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Some regularizing methods for transport equations and the regularity of solutions to scalar conservation laws

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Abstract. We study several regularizing methods, stationary phase or averaging lemmas for instance. Depending on the regularity assumptions that are made, we show that they can either be derived one from the other or that they lead to different results. Those are applied to Scalar Conservation Laws to precise and better explain the regularity of their solutions.

1 Introduction

We investigate three kind of regularizing results and their connections. The first is connected to scalar conservation laws, namely

$$\begin{aligned} &\partial_t u + \nabla\cdot(A(u)) = 0, \quad u(t,x)\in\mathbb{R}, \quad t\in\mathbb{R}_+,\ x\in\mathbb{R}^d,\\ &u(t=0,x)=u^0(x)\in L^1\cap L^\infty(\mathbb{R}^d). \end{aligned} \tag{1.1}$$

The flux A is a function from $\mathbb{R}$ to $\mathbb{R}^d$, which we assume to be C^2 for simplicity.

We refer to [16] for the original well posedness argument for this equation and the most important notion of entropy solutions; to [24] for a previous uniqueness result for BV initial data; to [19] for a presentation of this class of equations (and especially the kinetic formulations related to the approach in this paper). For a more general introduction to conservation laws, see for example [7] or [20].

It is known that the solution u to (1.1) immediately becomes more regular but it is still unclear exactly how much. The first argument in [18] proves that in dimension 1 and for a strictly convex flux A, then $u(t,.) \in BV_{loc}(\mathbb{R}^d)$ with a norm behaving like $1/t$. A natural generalization is the conjecture that

$$\|t\,\nabla_x \cdot (a(u(t,.))\|_{M^1_{loc}(\mathbb{R}^d)} \le C\,(\|u^0\|_{L^\infty} + \|u^0\|_{L^1}), \tag{1.2}$$

where a is the derivative of the flux A with respect to u.

Unfortunately (1.2) is not proved except in some particular situations, namely in $1d$ (or with a $1d$ structure) and for convex fluxes in [13] (see also [22] for a geometric argument or [23] for more references) and in [4] still in $1d$ for fluxes having only a finite number of critical points $A''(\xi) = 0$, and non degenerate $A^{(k)}(\xi) \neq 0$ for some finite k if $A''(\xi) = 0$.

However, even as an essentially still open problem, a natural question is which kind of regularity (1.2) would imply for the solution u.

Another way of obtaining a regularizing effect for (1.2) is through the kinetic formulation and averaging lemmas. Introducing

$$f(t,x,v) = \begin{cases} 1 \quad \text{if } 0 \le v \le u(t,x), \\ -1 \quad \text{if } u(t,x) \le v \le 0, \\ 0 \quad \text{in the other cases.} \end{cases} \tag{1.3}$$

It was proved first in [17] (see also [19] for an overview of kinetic formulation) that u is an entropy solution to (1.1) iff there exists a non negative measure m s.t.

$$\partial_t f + a(v) \cdot \nabla_x f = \partial_v m. \tag{1.4}$$

The solution u can be obtained as an average of f

$$u(t,x) = \int_{\mathbb{R}} f(t,x,v)\,dv,$$

and therefore one may use "averaging lemmas" to gain regularity on u. Averaging lemmas are a characteristic feature of kinetic equations : Averages in velocity of the solution are more regular than the solution itself. This was first noticed in a L^2 framework in [12] and [11] and later much extended in very different situations : See [9], [1] or [3] for the L^p theory for example and [10] for a generalization to other PDE's.

With a bootstrap argument, it was shown in [17] that

$$u \in W^{s,p}_{loc}(\mathbb{R}_+ \times \mathbb{R}^d), \quad s < \frac{\theta}{\theta+2}, \quad p = \frac{4+\theta}{2+\theta}, \tag{1.5}$$

where θ is given by a non degeneracy condition on a

$$\forall v \in I,\ \forall \xi \in S^{d-1}, \quad |\{w \in I,\ |\xi \cdot a(w) - \xi \cdot a(v)| \le \varepsilon\}| \le L\,\varepsilon^\theta. \tag{1.6}$$

In [14] and [15], it was shown that if $a(v) = v$ (or a diffeomorphism of the identity), then the regularity on u can be obtained directly by combining averaging lemmas with a sort of hypoelliptic argument (not a true hypoellipticity which would work on f as in [2] since it is done on the average). Nothing was known however if instead of the specific form, one only assumes (1.6) on a.

Summing up, we wish to give an answer to

Problem 1 : Assume (1.6) and that $u \in L^1 \cap L^\infty$ satisfies (1.2), find the best s such that $u \in W^{s,1}$.

Problem 2 : Assume (1.6) and that f given by (1.3) solves (1.4), show with a direct estimate that u satisfies (1.5).

It should be noted that there is a wide gap between the regularity provided by the argument in [18] (1 derivative) and the one of [17] (1/3 derivative in the best case $\alpha = 1$). Indeed using the kinetic formulation and averaging lemmas it is not known how to use the sign of the measure in (1.4).

In fact the regularity given by (1.5) is true for all solutions to (1.1) with bounded entropy production and not only for the entropy solution. It was proved in [8] that if one considers this more general class of solutions, then (1.5) is indeed the optimal result.

Finally let us point out that looking for the regularity of the solution u to (1.1) in terms of Sobolev spaces is probably not the most subtle or efficient way. See for instance [6] for a more precise structure on the solution. and [5] for more on regularity of scalar conservation laws.

2 The results and relation with a stationary phase argument

2.1 The connection between the two problems

The first problem may in fact be seen as a weaker form of the second. Indeed assume that u satisfies (1.2) at some time t_0 and define f at time t_0 by (1.3). Now for any $t \leq t_0$ put

$$g(t,x,v) = f(t_0, x + (t_0 - t)\, a(v), v).$$

Then g solves the free kinetic equation

$$\partial_t g + a(v) \cdot \nabla_x g = 0.$$

Moreover (1.2) and the definition of g and f exactly means that

$$\|g(t=0,.,.)\|_{M^1(\mathbb{R}^d,\ BV_v(\mathbb{R}))} \leq C\,(\|u^0\|_{L^\infty} + \|u^0\|_{L^1}).$$

This is exactly the regularity on f used in [14] and [15] for the modified averaging lemma.

More precisely, from [15] it is known that averaging lemmas can easily be deduced from the properties of the operator

$$T\,f = \int_0^t \int_{\mathbb{R}} f(t-s, x-s\,a(v), v)\, e^{-s}\, \phi(v)\, dv\, ds, \tag{2.1}$$

for a regular and compactly supported function Φ. Notice that the initial value problem corresponds to f having a Dirac mass at $t=0$ and that correspondingly the properties of T can be deduced from this initial value operator

$$T_0\, h(t,x) = \int_{\mathbb{R}} h(x-t\,a(v), v)\, \phi(v)\, dv, \tag{2.2}$$

simply by integrating then in time (and doing Hölder estimates if necessary). It is possible to prove the following result for T_0

Proposition 2.1 *Assume that a satisfies* (1.6) *and that I is a closed interval of $\mathbb{R}_+$ not containing 0 if $s \geq 1$. Then T_0 is continuous from $L^p(\mathbb{R}^d,\ W^{\alpha,p}(\mathbb{R}))$ to $W^{s,p}(I\times\mathbb{R}^d)$ for any $s < \theta(1-1/\bar{p}+\alpha)$, where $\bar{p} = min(p,p^*)$ and $1/p^* = 1-1/p$.*

With Prop. 2.1 it is possible to answer both problems

Theorem 2.1 *Take $u \in L^\infty(\mathbb{R}_+,\ L^1\cap L^\infty(\mathbb{R}^d))$ satisfying* (1.2) *for a function a with the property* (1.6)*. Then $u(t,.) \in W^{s,1}(\mathbb{R}^d)$ for any $s<\theta$, $t>0$.*

And for the second, directly by averaging lemmas without bootstrap

Theorem 2.2 *Assume that a satisfies* (1.6) *and that $u \in L^\infty(\mathbb{R}_+,\ L^1\cap L^\infty(\mathbb{R}^d))$ is a solution to* (1.1) *with bounded entropy production :* i.e. *f defined through* (1.3) *solves* (1.4) *for a bounded (but not necessarily non negative) measure m. Then u has the regularity given by* (1.5)*.*

2.2 The proof of Prop. 2.1 in the very regular case by a stationary phase argument

When a is very regular then the properties of T_0 may be deduced from the usual stationary phase argument. Indeed Prop. 2.1 is true if $\alpha=0$ by standard averaging lemmas (see again [9] for instance). By interpolation, it is enough to prove it for $\alpha = 1$. On the other hand in that case, denoting $\hat f = \mathcal{F} f$ the Fourier transform of a function f in x one has

$$\begin{aligned}|\mathcal{F}\, T_0 h| &= \left|\int_{\mathbb{R}} \hat h(\xi, v)\, e^{ita(v)\cdot\xi}\, \phi(v)\, dv\right|\\ &= \left|\int_{\mathbb{R}} \int_{-\infty}^{v} (\partial_v \hat h(\xi, w)\phi(w) + \hat h(\xi, w)\partial_v\phi(w))\, dw\, e^{ita(v)\cdot\xi}\, dv\right|\\ &\leq \int_{\mathbb{R}} Q(\xi)\, (\partial_v \hat h(\xi, w)\phi(w) + \hat h(\xi, w)\partial_v\phi(w))\, dw,\end{aligned}$$

where

$$|Q(\xi)| \leq \sup_I \left|\int_I e^{ita(v)\cdot\xi}\, dv\right|,$$

for all intervals I included in the convex hull of the support of ϕ.

Therefore as soon as one can bound Q then it is enough to apply again the result of [9] to conclude on the regularity of T_0. In particular if a is regular enough then Prop. 2.1 can be deduced from

Proposition 2.2 *Let $a \in C^k(I)$ for $I \subset \mathbb{R}$ with* (1.6) *and $k \geq 1/\theta$. Then the following holds true for a constant C depending only on θ, L, $\|a''\|_{L^1(I)}$ and $\|a^{(k)}\|_{L^\infty(I)}$*

$$\left|\int_I e^{i\xi\cdot a(v)}\, dv\right| \leq \frac{C}{|\xi|^\theta}.$$

2.3 The not so regular case

The main problem with the previous approach is of course that $a \in C^k$ with $k \geq 1/\theta$. Indeed θ may be very small, making this a quite unreasonable assumption, whereas $A \in C^2$ *i.e.* $a \in C^1$ is enough for the theory of scalar conservation laws for instance.

Moreover in high dimension, if $a \in C^\infty$, then $\theta \leq 1/d$ as for any v_0 it is always possible to find a direction ξ s.t. $\xi \cdot a'(v_0) = \ldots = \xi \cdot a^{(d-1)}(v_0) = 0$ and hence $|\{v,\ |\xi \cdot (a(v) - a(v_0))|\}| \leq C\, |v - v_0|^d$. Therefore the coefficient in (1.6) can sometimes be much better for a non regular a than for a regular one...

Unfortunately when a is not regular enough then condition (1.6) is no more the right one for the stationary phase argument (see the discussion at the end of the last section). Nevertheless Prop. 2.1 is still true and therefore Th. 2.1 and 2.2 as well.

In fact instead of studying T_0 only in dimension 1 for the velocities, it is possible to generalize it to higher dimensions

$$T_0h(x,v)=\int_{\mathbb{R}^N} h(x-t\,a(v),v)\,\phi(v)\,dv, \tag{2.3}$$

where now $a\ :\mathbb{R}^N\rightarrow\mathbb{R}^d$. In that case the gain in regularity depends on the dimension and more precisely

Proposition 2.3 *Assume $a\in C^1(\mathbb{R}^N,\ \mathbb{R}^d)$, and that a satisfies (1.6). Take I any closed interval of $\mathbb{R}_+$. Then for any $\alpha<N/\bar p$, T_0 is continuous from $L^p(\mathbb{R}^d,\ W^{\alpha,p}(\mathbb{R}))$ to $L^p(I,\ W^{s,p}(\mathbb{R}^d))$ for any $s<\theta(1-1/\bar p+\alpha/N)$, where $\bar p=min(p,p^*)$ and $1/p^*=1-1/p$.*

The dependency on N may seem strange and for instance in the simplest case $N=d$, $a(v)=v$, it was shown in [14] and [15] that the gain in regularity is $\theta(1-1/\bar p+\alpha)$ (*i.e.* the same as for $N=1$ in the proposition). In this particular case there are however many more symmetries than what condition (1.6) explicits. In the general case, this could in fact be precised by replacing (1.6) by a condition on the regularity and dimension of the set $\{v,\ a(v)\cdot\xi=\tau\}$. This would however typically involve many derivatives of a, which we are trying to avoid. Notice finally that for scalar conservation laws, we are in the case $N=1$ and this problem does not exist at all.

Proof of Prop. 2.3. From usual results on averaging lemmas, it is known that T_0 is continuous from $L^p(\mathbb{R}^d\times\mathbb{R}^N)$ to $W^{s,p}$ with $s<\theta(1-1/\bar p)$. Because ϕ is regular and compactly supported, one may also freely assume that h is compactly supported in velocity.

By interpolation it is enough to do the case $\alpha=1$ (or any other integer) and again by interpolation the cases $p=1$ and $p=2$ are enough.

Start with $p=1$ and for the usual technical reason take instead any $p>1$. After Fourier transform in x and t, one gets

$$\mathcal{F}_{t,x}\left(\mathbb{I}_{t\in I}\,T_0h\right)(\tau,\xi)=\int_{\mathbb{R}^N}\hat h(\xi,v)\,\chi(\tau-a(v)\cdot\xi)\,\phi(v)\,dv,$$

with χ regular and

$$|\chi(\eta)|\leq\frac{\sin(|I|\eta)}{\eta}.$$

Using that

$$\hat h(\xi,v)=(\nabla_v\hat h(\xi,v))\star_v\gamma,$$

with $\gamma(v)=Cv/|v|^N$, we get

$$\mathcal{F}_{t,x}\left(\mathbb{I}_{t\in I}\,T_0h\right)(\tau,\xi)=-\int_{\mathbb{R}^N}\nabla_v\hat h(\xi,v)\,M(\tau,\xi,v),$$

for

$$M = \gamma \star_v (\phi(v)\, \chi(\tau - a(v)\cdot\xi))\, dv.$$

On the other hand M is regular in τ and ξ and for example assuming that $\operatorname{supp}\phi \subset B(0,R)$

$$\begin{aligned}|M| &\leq C \int_{|v|\leq R} \frac{|\chi(\tau - a(w)\cdot\xi)|}{|v-w|}\, dw\\ &\leq C\left(\int_{\mathbb{R}^N} \frac{\sin^N(|I||\tau - a(w)\cdot\xi|)}{|\tau - a(w)\cdot\xi|^N}\, dw\right)^{1/N}.\end{aligned}$$

Denote for $n < n_0$ with $2^{-n_0} \leq |\xi| < 2^{-n_0+1}$

$$\Omega_n = \{v,\ R\,2^{-n-1} \leq |a(v)\cdot\xi/|\xi| - \tau/|\xi|| < R\,2^{-n}\},$$

and decompose using (1.6)

$$\begin{aligned}|M| =& C\left(\sum_{n<n_0} \frac{2^{N(n+1)}}{|\xi|^N\, R^N}\, 2^{-n\theta}\, R^\theta\right)^{1/N}\\ =& \frac{C}{|\xi|}\, 2^{(1-\theta/N)\,n_0} = \frac{C}{|\xi|^{\theta/N}}.\end{aligned}$$

The same kind of estimates may be derived for derivatives of M in ξ or τ (with the corresponding loss of exponent in the bound). Consequently by usual Calderon-Zygmund theory, one has

$$\||\xi|^{\theta/N}\, \mathcal{F}_{t,x}\,(\mathbb{I}_{t\in I}\, T_0 h)(\tau,\xi)\|_{L^p(I\times\mathbb{R}^d)} \leq C\|h\|_{L^p(\mathbb{R}^d,\ W^{1,p}(\mathbb{R}^N))},$$

which is the desired estimate for p close to 1.

For $p = 2$, things are even simpler as one may compute the norm directly in Fourier transform

$$\begin{aligned}&\int_{\mathbb{R}} |\mathcal{F}_{t,x}\,(\mathbb{I}_{t\in I}\, T_0 h)(\tau,\xi)|^2\, d\tau\\ &\quad\leq \int_{\mathbb{R}^N} |\Delta_v^{\alpha/2}\hat{h}|^2\, dv\, \left(\int_{\mathbb{R}^N} \left|\int_{\mathbb{R}} \chi(\tau - a(v)\cdot\xi)\, d\tau\right|^{2N/(N-2\alpha)} dv\right)^{(N-2\alpha)/N},\end{aligned}$$

again by Sobolev embedding (and necessarily limiting ourselves to the case $2\alpha \leq N$).

Now $\left|\int \chi(\tau - a(v)\cdot\xi)\, d\tau\right| \leq C/|a(v)\cdot\xi|$ and the same dyadic decomposition in v as before yields that

$$\int_{\mathbb{R}} |\mathcal{F}_{t,x}\,(\mathbb{I}_{t\in I}\, T_0 h)(\tau,\xi)|^2\, d\tau \leq \frac{C}{|\xi|^{\theta(1-2\alpha/N)}} \int_{\mathbb{R}^N} |\Delta_v^{\alpha/2}\hat{h}|^2\, dv,$$

which again is the estimate of the proposition after integration in ξ.

3 Appendix : The stationary phase argument

3.1 The regular case: Proof of Prop. 2.2.

First recall the usual stationary phase result in $1d$ (see Stein [21] for instance)

Proposition 3.1 *Let $\psi \in C^k(\mathbb{R})$, $I \subset \mathbb{R}$, $v_0 \in I$, $l+1 \leq k$ s.t.*

$$\psi'(v_0) = \ldots = \psi^{(l)}(v_0) = 0, \quad \psi^{(l+1)}(v_0) \neq 0, \quad \psi'(v) \neq 0 \ \forall v \neq v_0.$$

Then $\exists C$, depending only on k, ψ', $\|\psi''\|_{L^1(I)}$, $\psi^{(l+1)}(v_0)$ and $|I|$ s.t.

$$\left| \int_I e^{i|\xi|\,\psi(v)}\, dv \right| \leq \frac{C}{|\xi|^{1/(l+1)}}.$$

Turning to the proof of Prop. 2.2, one only has to prove that the property (1.6) implies that $\psi = \frac{\xi}{|\xi|} \cdot a(v)$ satisfies the assumptions of Prop. (3.1) on intervals $I_1 \cup \ldots \cup I_N = I$ with $l+1 = 1/\theta$ and N depending only on the constants in (1.6).

First observe that if

$$\psi'(v_0) = \ldots = \psi^{(l)}(v_0) = 0,$$

with $l \leq k$ then for v close enough to v_0

$$\left| \frac{\xi}{|\xi|} \cdot (a(v) - a(v_0)) \right| \leq \frac{2|a^{(l+1)}(v_0)|}{l!}\, |v - v_0|^{l+1}.$$

So

$$|\{v,\ |\xi \cdot a(v) - \xi \cdot a(v_0)| \leq |\xi|\,\varepsilon\}| \geq C\,\varepsilon^{1/(l+1)}.$$

From (1.6), this implies that $1/(l+1) \geq \theta$ or $l \leq l_0 = 1/\theta - 1$. If $l = l_0$ it shows that $|\psi^{(l_0+1)}(v_0)| \geq CL^{l_0+1}$.

Now let us bound the maximal number of such points v_0 where ψ' may vanish. Denote

$$n_l = \#V_l = \#\{v \in I,\ \psi'(v) = \ldots = \psi^{(l)}(v) = 0,\ \psi^{(l+1)}(v) \neq 0\}.$$

For $l \leq l_0$, assume that $n_l > 0$ otherwise there is nothing to do. In some interval J of length η one finds $v_1 < \ldots < v_m$ in V_l. As $\psi^{(l)}(v_i) = 0$ there exist $m-1$ $v_i^2 \in [v_i,\ v_{i+1}]$ s.t. $\psi^{(l+1)}(v_i^2) = 0$. Applying the same for the v_i^2 and so on, one finally gets $z \in J$ s.t. $\psi^{(l+m-1)}(z) = \phi(0)$. Taking $m = l_0 + 2 - l$, this would entail

$$\psi^{(l_0+1)}(z) = 0,$$

which we showed is impossible at any v_i and thus at any z for η small enough with respect to continuity modulus of $a^{(l_0+1)}$.

Therefore in any interval of size η, there is less than $m = l_0 + 1 - l$ points of V_l and then necessarily

$$n_l \leq (l_0 + 1 - l)\, |I|/\eta.$$

The total number of points in $V = \cup_l V_l = \{v,\ \psi'(v) = 0\}$ is less than $C\,(l_0+1)^2\,|I|/\eta$. Denoting by v_i those points and defining

$$I_i = [v_i/2 + v_{i-1}/2,\ v_i/2 + v_{i+1}/2],$$

one may apply Prop. 3.1 on each I_i and conclude the proof of Prop. 2.2.

3.2 The stationary phase in the not regular case

Independently of the two problems mainly considered here, one could wonder if something remains of the stationary phase argument itself under the only assumption (1.6) and $a \in C^1$ for instance.

There could be some reasons to hope still for a result as for instance the following simple remark due to P. Gérard suggests. Denote again

$$Q(\xi) = \int_I e^{ia(v)\cdot\xi}\, dv,$$

and compute

$$\int_{\mathbb{R}^d} |Q(\xi)|^2 |\xi|^{-d+\theta}\, d\xi = \int_{I^2}\int_{\mathbb{R}^d} e^{i(a(v)-a(w))\cdot\xi}\, |\xi|^{-d+\theta} = C\int_{I^2} |a(v)-a(w)|^{-\theta},$$

which is bounded directly by condition (1.6).

This would suggest a behaviour of Q like $|\xi|^{-\theta/2}$ in the low regularity case instead of $|\xi|^{-\theta}$. This is however not true if one still wants a pointwise estimate.

In fact, instead of (1.6), the correct assumption for the stationary phase argument is

$$|\{v,\ |a'(v)| \leq \varepsilon\}| \leq C\,\varepsilon^\nu. \tag{3.1}$$

In the case where a has enough regularity then (3.1) and (1.6) are equivalent with the relation between the exponents

$$\nu = \frac{\theta}{1-\theta}.$$

This is not true if a is not regular and for instance for $\theta \leq 1/2$, it is very easy to find $a \in C^1(\mathbb{R})$ satisfying (1.6) but s.t.

$$|\{v,\ a'(v) = 0\}| > 0.$$

In that case, it is in general not possible to recover a stationary phase argument at all, showing that (1.6) is no longer the right assumption. More precisely

Lemma 3.1 *Take $d = 1$ and $a \in C^1(I)$ an increasing (or decreasing) function. Assume that there exists a constant C and an exponent $\gamma > 0$ s.t.*

$$|Q(\xi)| = \left| \int_I e^{i\xi\, a(v)}\, dv \right| \leq C\, |\xi|^{\gamma}.$$

Then necessarily

$$|\{v,\ a'(v) = 0\}| = 0.$$

Proof of Lemma 3.1. If $|Q(\xi)| \leq C\, |\xi|^{-\gamma}$, then as Q is on the other hand bounded, there exists an exponent k s.t.

$$\int_{\mathbb{R}} |Q(\xi)|^{2k}\, d\xi < \infty.$$

But of course

$$\begin{aligned}\int_{\mathbb{R}} |Q(\xi)|^{2k}\, d\xi &= \int_{\mathbb{R}} \int_{I^{2k}} e^{i\xi(a(v_1)-a(w_1)+\ldots a(v_k)-a(w_k))}\, dv_1\, dw_1\, \ldots\, dv_k\, dw_k\, d\xi \\ &= \int_{I^{2k}} \delta(a(v_1) - a(w_1) + \ldots a(v_k) - a(w_k))\, dv_1\, dw_1\, \ldots\, dv_k\, dw_k.\end{aligned}$$

This last integral can easily be shown to be $+\infty$ if a is increasing and $|\{v,\ a'(v) = 0\}| > 0$. $\square$

With (3.1) however, it is easy to recover a stationary phase argument, for instance

Proposition 3.2 *For any $a \in W^{2,\infty}(I)$ satisfying* (3.1) *with $\nu < 2$, there exists a constant C s.t. $\forall \xi \in \mathbb{R}^d$*

$$|Q(\xi)| = \left| \int_I e^{i\xi\cdot a(v)}\, dv \right| \leq C\, |\xi|^{-\nu/2}.$$

Proof of Prop. 3.2. Fixing the direction of ξ and considering $a(v)\cdot\xi/|\xi|$ instead of a, we may reduce ourselves to the case $d=1$.

For any ε, find all intervals $J_i=[a_i,\ b_i]$ s.t.

$$|a'(a_i)|=|a'(b_i)|=\varepsilon,\quad \inf_{J_i}|a'|<\varepsilon/2,\quad \sup_{J_i}|a'|\le\varepsilon,$$
$$\sup_{[a_i-\eta,\ a_i]}|a'|>\varepsilon,\quad \sup_{[b_i,\ b_i+\eta]}|a'|>\varepsilon\quad \forall\eta>0.$$

Note that the number n_ε of such intervals is finite as $|J_i|\ge\varepsilon/\|a''\|_{L^\infty}$. Now define

$$I^\varepsilon=I\setminus(J_1\cup\cdots\cup J_n)=I_1\cup\cdots\cup I_n\subset\{v,\ |a'(v)|>\varepsilon/2\}.$$

Note in addition that by condition (3.1)

$$|I\setminus I^\varepsilon|\le|\{v,\ |a'(v)|\le\varepsilon\}|\le C\,\varepsilon^\nu.$$

As a consequence

$$|Q(\xi)|\le\left|\sum_{i=1}^n\int_{I_i}e^{i\xi a(v)}\,dv\right|+C\,\varepsilon^\nu.$$

Now on each $I_i=[b_i,\ a_{i+1}]$, a is increasing or decreasing, so that, making the change of variable

$$\begin{aligned}\int_{I_i}e^{i\xi a(v)}\,dv&=\int_{a(I_i)}e^{i\xi\,w}\,\frac{dw}{|a'(a^{-1}(w))|}\\&=\frac{1}{i\,\xi}\left(\frac{e^{i\xi\,a(a_{i+1})}}{|a'(a_{i+1})|}-\frac{e^{i\xi a(b_i)}}{|a'(b_i)|}\right)-\frac{1}{i\xi}\int_{a(I_i)}e^{i\xi\,w}\,\frac{a''(a^{-1}(w)}{|a'(a^{-1}(w))|^3}\,dw,\end{aligned}$$

by integration by parts. Note that

$$|a(a_i)-a(b_i)|\le|J_i|\,\varepsilon,\ |a'(a_i)|=\varepsilon=|a'(b_i)|.$$

Therefore

$$|Q(\xi)|\le\sum_i\frac{|\xi|\,|J_i|\,\varepsilon}{|\xi|\,\varepsilon}+\frac{C}{|\xi|}\int_{I^\varepsilon}\frac{dv}{|a'(v)|^2}+C\,\varepsilon^\nu.$$

Denote

$$\omega_k=\{v,\ 2^{-k-1}<|a'(v)|\le 2^{-k}\}.$$

Then for $k_\varepsilon=-\log_2\varepsilon$

$$\int_{I^\varepsilon}\frac{dv}{|a'(v)|^2}\le\sum_{k\le k_\varepsilon}2^{2k+2}\,|\omega_k|\le C\sum_{k\le k_\varepsilon}2^{2k+2-k\,\nu}\le C\,\varepsilon^{\nu-2}.$$

Finally recalling that $\sum_i |J_i| \leq C\,\varepsilon^\nu$

$$|Q(\xi)| \leq C\,\varepsilon^\nu + \frac{C}{|\xi|\,\varepsilon^{2-\nu}} + C\,\varepsilon^\nu,$$

and we conclude by minimization in ε that

$$|Q(\xi)| \leq C\,|\xi|^{-\nu/2}.$$

□

Prop. 3.2 might seem disappointing since from the regular case, we would like a behaviour in $|\xi|^{-\theta}$ with $\nu = \theta/(1-\theta)$, *i.e.* in $|\xi|^{-\nu/(1+\nu)}$.

Unfortunately it is easy to find examples demonstrating that a bound in $|\xi|^{-\nu/(1+\nu)}$ is not possible.

For instance, define $v_n = -n^{-k}$ for any $n \geq 0$. Take $a(0) = 0$ and on $[v_n,\ (v_n + v_{n+1})]$,

$$a'(v) = v - v_n,$$

whereas on $[(v_n + v_{n+1}),\ v_{n+1}]$ put

$$a'(v) = v_{n+1} - v.$$

The function a' is lipschitz and vanishes at every v_n; as a consequence for $n_0 = \varepsilon^{-1/k}$

$$|\{v,\ |a'(v)| \leq \varepsilon\}| \leq n_0 \times \varepsilon + n_0^{-k} \leq C\,\varepsilon^{1-1/k}.$$

The function a satisfies (3.1) provided $1 - 1/k \geq \nu$.

There remains to estimate $Q(\xi)$. Noting that $a(v_{n+1}) - a(v_n) \sim (v_{n+1} - v_n)^2 \sim n^{-2k-2}$, one sees three regions.

The first one corresponds to $n \geq n_1$ with $n_1^{-2k-1}\,\xi = 1$. On the interval $[v_{n_1},\ 0]$, $\xi a(v)$ almost does not change, making the contribution of this region to Q of order

$$n_1^{-k} = \xi^{-k/(2k+1)} \leq \xi^{-1/(3-\nu)} \leq \xi^{-\nu/(1+\nu)},$$

for ν small.

In the second region, $\xi a(v)$ is almost constant on one interval $[v_n,\ v_{n+1}]$ but not on several of them. This gives the range $n_2 \leq n < n_1$ with $n_2^{-2k-2}\,\xi = 1$. The part of Q corresponding to this interval may be approximated by

$$\sum_{n=n_2}^{n_1} (v_{n+1} - v_n)\, e^{i\xi a(v_n)} = \sum_{n=n_2}^{n_1} (v_{n+1} - v_n)\, e^{i\xi b(v_n)},$$

with $b \in C^\infty$, coinciding with a at the v_n but with b' non vanishing and of order n^{-k-1} on each $[v_n,\ v_{n+1}]$. b satisfies (3.1) but is regular and so the corresponding Q is exactly the one given by the regular case or

$$\xi^{-\nu/(1+\nu)}.$$

The last interval in v is for $v < -n_2^{-k}$. On each subinterval

$$\int_{v_n}^{v_{n+1}} e^{i\xi\, a(v)}\, dv = e^{i\xi a(v_n)}\,(v_{n+1} - v_n)\,\tilde{Q}((v_{n+1} - v_n)^2\,\xi),$$

with

$$\tilde{Q}(\xi) = \int_0^1 e^{i\xi\,\tilde{a}(v)}\, dv,$$

and $\tilde{a}$ defined by $\tilde{a}(0) = 0$, $\tilde{a}'(v) = v$ for $v \in [0,\ 1/2]$, $\tilde{a}'(v) = 1 - v$ for $v \in [1/2,\ 1]$. $\tilde{Q}(\xi)$ behaves exactly like $\xi^{-1/2}$. Hence

$$Q(\xi) \sim \sum_{n \le n_2} \frac{e^{i\,\xi\, a(v_n)}}{\xi^{1/2}} + O(\xi^{-\nu/(1+\nu)}) \sim \sum_{n \le n_2} \frac{e^{-i\,\xi\, n^{-2k-1}}}{\xi^{1/2}} \sim \xi^{-k/(2(k+1))},$$

which as one may choose k freely as long as $1 - 1/k = k/(k+1) > \nu$ is the behaviour predicted by Prop. 3.2.

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