless trip containing } p \text{ meets } S \Big\}.$$

A spacelike hypersurface S satisfying $\mathcal{D}^+(S) = M$ is called a future Cauchy hypersurface in M.

We are interested in matter spacetimes, (M, g), satisfying Einstein's field equations

$$G_{\alpha\beta} = \kappa \, T_{\alpha\beta}, \tag{2.1}$$

where the Einstein tensor

$$G_{\alpha\beta} := R_{\alpha\beta} - (R/2) \, g_{\alpha\beta}$$

is determined from the Ricci tensor $R_{\alpha\beta}$ and the scalar curvature $R := R_\alpha{}^\alpha$ of the manifold. In the right-hand side of (2.1), the energy-momentum tensor $T_{\alpha\beta}$ describes the matter content of the spacetime. Perfect fluids under consideration in the present work correspond to

$$T_{\alpha\beta} := (\mu + p) \, u_\alpha \, u_\beta + p \, g_{\alpha\beta}. \tag{2.2}$$

The main two unknowns describing the matter are the time-like unit velocity u^α and the energy density $\mu \geq 0$. Importantly, the (second contracted) Bianchi identities imply the Euler equations

$$\nabla^\beta T_{\alpha\beta} = 0. \tag{2.3}$$

Finally, the Einstein-Euler equations (2.1)–(2.3) must be supplemented by an equation of state for the pressure $p = p(\mu)$, typically restricted by the dominant energy condition: for every future-oriented time-like vector X, the energy density $T(X, X)$ is non-negative and, moreover, for every future-oriented causal (i.e. time-like or null) vector X, the energy flux $T(X, \cdot)$ is future-oriented and causal.

The initial value problem for the Einstein equations is formulated as follows. An initial data set consists of a Riemannian 3-manifold $(\overline{M}, \overline{g})$ (with covariant derivative denoted by $\overline{\nabla}$), a symmetric 2-covariant tensor field $\overline{k}$, matter fields (energy density and current) $\overline{\rho}, \overline{J}$, satisfying Einstein's constraint equations

$$\overline{R} + (\mathrm{tr}\, \overline{k})^2 - |\overline{k}|^2 = 16\pi\overline{\rho}, \qquad \overline{\nabla}_j \overline{k}_i^j - \overline{\nabla}_i (\mathrm{tr}\, \overline{k}) = 8\pi \overline{J}_i, \tag{2.4}$$

where $\mathrm{tr}\, \overline{k} := \overline{k}_j^j$. Then, one searches for a future globally hyperbolic development of this initial data set, consisting of a Lorentzian manifold (M, g) satisfying the Einstein equations, together with matter fields ρ, J. The development is foliated by spacelike hypersurfaces with normal vector field n^α

$$M = \bigcup_{t \geq 0} \mathcal{H}_t = \mathcal{D}^+(\mathcal{H}_0),$$

and there exists an embedding $\psi : \overline{M} \to \mathcal{H}_0 \subset M$ such that $\overline{g}$ is the induced metric and $\overline{k}$ the second fundamental form, with moreover

$$\rho = T_{\alpha\beta} n^\alpha n^\beta, \qquad J^\alpha = (g^{\alpha\beta} + n^\alpha n^\beta) T_{\beta\gamma} n^\gamma,$$

$\overline{\rho}, \overline{J}$ being their restrictions to $\mathcal{H}_0$ (and $\overline{J}$ being tangent).

A large literature is available on the above problem, and we will not try to review it here. Let us only mention that, concerning the Einstein equations with arbitrarily large data and for large classes of matter models, the existence of a unique maximal, globally hyperbolic development of given initial data set was established by Choquet-Bruhat [6], Choquet-Bruhat and Geroch [8], and followers: for an account of the historical background as well as recent references, see [7] and the references therein. In the vacuum case, the maximal development is known to be future geodesically complete when the initial data set is asymptotically flat and sufficiently close to a spacelike hypersurface of the Minkowski spacetime [9]. On the other hand, models with symmetries can be handled under much weaker assumptions and large data can be treated.

3 Global causal structure of Gowdy-symmetric matter spacetimes

Under the assumption of T^3 Gowdy symmetry, the Einstein field equations take the form of a coupled system of nonlinear wave equations with differential constraints. Since the pioneering work by Gowdy [14], (vacuum) Gowdy symmetric spacetimes have been extensively studied [5, 4, 10, 11, 13, 16, 17, 22] and the strong cosmic censorship conjecture [15, 23] was established in [26, 27]. For a generalization of these spacetimes to matter governed by the Vlasov equation, see [1, 2, 12, 24, 25, 28].

On the other hand, the mathematical investigation of Gowdy-type spacetimes with compressible matter was initiated by LeFloch and Stewart ([20] and also [3]), who introduced a converging approximation scheme for the initial value problem, and derived several a priori bounds in suitably chosen local coordinates. Therein, it was found necessary to cope with *weak solutions* to the Einstein equations, understood in the distributional sense and containing propagating discontinuities (shock waves). These authors established a *local-in-time* existence result, while in the present paper we aim at constructing a *global* foliation.

Gowdy symmetry is well-adapted to describe inhomogeneous cosmologies with a "big bang" or "big crunch" and also allows for the propagation of gravitational waves —contrary to radial symmetry which excludes all dynamical modes of gravitation. The existence of global foliations of Gowdy symmetric spacetimes can be established and the long-time behavior of solutions to the Einstein equations be addressed. The novelty of the present work lies in the low regularity assumed on the spacetimes, which may contain, both, impulsive gravitational waves propagating at the speed of light and shock waves propagating at (about) the sound speed.

We assume here that the matter is a perfect fluid governed by the isothermal pressure equation

$$p = k^2 \mu,$$

where μ denotes the energy density and $k \in (0, 1)$ is the sound speed (the light speed being normalized to be unit).

Gowdy symmetry on T^3 for a (3 + 1)-dimensional Lorentzian manifold (M, g) is defined as follows. The spacetime admits the Lie group T^2 as an isometry group acting on the torus T^3 and generated by two (linearly independent) spacelike vector fields X, Y, satisfying by definition the Killing property $\mathcal{L}_X g = \mathcal{L}_Y g = 0$ and the commutation property $[X, Y] = 0$. We also impose that the action is orthogonally transitive, in the sense that the distribution of 2-planes $\left\{X, Y\right\}^{\perp}$ is Frobenius integrable. In the vacuum, these conditions define precisely the class of Gowdy spacetimes [14].

To make the latter condition more precise, we introduce two vectors Z, T orthogonal to X, Y, and we observe that (in dimension 2 and after normalization)

$$\epsilon^{\alpha\beta\gamma\delta} X_\gamma Y_\delta = Z^\alpha T^\beta - T^\alpha Z^\beta,$$

where ϵ denotes the Lorentzian volume form. Then, the distribution of covectors $g(X, \cdot)$, $g(Y, \cdot)$ is Frobenius integrable if and only if $Z, T, [Z, T]$ are linearly dependent, i.e.

$$\epsilon^{\alpha\beta\gamma\delta} Z_\alpha T_\beta [Z, T]_\gamma = 0.$$

After some calculations, the condition is found to be equivalent to saying that the "twist constants" vanish:

$$\epsilon_{\alpha\beta\gamma\delta} X^\alpha Y^\beta \nabla^\gamma X^\delta = \epsilon_{\alpha\beta\gamma\delta} X^\alpha Y^\beta \nabla^\gamma Y^\delta = 0. \tag{3.1}$$

More generally, in the special vacuum case, the Einstein field equations imply that the above (twist) quantities are constants and, without the requirement (3.1) one refers to the above class as T^2 symmetric vacuum spacetimes.

We define the class of weakly regular Gowdy symmetric T^3-spacetimes by the following conditions: the metric coefficients (in the areal gauge, see below) belong to $H^1(\Sigma)$ on every space-like slice Σ; the fluid variables $\rho \geq 0$ (scalar field) and J (vector field) belong to $L^1(\Sigma)$ (Prescribing ρ and J is equivalent to prescribing μ, u.) Furthermore, these functions satisfy the Einstein equations in the distributional sense and the entropy inequalities associated with the Euler equations hold. A precise statement of these equations is provided at the end of the present section. The notion of Gowdy symmetric initial data set on T^3 is defined in a similar way. Observe that the Einstein equation do make sense since, under our assumptions, the curvature is well-defined as a distribution in $H^{-1}(\Sigma)$; see LeFloch and Mardare [18]. Actually, an even weaker regularity (see below) will be assumed.

Our main result is as follows.

Theorem 3.1 (Global structure of Gowdy-symmetric matter spacetimes)**.** *Given any weakly regular, Gowdy-symmetric T^3-initial data set $(\overline{g}, \overline{k}, \overline{\rho}, \overline{J})$: there exists a weakly regular, Gowdy-symmetric T^3-spacetime (M, g, ρ, J) that is a maximal, globally hyperbolic, future development of $(\overline{g}, \overline{k}, \overline{\rho}, \overline{J})$. This spacetime is globally covered by a single chart in areal coordinates and admits a foliation by space-like hypersurfaces; the corresponding time variable t coincides with (plus or minus) the area of the (two-dimensional space-like) orbits of the T^2 isometry group. Furthermore, one can distinguish two cases, whether R is increasing or decreasing toward the future: for future expanding spacetime one has $t = R \in [h(\theta), +\infty)$ where $h(\theta) > 0$ denotes a parameterization of the initial hypersurface; for future contracting spacetimes, for some $R_0 \leq 0$ one has $t = -R \in [h(\theta), R_0]$ with $h(\theta) < R_0 \leq 0$.*

The above theorem is established in [19] and provides the maximal development of the given initial data set. In contrast, the earlier work [20] provided a "small time" existence result, only, but in a somewhat more regular function space: the fluid variables in [20] have bounded variation (BV) and the proof is based in Glimm's random choice scheme. Theorem 3.1 above is established via the technique of compensated compactness, and the regularity of the spacetime is weaker than BV, while the conclusion is more precise.

It should be noted that the time-function is tight to the geometry (areal coordinates) but is based on the Gowdy symmetry assumption. It would be interesting to search for a global CMC foliation for these spacetimes.

The main difficulty in the above theorem lies in the low regularity which we must assume on the metric coefficients, due to the coupling with the Euler equations. We allow here for impulsive gravitational waves, i.e. curvature singularities propagating at light speed. Fluid variables have solely finite total mass-energy and arbitrary large amplitude, and may contain shock waves which in turn create curvature discontinuities propagating at about the sound speed.

The main open question concerns the structure of the boundary of the Cauchy development. First of all, it need not hold that $R_0 = 0$ and explicit counter-examples can be constructed for the Einstein-Euler equations. Establishing the strong cosmic censorship (in-extendibility of the future development) for generic initial data at least, is the main challenge for this class of spacetimes. In the expanding case, this property can be established thanks to an argument originally due to Rendall for the Vlasov model. In the contracting case, the question is open in general and was settled in the vacuum case only [26, 27].

In the rest of this section, we explain how to reformulate in areal coordinates the problem under consideration in the quotient manifold M/T^2. Since $(X, Y) =: (X_1, X_2)$ are linearly independent and space-like, we have

$$R^2 := \det\left(\lambda_{ab}\right) > 0, \qquad \lambda_{ab} := g(X_a, X_b).$$

Since they are Killing fields, the area $R > 0$ is a constant on each orbit. The formula

$$h_{\mu\nu} := g_{\mu\nu} - \lambda^{ab} X_{a\mu} X_{b\nu}$$

defines a Lorentzian metric on M/T^2, and $h_\mu^\nu := g_\mu^\nu - \lambda^{ab} X_{a\mu} X_b^\nu$ is the natural projection operator. With these notation, the Einstein equations can be expressed on the quotient manifold. Coordinates (t, θ, x, y) are chosen such that (x, y) spans the orbits of symmetry, and the metric coefficients depend on (t, θ) and are periodic in θ, and the Einstein equations take the form of a system of partial differential equations in the variables (t, θ).

An important feature of Gowdy-symmetric spacetimes is the timelike property of the gradient ∇R, which allows one to choose $t = R$ as a time coordinate. In this setting, the metric reads

$$g = e^{2(\nu - U)} (-dt^2 + \alpha^{-1} d\theta^2) + e^{2U}(dx + A\, dy)^2 + e^{-2U}\, t^2\, dy^2,$$

in which the metric coefficients U, A, ν, α depend on t, θ and, by construction, t coincides with the area of the orbits of the T^2 symmetry group. The field equations consist of second-order nonlinear wave equations (for the functions U, A, ν), a constraint equation (a nonlinear differential equation for α), and the Euler equations. The latter consist of a system of two nonlinear hyperbolic equations for the matter variables ρ, J^1, with $J^2 = J^3 = 0$. Equivalently, a (normalized) component $V \in (-1, 1)$ of the velocity and the (normalized) energy density

$$M := \frac{\mu}{1 - V^2}.$$

are used. Indeed, it is necessary to allow V to be arbitrarily close to the light speed and rescale the density accordingly.

The Einstein equations imply the following conditions.

- The evolution equations for U, A read

$$(t\, \alpha^{-1/2} U_t)_t - (t\, \alpha^{1/2}\, U_\theta)_\theta = \frac{e^{4U}}{2t\alpha^{1/2}} (A_t^2 - \alpha\, A_\theta^2) + t\, \alpha^{1/2}\, \Pi_1,$$

$$(t^{-1}\, \alpha^{-1/2} A_t)_t - (t^{-1}\, \alpha^{1/2}\, A_\theta)_\theta = -\frac{4}{t\, \alpha^{1/2}} (U_t A_t - \alpha\, U_\theta A_\theta) + \alpha^{1/2}\, \Pi_2,$$

where Π_1, Π_2 are certain nonlinear expressions depending on the geometric and fluid variables. The equation for ν is analogous.

- The constraint equation for α reads

$$\alpha(t,\theta) = \overline{\alpha}(\theta) \exp\Big(-2(1-k^2) \int_1^t t'\, e^{2(\nu-U)} M\,(1-V^2)(t',\theta)\, dt'\Big),$$

where $\overline{\alpha} > 0$ is some prescribed initial data. (We assume here for simplicity that the area of symmetry is the constant $t = 1$ on the initial hypersurface.) Note that the function α is globally bounded in the future, that is, $0 < \alpha \le \overline{\alpha}$ for all $t \ge 1$.

- The Euler equations for μ (or equivalently M) and V read

$$\Big(a_1\Big(\mu + (\mu + p(\mu))\frac{V^2}{1-V^2}\Big)\Big)_t + \Big(a_2\,(\mu + p(\mu))\,\frac{V}{1-V^2}\Big)_\theta = \Sigma_1,$$
$$\Big(a_3\,(\mu + p(\mu))\,\frac{V}{1-V^2}\Big)_t + \Big(a_4\Big((\mu + p(\mu))\frac{V^2}{1-V^2} + p(\mu)\Big)\Big)_\theta = \Sigma_2,$$

where $a_1, \dots, a_4$ are nonlinear expressions depending on the geometric variables and Σ_1, Σ_2 depend on both the geometric and the fluid variables.

We are now in a position to state the regularity of the metric and matter coefficients in areal coordinates. For definiteness, we consider the case of expanding spacetimes, described by $t \in [1, +\infty)$. By definition, a weakly regular solution to the Einstein-Euler equations (in areal coordinates) consists of measurable functions U, A, ν, α, M, V defined on $I \times S^1 := [1, +\infty) \times S^1$ and satisfying the following conditions:

- The geometry and matter coefficients have the following regularity: $U_t, A_t, U_\theta, A_\theta \in L^\infty_{\text{loc}}(I, L^2(S^1))$, $\nu_t, \nu_\theta \in L^\infty_{\text{loc}}(I, L^1(S^1))$, $\alpha > 0$, $\alpha, \alpha^{-1} \in L^\infty_{\text{loc}}(I, L^\infty(S^1))$, $M \in L^\infty_{\text{loc}}(I, L^1(S^1))$, with $M \ge 0$ and $|V| \le 1$.
- The evolution and constraint equation hold in the distributional sense.
- The entropy inequalities

$$\nabla_\alpha \mathcal{F}^\alpha(M, V) \le \mathcal{G}(M, V)$$

hold for all convex weak entropy flux $\mathcal{F}^\alpha$ and associated source $\mathcal{G}$ defining a formal conservation law to the relativistic Euler equations.

Finally, we point out that weak regularity of the metric can be stated in a purely geometric way that does not require local coordinates but involves the Killing fields, only.

4 Censorship conjecture for polarized Gowdy symmetric matter spacetimes

The spacetimes of interest now are "polarized" Gowdy-symmetric spacetimes (with weak regularity), filled with an irrotational fluid whose sound speed coincides with the light speed. We formulate a characteristic value problem with data prescribed on two null hypersurfaces intersecting along a 2-plane. This describes colliding spacetimes (or interacting gravitational plane waves), following an explicit example discovered by Khan and Penrose in the vacuum case. Importantly, in the proposed setting we are able to establish the strong censorship conjecture.

Taking into account that each Killing field is hypersurface orthogonal ("plane symmetry"), we find the following expression of the metric:

$$\begin{aligned} g &= e^{2a}\,(-dt^2 + dx^2) + e^{2b}\,(e^{2c}\,dy^2 + e^{-2c}\,dz^2) \\ &= -e^{2a}\,dudv + e^{2b}\,(e^{2c}\,dy^2 + e^{-2c}\,dz^2), \end{aligned} \tag{4.1}$$

in which the coefficients a, b, c depend upon the characteristic variables

$$u = t - x, \qquad v = t + x.$$

In these coordinates, the relevant components of the Einstein tensor read

$$\begin{aligned} G_{00} &= 2\left(-2\,a_u b_u + b_{uu} + b_u^2 + c_u^2\right), \\ G_{01} &= 2\left(-b_{uv} - 2\,b_u b_v\right), \\ G_{11} &= 2\left(-2\,a_v b_v + b_{vv} + b_v^2 + c_v^2\right), \\ G_{22} &= 4e^{-2a+2b+2c}\left(a_{uv} + b_{uv} + b_u b_v - b_u c_v - b_v c_u - c_{uv} + c_u c_v\right), \\ G_{33} &= 4e^{-2a+2b-2c}\left(a_{uv} + b_{uv} + b_u b_v + b_u c_v + b_v c_u + c_{uv} + c_u c_v\right), \end{aligned}$$

which are at most quadratic expressions involving up to second-order derivatives.

The Einstein equation driving the coefficient b,

$$b_{uv} + 2\,b_u b_v = 0$$

is equivalent to the wave equation $(e^{2b})_{uv} = 0$, and allows us to write

$$e^{2b} = f(u) + g(v) > 0$$

for some arbitrary functions f, g. Assuming that f and g are invertible and observing that the transformations $u \mapsto f^{-1}(u/2)$ and $v \mapsto g^{-1}(v/2)$ do not change the form of the metric, we impose

$$f(u) = \frac{1}{2}\, u, \qquad g(v) = \frac{1}{2}\, v.$$

So, our choice of coordinates leads to

$$e^{2b} = \frac{1}{2}\,(u + v). \tag{4.2}$$

The region of interest is the past, $\Big\{u + v < 0\Big\}$, of the spacelike hypersurface

$$\mathcal{H}_0 := \Big\{u + v = 0\Big\},$$

the latter corresponding to a genuine singularity of the spacetime (as we will show it for generic initial data in Theorem 4.2, below).

The matter model under consideration is a perfect fluid (2.2) with pressure p equal to its mass-energy density μ,

$$p = \mu. \tag{4.3}$$

The energy-momentum tensor (2.2) involves the fluid velocity vector u satisfying $u^\alpha u_\alpha = -1$ and, with the equation of state (4.3), the fluid sound speed coincides with the light speed, normalized to 1. The (second) contracted Bianchi identities (implied by the geometry) yield the Euler equations $\nabla_\alpha T^{\alpha\beta} = 0$, which read

$$(u^\alpha\, \nabla_\alpha \mu)\, u^\beta + \mu\, (\nabla_\alpha u^\alpha)\, u^\beta + \mu\, u^\alpha\, \nabla_\alpha u^\beta - \frac{1}{2} \nabla^\beta \mu = 0. \tag{4.4}$$

Multiplying the above equations by u_β, we obtain the scalar equation

$$2\, \nabla_\alpha \mu\, u^\alpha - 2\, \mu\, \nabla_\alpha u^\alpha + 2\, \mu\, u^\alpha\, u_\beta\, \nabla_\alpha u^\beta - \nabla_\alpha \mu\, u^\alpha = 0,$$

which, in view of $u_\beta\, \nabla_\alpha u^\beta = 0$, simplifies into

$$u^\alpha\, \nabla_\alpha \mu + 2\, \mu\, \nabla_\alpha u^\alpha = 0.$$

Assuming that the density is bounded away from zero and setting

$$\Sigma = \frac{1}{2} \log \mu,$$

we obtain the scalar equation

$$u^\alpha \nabla_\alpha \Sigma + \nabla_\alpha u^\alpha = 0. \tag{4.5}$$

On the other hand, multiplying the Euler equations (4.4) by the projection operator $H_{\beta\gamma} = g_{\beta\gamma} - u_\beta u_\gamma$, we obtain the vector equation

$$H^{\alpha\gamma} \nabla_\alpha \mu - 2\mu \, u^\alpha \nabla_\alpha u^\gamma = 0,$$

or equivalently

$$H^{\alpha\gamma} \nabla_\alpha \Sigma - u^\alpha \nabla_\alpha u^\gamma = 0. \tag{4.6}$$

We assume that the fluid is irrotational, in the sense that there exists a (scalar) potential ψ with timelike gradient such that

$$\nabla_\beta \psi \, \nabla^\beta \psi < 0, \qquad u_\alpha = \frac{\nabla_\alpha \psi}{\sqrt{-\nabla_\beta \psi \, \nabla^\beta \psi}} \tag{4.7}$$

and we note that the anti-symmetric part of $u_\alpha \nabla_\beta u_\gamma$ vanishes.

From now on, subscripts u and v denote partial derivatives with respect to the characteristic coordinates. After normalization (by replacing ψ with $F(\psi)$, if necessary), the Euler equations imply the so-called Bernouilli's equation

$$\mu = -\nabla_\beta \psi \, \nabla^\beta \psi = -4 e^{-2a} \, \psi_u \psi_v, \tag{4.8}$$

which expresses the energy density in terms of the potential. Finally, we end up having to solve a single matter equation for the scalar field ψ which, in characteristic coordinates, takes the form

$$\psi_{uv} + b_v \, \psi_u + b_u \psi_v = 0.$$

In turn, we have arrived at the following essential field equations:

- The evolution equations for the metric component c and the fluid variable ψ are singular wave equations of the Euler-Poisson-Darboux type:

$$c_{uv} + \frac{1}{2(u+v)} \left(c_u + c_v \right) = 0, \tag{4.9}$$

$$\psi_{uv} + \frac{1}{2(u+v)} \left(\psi_u + \psi_v \right) = 0. \tag{4.10}$$

- The constraint equations for the metric coefficient a reads

$$\begin{aligned} a_u &= \Big(c_u^2 + \frac{1}{2}\,\psi_u^2\Big)(u+v) - \frac{1}{4\,(u+v)}, \\ a_v &= \Big(c_v^2 + \frac{1}{2}\,\psi_v^2\Big)(u+v) - \frac{1}{4\,(u+v)}. \end{aligned} \tag{4.11}$$

We will seek for solutions in the Sobolev space H^1 on each spacelike hypersurface, and we observe that the curvature and therefore each term in the field equations (2.1) are well-defined in the distributional sense [18].

We formulate the characteristic initial value problem, as follows. Fixing some (u_0, v_0) with $u_0+v_0 < 0$, we consider the 2-plane P_0 identified with this point and the spacetime region $\mathcal{T}(u_0, v_0)$ limited by the null hypersurfaces

$$\mathcal{N}_{v_0} := \Big\{v = v_0\Big\}, \qquad \mathcal{N}_{u_0} := \Big\{u = u_0\Big\},$$

and the spacelike hypersurface $\mathcal{H}_0$. We then prescribe initial data for the geometry and the matter on $\mathcal{N}_{v_0} \cup \mathcal{N}_{u_0}$, which may have a jump discontinuity on the 2-plane P_0. That is, let $\overline{\psi}|_{v_0}, \overline{c}|_{v_0}$ and $\psi|_{u_0}, \overline{c}|_{u_0}$ be data on the null hypersurfaces $\mathcal{N}_{v_0}$ and $\mathcal{N}_{u_0}$, respectively. The region of interest is covered by the double-null foliation:

$$\mathcal{N}_v := \Big\{(u', v') \,/\, v' = v\Big\}, \qquad \mathcal{N}_u := \Big\{(u', v') \,/\, u' = u\Big\}.$$

Definition 4.1. *A* weak solution *to the Einstein equations of self-gravitating irrotational fluids is determined by measurable metric coefficients a, c entering in the expression* (4.1) *of the metric, and a fluid potential ψ, defined in a characteristic region $\mathcal{T}(u_0, v_0)$ for some (u_0, v_0) and such that —after excluding an arbitrary small neighborhood of the singularity hypersurface $\mathcal{H}_0$— the expression*

$$\sup_v \int_{\mathcal{N}_v} \Big(\psi_u^2 + c_u^2\Big)\, du + \sup_u \int_{\mathcal{N}_u} \Big(\psi_v^2 + c_v^2\Big)\, dv + \sup_{(u,v)} |a(u,v)|$$

is finite and, moreover, the evolution equations (4.9)-(4.10) *for the coefficients c, ψ and the constraint equations* (4.11) *for the coefficient a hold in the distributional sense.*

Without loss of generality, we impose the normalization

$$a(u_0, v_0) = 1$$

of the metric coefficient a on the 2-plane. Our main result for polarized spacetimes is as follows.

Theorem 4.2 (Global causal structure of plane symmetric matter spacetimes)**.** *Given (u_0, v_0) satisfying $u_0 + v_0 < 0$, consider the characteristic region $\mathcal{T}(u_0, v_0)$ limited by the null hypersurfaces $\mathcal{N}_{v_0} = \{v = v_0\}$, $\mathcal{N}_{u_0} = \{u = u_0\}$, and the (spacelike, coordinate singularity) hypersurface $\mathcal{H}_0 := \{u + v = 0\}$. Denote by $\overline{\psi}|_{v_0}, \overline{c}|_{v_0}$ and $\psi|_{u_0}, \overline{c}|_{u_0}$ some initial data prescribed on the null hypersurfaces $\mathcal{N}_{v_0}$ and $\mathcal{N}_{u_0}$, respectively. Then, the corresponding characteristic value problem admits a unique weak solution to the Einstein equations for self-gravitating irrotational fluids. Furthermore, for "generic" initial data the curvature of the spacetime blows-up on the hypersurface $\mathcal{H}_0$ along future-oriented timelike geodesics, and the spacetime can not be continued beyond $\mathcal{H}_0$.*

We refer to [21] for a proof of this theorem. The main equations to be solved are singular wave equations, whose solutions are expressed in terms of an associated Riemann function. Only weak regularity is imposed on the geometric and matter variables, and so our theorem allows for the propagation of curvature singularities: the Ricci part of the curvature is solely integrable and may contain jump discontinuities propagating along null hypersurfaces, while the Weyl part is even more singular and may contain Dirac masses propagating along null hypersurfaces.

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