



Centre de
Mathématiques
Laurent Schwartz
X
ECOLE
POLYTECHNIQUE

SEMINAIRE

**Equations aux
Dérivées
Partielles**

2008-2009

Jakob Yngvason

Bosons in Rapid Rotation: From the Quantum Many-Body Problem to Effective Equations

Séminaire É. D. P. (2008-2009), Exposé n° XI, 12 p.

<http://sedp.cedram.org/item?id=SEDP_2008-2009____A11_0>

U.M.R. 7640 du C.N.R.S.
F-91128 PALAISEAU CEDEX

Fax : 33 (0)1 69 33 49 49

Tél : 33 (0)1 69 33 49 99

cedram

*Exposé mis en ligne dans le cadre du
Centre de diffusion des revues académiques de mathématiques*
<http://www.cedram.org/>

Bosons in Rapid Rotation: From the Quantum Many-Body Problem to Effective Equations*

Jakob Yngvason

*Faculty of Physics, University of Vienna,
Boltzmanngasse 5, 1090 Vienna, Austria
and*

*Erwin Schrödinger Institute for Mathematical Physics,
Boltzmanngasse 9, 1090 Vienna, Austria*

Email: jakob.yngvason@univie.ac.at

1 Introduction

The Gross-Pitaevskii (GP) Equation is a non-linear Schrödinger equation that is widely used for describing the properties of Bose-Einstein condensates. Its stationary version is

$$-\Delta\varphi(\mathbf{x}) + V(\mathbf{x})\varphi(\mathbf{x}) + 2g|\varphi(\mathbf{x})|^2\varphi(\mathbf{x}) = \mu\varphi(\mathbf{x}) \quad (1)$$

and the time-dependent version

$$i\partial_t\psi(\mathbf{x}, t) = -\Delta\psi(\mathbf{x}, t) + V(\mathbf{x})\psi(\mathbf{x}, t) + 2g|\psi(\mathbf{x}, t)|^2\psi(\mathbf{x}, t). \quad (2)$$

Here φ and ψ are complex-valued functions with variables $\mathbf{x} \in \mathbb{R}^d$, $d = 1, 2, 3$ and $t \in \mathbb{R}$, V an external potential, g a coupling constant and μ a Lagrange multiplier (chemical potential) determined by the normalization condition $\|\varphi\|_2^2 = 1$. In dimensions $d = 2, 3$ it is also of interest to consider the equations for rotating systems obtained by the substitutions

$$-\Delta \rightarrow (i\nabla + \mathbf{A}(\mathbf{x}))^2 \quad \text{and} \quad V(\mathbf{x}) \rightarrow V(\mathbf{x}) - |\mathbf{A}(\mathbf{x})|^2 \quad (3)$$

with $\mathbf{A}(\mathbf{x}) = \frac{1}{2}\boldsymbol{\Omega} \wedge \mathbf{x}$, where $\boldsymbol{\Omega}$ is the angular velocity of the container enclosing the condensate.

In this lecture the following points will be discussed:

- The meaning of $\varphi(\mathbf{x})$ resp. $\psi(\mathbf{x}, t)$ in the many-body context.
- The derivation of the GP equation from the full many-body Hamiltonian.
- Some properties of rapidly rotating systems.

The focus will be on the stationary situation, in fact on the ground state. A general reference on the first two point, discussed in Sections 2-5 is [1]; rotating systems will be discussed in the last Section 6.

*Lecture given at the Ecole Polytechnique, Palaiseau, January 20, 2009

2 The Mathematical Setting

The basic quantum mechanical Hamiltonian for N spinless bosons in \mathbb{R}^3 that are trapped in an external potential V and interact via a pair potential v is, in a frame that rotates with angular velocity $\boldsymbol{\Omega}$

$$H = \sum_{j=1}^N \left\{ (i\nabla_j + \mathbf{A}(\mathbf{x}_j))^2 + V(\mathbf{x}_j) - \frac{1}{4}\Omega^2 r_j^2 \right\} + \sum_{1 \leq i < j \leq N} v(\mathbf{x}_i - \mathbf{x}_j). \quad (4)$$

Here $\mathbf{x}_i \in \mathbb{R}^3$, $i = 1, \dots, N$ are the positions of the particles. We shall assume that the interaction potential v is nonnegative and spherically symmetric. Moreover, $V(\mathbf{x}) \rightarrow \infty$ for $|\mathbf{x}| \rightarrow \infty$. Units have been chosen so that the combination $\hbar^2/2m$ of Planck's constant \hbar and the particle mass m does not appear explicitly as a pre-factor of the Laplacian.

The Hamiltonian operates on symmetric wave functions in $L^2(\mathbb{R}^{3N}, d\mathbf{x}_1 \cdots d\mathbf{x}_N)$. For ultracold bosons the normalized ground state wave function $\Psi_0(\mathbf{x}_1, \dots, \mathbf{x}_N)$ of H is of particular interest.

The particle density associated with a wave function Ψ is

$$\rho(\mathbf{x}) = N \int_{\mathbb{R}^{3(N-1)}} |\Psi(\mathbf{x}, \mathbf{x}_2, \dots, \mathbf{x}_N)|^2 d\mathbf{x}_2 \cdots d\mathbf{x}_N \quad (5)$$

and the one-particle density matrix is

$$\rho^{(1)}(\mathbf{x}, \mathbf{x}') = N \int_{\mathbb{R}^{3(N-1)}} \Psi(\mathbf{x}, \mathbf{x}_2, \dots, \mathbf{x}_N) \Psi^*(\mathbf{x}', \mathbf{x}_2, \dots, \mathbf{x}_N) d\mathbf{x}_2 \cdots d\mathbf{x}_N. \quad (6)$$

These definitions can be extended to mixed states in an obvious way by convex combinations.

3 The Concept of Bose-Einstein Condensation

The general idea is that Bose-Einstein condensation (BEC) means “macroscopic occupation of a single one-particle state”. In the case of ideal gases without interaction between the particles, i.e., if $v \equiv 0$, the many-body ground state Ψ_0 of H has the form $\psi_0^{\otimes N}$ with ψ_0 the ground state of the one-particle operator $-\Delta + V$, or, more generally of $(i\nabla + \mathbf{A})^2 + V - \frac{1}{4}\Omega^2 r^2$. Thus BEC trivially holds for such a state: all particles occupy the same state ψ_0 . It is, however, still a nontrivial question whether BEC holds for thermal equilibrium states at (sufficiently low) positive temperatures. That this is true for an ideal Bose gas in the thermodynamic limit and $d = 3$ was discovered by Albert Einstein in 1924.

For interacting bosons the question is nontrivial even at temperature zero, i.e., in the many-particle ground state. To discuss it we must first define the concept of BEC precisely for states that can be highly correlated. Let φ be a single-particle wave function and denote the projector onto φ by P_φ . Then the average occupation of φ in a many-particle state $\langle \cdot \rangle$ (that can be pure or mixed) is

$$N_\varphi = \langle P_\varphi \otimes 1 \otimes 1 \cdots + 1 \otimes P_\varphi \otimes 1 \cdots \rangle. \quad (7)$$

In terms of creation and annihilation operators this can also be written as

$$N_\varphi = \langle a(\varphi)^\dagger a(\varphi) \rangle. \quad (8)$$

Now, by a definition, BEC in the many-particle state $\langle \cdot \rangle$ means that for some 1-particle state φ ,

$$N_\varphi = O(N) \quad (9)$$

as $N \rightarrow \infty$, or more precisely, $N_\varphi/N \geq c > 0$ for all (large enough) N . (Here $N = \sum_i \langle a(\varphi_i)^\dagger a(\varphi_i) \rangle$ with $\{\varphi_i\}$ an orthonormal basis of one-particle states.)

This definition can also be formulated as a property of the one-particle density matrix $\rho^{(1)}(\mathbf{x}, \mathbf{x}')$ because (8) can be written

$$N_\varphi = \int \int \varphi(\mathbf{x})^* \rho^{(1)}(\mathbf{x}, \mathbf{x}') \varphi(\mathbf{x}') d\mathbf{x} d\mathbf{x}'. \quad (10)$$

The density matrix has a spectral decomposition:

$$\rho^{(1)}(\mathbf{x}, \mathbf{x}') = \sum_i \lambda_i \varphi_i(\mathbf{x}) \varphi_i^*(\mathbf{x}') \quad (11)$$

with $\lambda_0 \geq \lambda_1 \geq \dots$ and orthonormal φ_i . Because $N_{\varphi_0} = \lambda_0$ is the maximal occupancy of any single-particle state, BEC in the many-body state to which $\rho^{(1)}(\mathbf{x}, \mathbf{x}')$ belongs means that

$$\lambda_0 = O(N), \quad (12)$$

i.e., the one-particle density matrix has a macroscopic eigenvalue. The eigenfunction $\varphi_0(\mathbf{x})$ to the highest eigenvalue of $\rho^{(1)}(\mathbf{x}, \mathbf{x}')$ is often referred to as the *wave function of the condensate*. Note that

$$\lambda_0 |\varphi_0(\mathbf{x})|^2 \quad \text{resp.} \quad \lambda_0 |\tilde{\varphi}_0(\mathbf{p})|^2 \quad (13)$$

is the spatial density resp. the momentum density of the condensate, where $\tilde{\varphi}_0(\mathbf{p})$ denotes the Fourier transform of the wave function. Now for a homogeneous gas in a large box Λ the wave function of the condensate can be expected to be constant, i.e., $\varphi_0 = |\Lambda|^{-1/2}$. Since

$$\lambda_0 = \int \int \varphi_0^*(\mathbf{x}) \rho^{(1)}(\mathbf{x}, \mathbf{x}') \varphi_0(\mathbf{x}') d\mathbf{x} d\mathbf{x}', \quad (14)$$

BEC for a homogeneous gas in the thermodynamic limit means that

$$|\Lambda|^{-2} \int \int \rho^{(1)}(\mathbf{x}, \mathbf{x}') d\mathbf{x} d\mathbf{x}' \geq c > 0 \quad (15)$$

rather than tending to zero as $N \rightarrow 0$. Eq. (15) is called “Off Diagonal Long Range Order”.

Important Remark: The definition of BEC is only precise if the N dependence of the parameters of the many particle state has been specified. Important cases are:

- Thermodynamic limit in a box Λ : $N \rightarrow \infty$, $|\Lambda| \rightarrow \infty$, $N/|\Lambda| = \text{const.}$
- Gross-Pitaevskii-limit: $N \rightarrow \infty$, $Na/L = \text{const.}$ with a the *scattering length* of v and L the length scale associated with $-\Delta + V$.
- ‘Thomas-Fermi’-limit: $N \rightarrow \infty$, $Na/L \rightarrow \infty$, but $Na^3/L^3 \rightarrow 0$.

4 The Ground State Energy

Consider a spherically symmetric pair interaction potential v of finite range. The zero energy scattering equation is

$$-\Delta\psi + \frac{1}{2}v\psi = 0. \quad (16)$$

For $r = |\mathbf{x}|$ larger than the range of v the solution has the form

$$\psi(r) = (\text{const.}) \left(1 - \frac{a}{r}\right) \quad (17)$$

with a constant a that is called the *scattering length* of v .

If $v \geq 0$ the scattering length determines completely the ground state energy $E^{\text{QM}}(2, L)$ of a pair of Bosons in a large box Λ of side length $L \gg a$:

$$E^{\text{QM}}(2, \Lambda) \approx \frac{8\pi a}{L^3}. \quad (18)$$

Consider now for $v \geq 0$ the Hamiltonian of N Bosons in a box Λ of side length L (with appropriate boundary conditions):

$$H = -\sum_{i=1}^N \Delta_i^2 + \sum_{1 \leq i < j \leq N} v(\mathbf{x}_i - \mathbf{x}_j) \quad (19)$$

We denote its ground state energy by $E^{\text{QM}}(N, L)$. The energy per particle in the thermodynamic limit, i.e., $N \rightarrow \infty$ and $L \rightarrow \infty$ with $\rho = N/L^3$ fixed, is

$$e_0(\rho) = \lim_{L \rightarrow \infty} E^{\text{QM}}(\rho L^3, L)/(\rho L^3). \quad (20)$$

We ask for the *low density asymptotics* of $e_0(\rho)$, where low density means $a \ll \rho^{-1/3}$ i.e., the scattering length is much smaller than the mean particle distance. This can also be written as

$$\rho a^3 \ll 1. \quad (21)$$

The basic formula for the energy is

THEOREM (Ground state energy of a dilute gas)

For $\rho a^3 \ll 1$

$$e_0(\rho) = 4\pi a\rho(1 + o(1)). \quad (22)$$

A heuristic argument for this formula can be given as follows: Since “for a dilute gas only two body scattering matters”,

$$E^{\text{QM}}(N, L) \approx \frac{N(N-1)}{2} E^{\text{QM}}(2, L) \approx \frac{N^2}{2} \frac{8\pi a}{L^3} = N 4\pi a\rho. \quad (23)$$

This heuristic argument is, however, very far from a rigorous proof and it gives a wrong answer in two dimensions [2].

The formula (22) has an interesting history and it took almost 70 years to establish it rigorously, see [1]. An upper bound (for a gas of hard spheres) was given by Dyson in 1957 [3] but a matching lower bound was not obtained by Lieb and Yngvason until 40 years later [4]. Besides the result itself the techniques of [4] turned out to be important for the subsequent developments [1].

Why is the lower bound so difficult? The basic reason is that we are looking for a very small energy if ρ is small. One can distinguish two regimes:

1. ‘Hard potential’, v large (in particular a hard sphere potential). The energy is here *mostly kinetic* and the ground state *highly correlated*. There is *no perturbation theory* in this region.
2. ‘Soft potential’, v small. The energy is *mostly potential*. Lowest order perturbation theory (with the *uncorrelated*, unperturbed state $\Psi_0 = L^{-3N/2}$) gives

$$e_0(\rho) \approx \frac{1}{2}\rho \int v(\mathbf{x})d^3\mathbf{x}. \quad (24)$$

This is a wrong answer (it is independent of \hbar and m !), but it is at least the first Born approximation to $4\pi a\rho$. (Note that a depends on \hbar and m through the pre-factor $\hbar^2/2m$ of the Laplacian.)

The ground state energy of dilute gases does not distinguish the two regimes according to (22). The question whether the same holds for BEC in the thermodynamic limit is still open.

Dyson [3] succeeded in transforming Regime 1 into Regime 2 (for hard spheres) by sacrificing the kinetic energy. In this way he obtained a lower bound $\sim \rho a$ but the factor in front was only about 1/14 of the optimal one. His idea of replacing a hard potential by a soft one was, however, taken up in [4] and the following lemma is a key element in the proof of the lower bound as well as for much of the subsequent developments, in particular the rigorous derivations of the GP equation in [5] and [6].

LEMMA (Dyson’s Lemma): *Let $v(r) \geq 0$ with finite range R_0 . Let $U(r) \geq 0$ satisfy $\int U(r)r^2dr \leq 1$, $U(r) = 0$ for $r < R_0$. Then for all ψ and domains $\mathcal{B} \subset \mathbb{R}^3$ that are star shaped w.r.t. 0*

$$\int_{\mathcal{B}} [|\nabla\psi|^2 + \frac{1}{2}v|\psi|^2] \geq \int_{\mathcal{B} \cap \text{supp } U} [|\nabla\psi|^2 + aU|\psi|^2]. \quad (25)$$

Proof: By superposition, it is sufficient to consider $U(r) = \frac{1}{R^2}\delta(r-R)$ with $R > R_0$. Write $\psi(r) = u(r)/r$ and minimize $\int_0^R \{\mu[u'(r) - (u(r)/r)]^2 + \frac{1}{2}v(r)|u(r)|^2\}dr$ with $u(0) = 0$ and $u(R) = R - a$ (normalization). This leads to the zero energy scattering equation $-\mu u'' + \frac{1}{2}vu = 0$ with and $u(r) = r - a$ for $r > R_0$. The Lemma now follows by partial integration.

Remark. For the proof of BEC in the GP limit one makes use of the fact that the full kinetic energy outside the support of U , can be retained, i.e., one can add $\int_{\mathcal{B} \setminus \text{supp } U} |\nabla\psi|^2$ to the right side of (25).

Applying Dyson’s Lemma to the many-body problem implies that the interaction $\sum_{i < j} v(\mathbf{x}_i - \mathbf{x}_j)$ can, for the purpose of a lower bound, be replaced by a nearest neighbor interaction

$$a \sum_i U(\min_{j, j \neq i} |\mathbf{x}_i - \mathbf{x}_j|).$$

Hence configurations where three-particles come close together are ignored but the error is small for dilute gases. The proof of the energetic lower bound also involves the following ingredients:

- Retaining part of the kinetic energy inside the support of the potential U .
- Splitting the big box Λ into smaller Neumann boxes with of side length that stays fixed in the thermodynamic limit.

5 Gross-Pitaevskii Theory

Consider now the N -body Hamiltonian (4) with an external Potential V representing a confining trap, but, to begin with, no rotation:

$$H = \sum_{i=1}^N \{-\Delta_i + V(\mathbf{x}_i)\} + \sum_{1 \leq i < j \leq N} v(|\mathbf{x}_i - \mathbf{x}_j|). \quad (26)$$

The external potential comes with a natural length scale $L_{\text{osc}} = e_V^{-1/2}$ where e_V is the spectral gap of $-\Delta + V$.

One would like to study the ground state properties of H , and in particular BEC, in the *Gross-Pitaevskii (GP) limit* where $N \rightarrow \infty$ with a fixed value of the GP interaction parameter

$$g \equiv 4\pi N a / L_{\text{osc}} = e_0(\rho) / e_V \quad (27)$$

with $\rho = N/L_{\text{osc}}^3$. Note that $\rho a^3 \sim g/N^2 = O(1/N^2)$ if g is fixed, so the GP limit is a special case of a dilute limit.

The GP limit can be achieved either by keeping a fixed and scaling the external potential V so that $L_{\text{osc}} \sim N$, or, by keeping V fixed and taking $a \sim N^{-1}$. The latter can formally be regarded as a scaling of the interaction potential:

$$v(r) = N^2 v_1(Nr) \quad (28)$$

with v_1 fixed. Note that this is the opposite of the usual mean field limit where the potential is scaled with N as $v(r) = N^{-3} v_1(r/N)$. In fact, the technique for deriving the GP equation from the many-body Hamiltonian is quite different from mean field techniques.

It turns out that in the GP limit the ground state can be described by minimizing a functional of functions on \mathbb{R}^3 , the *GP energy functional*

$$\mathcal{E}^{\text{GP}}[\varphi] = \int_{\mathbb{R}^3} (|\nabla\varphi|^2 + V|\varphi|^2 + g|\varphi|^4) d^3\mathbf{x} \quad (29)$$

with the subsidiary condition $\int |\varphi|^2 = 1$. The corresponding Euler-Lagrange equation is precisely the GP equation (1).

The term $g|\varphi|^4$ is motivated by the energy formula (22): With $\rho(\mathbf{x}) = N|\varphi(\mathbf{x})|^2$ the local particle density, we have $Ng \int |\varphi(\mathbf{x})|^4 = 4\pi a \int \rho(\mathbf{x})^2$, and $4\pi a \rho(\mathbf{x})^2$ is the interaction energy per unit volume.

The minimizer $\varphi^{\text{GP}}(\mathbf{x})$ of the GP functional is the unique, nonnegative solution of the GP equation (1). The corresponding energy is

$$E_g^{\text{GP}} = \mathcal{E}^{\text{GP}}[\varphi^{\text{GP}}] = \inf\{\mathcal{E}^{\text{GP}}[\varphi] : \int |\varphi|^2 = 1\}. \quad (30)$$

The GP energy functional can be obtained formally from the many body Hamiltonian by replacing $v(\mathbf{x}_i - \mathbf{x}_j)$ by $8\pi a \delta(\mathbf{x}_i - \mathbf{x}_j)$ and making a Hartree-type product ansatz for the many body wave function, i.e., writing

$$\Psi(\mathbf{x}_1, \dots, \mathbf{x}_N) = \varphi(\mathbf{x}_1) \cdots \varphi(\mathbf{x}_N). \quad (31)$$

This is not a proof, however, and the true ground state is not of this form. This is particularly obvious if iv is a hard sphere potential since $\langle \Psi, H\Psi \rangle = \infty$ for

all product wave functions (31). Finite energy can in this case only be obtained for functions of the form

$$\Psi(\mathbf{x}_1, \dots, \mathbf{x}_N) = \varphi(\mathbf{x}_1) \cdots \varphi(\mathbf{x}_N) F(\mathbf{x}_1, \dots, \mathbf{x}_N) \quad (32)$$

with $F(\mathbf{x}_1, \dots, \mathbf{x}_N) = 0$ if $|\mathbf{x}_i - \mathbf{x}_j| \leq a$ for some $i \neq j$. The upper bound on the energy is, in fact, proved by using trial functions of this form with a judiciously chosen F involving the zero-energy scattering solution of the two-body problem.

The basic results in GP theory are the following theorems, proved in [5] and [6] respectively:

THEOREM (GP energy asymptotics) *If $N \rightarrow \infty$ with g fixed, then*

$$\frac{E^{\text{QM}}(N, a)}{NE_g^{\text{GP}}} \rightarrow 1. \quad (33)$$

THEOREM (BEC in GP limit) *If $N \rightarrow \infty$ with g fixed (i.e., $a \sim N^{-1}L_{\text{osc}}$), then*

$$\frac{1}{N}\rho^{(1)}(\mathbf{x}, \mathbf{x}') \rightarrow \varphi^{\text{GP}}(\mathbf{x})\varphi^{\text{GP}}(\mathbf{x}'). \quad (34)$$

In other words: There is complete BEC in the GP limit and the solution of the GP equation is the wave function of the condensate.

COROLLARY (Momentum density in GP limit) *In the GP limit the normalized particle density in the many-body ground state converges to $N|\varphi^{\text{GP}}(\mathbf{x})|^2$ and the momentum density to $N|\tilde{\varphi}^{\text{GP}}(\mathbf{p})|^2$.*

6 The Rotating Case

The GP functional in the rotating case is

$$\mathcal{E}^{\text{GP}}[\varphi] = \int_{\mathbb{R}^3} \left\{ |(i\nabla + \mathbf{A}(\mathbf{x}))\varphi|^2 + (V - \frac{1}{4}\Omega^2 r^2)|\varphi|^2 + g|\varphi|^4 \right\} d\mathbf{x}. \quad (35)$$

with corresponding GP equation

$$-(\nabla - i\mathbf{A}(\mathbf{x}))^2\varphi(\mathbf{x}) + V(\mathbf{x})\varphi(\mathbf{x}) + 2g|\varphi(\mathbf{x})|^2\varphi(\mathbf{x}) = \mu\varphi(\mathbf{x}). \quad (36)$$

The infimum of (35) over normalized wave functions is denoted $E_{g,\Omega}^{\text{GP}}$. Contrary to the nonrotating case the minimizer, i.e., the solution of (36), need not be unique due to the appearance of vortices. The following basic results on the relation to ground state of the quantum mechanical Hamiltonian (4) were proved in [7] and [6]:

THEOREM (Energy asymptotics in GP limit, Ω fixed) *If $N \rightarrow \infty$ with g and Ω fixed, then*

$$\frac{E^{\text{QM}}(N, a, \Omega)}{NE_{g,\Omega}^{\text{GP}}} \rightarrow 1. \quad (37)$$

THEOREM (BEC in GP limit, Ω fixed) *If $N \rightarrow \infty$ with g and Ω fixed, then the convex hull of the projectors onto GP minimizers coincides with the*

possible $N \rightarrow \infty$ limits of one particle density matrices of N -particle ground states.

The technique of proof is by necessity rather different from the one originally used in the non-rotating case. The reason is that the many-body wave functions are no longer real valued (in general), and the phase factor prevents localization of the system in small Neumann boxes as in the original proof.

Basic ingredients for the proof of the energetic lower bound in [6] are:

- Coherent states
- Dyson's Lemma, leading to a 'soft' potential, but sacrificing the high frequency part of the kinetic energy
- A bound on the three-particle density in the ground state, using a functional integral representation.

The theorems above are stated for fixed coupling and rotational velocity. What can be said if $g \rightarrow \infty$ as $N \rightarrow \infty$ and/or if Ω varies with N ? For the latter one should distinguish two cases:

- Anharmonic trap, $V(\mathbf{x}) \geq (\text{const.})|\mathbf{x}|^s$ with $s > 2$.
- Harmonic trap, $V(\mathbf{x})$ quadratic, e.g. $V(\mathbf{x}) = \frac{1}{2}\Omega_{\text{osc}}|\mathbf{x}|^2$.

For harmonic traps it is necessary that $\Omega < \Omega_c = \sqrt{2}\Omega_{\text{osc}}$, but very interesting phenomena, related to a fractional quantum Hall effect for bosons in the lowest Landau level, are expected when $\Omega \rightarrow \Omega_c$. For recent results on this regime and further references we refer to [8] and [9].

In the remainder of the lecture we shall discuss the case of an anharmonic trap, where it is possible to take $\Omega \rightarrow \infty$. For simplicity we shall assume that V is homogeneous of degree $s > 2$, i.e., $V(\lambda\mathbf{x}) = \lambda^s V(\mathbf{x})$ for $\lambda > 0$.

The Thomas-fermi (TF) functional is obtained from the GP functional by dropping the 'magnetic' kinetic energy term:

$$\mathcal{E}^{\text{TF}}[\rho] \equiv \int_{\mathbb{R}^3} \left\{ V\rho - \frac{1}{4}\Omega^2 r^2 \rho + g\rho^2 \right\} \quad (38)$$

defined for nonnegative densities $\rho(\cdot)$, with the TF energy

$$E_{g,\Omega}^{\text{TF}} \equiv \inf \{ \mathcal{E}^{\text{TF}}[\rho] : \|\rho\|_1 = 1 \}. \quad (39)$$

There is a unique minimizer:

$$\rho_{g,\Omega}^{\text{TF}}(\mathbf{x}) = \frac{1}{2g} \left[\mu_{g,\Omega}^{\text{TF}} + \frac{1}{4}\Omega^2 r^2 - V(\mathbf{x}) \right]_+ \quad (40)$$

where $[\cdot]_+$ denotes the positive part and $\mu_{g,\Omega}^{\text{TF}}$ is the TF chemical potential determined by the normalization $\|\rho_{g,\Omega}^{\text{TF}}\|_1 = 1$.

As $g \rightarrow \infty$ and/or $\Omega \rightarrow \infty$ for $N \rightarrow \infty$, the TF energy and density give the leading asymptotics of the (suitably scaled) many-body QM energy and density, provided the gas remains dilute, i.e. $\bar{\rho}a^3 \rightarrow 0$, where $\bar{\rho}$ is the average density.

It is convenient to define a scaled rotational velocity

$$\omega \equiv g^{-\frac{s-2}{2(s+3)}} \Omega \quad (41)$$

and we distinguish the following three cases:

- **Slow rotation**, $\omega \ll 1$: The effect of the rotation is negligible to leading order.
- **Rapid rotation**, $\omega \sim 1$: Rotational effects are comparable to those of the interactions.
- **Ultrarapid rotation**, $\omega \gg 1$: Rotational effects dominate.

The precise statement for the energy asymptotics is the following theorem proved in [10]:

THEOREM (Leading QM Energy Asymptotics)

Assume that $a^3 \|\rho_{g,\Omega}^{\text{TF}}\|_\infty \rightarrow 0$ as $N \rightarrow \infty$.

(i) If $g \rightarrow \infty$ and $\omega \rightarrow 0$ as $N \rightarrow \infty$, then

$$\lim_{N \rightarrow \infty} \left\{ g^{-\frac{s}{s+3}} N^{-1} E_{g,\Omega}^{\text{QM}}(N) \right\} = E_{1,0}^{\text{TF}} \quad (42)$$

(ii) If $g \rightarrow \infty$ and $\omega > 0$ is fixed as $N \rightarrow \infty$, then

$$\lim_{N \rightarrow \infty} \left\{ g^{-\frac{s}{s+3}} N^{-1} E_{g,\Omega}^{\text{QM}}(N) \right\} = E_{1,\omega}^{\text{TF}} \quad (43)$$

(iii) If $\Omega \rightarrow \infty$ and $\omega \rightarrow \infty$ as $N \rightarrow \infty$, then

$$\lim_{N \rightarrow \infty} \left\{ \Omega^{-\frac{2s}{s-2}} N^{-1} E_{g,\Omega}^{\text{QM}}(N) \right\} = E_{0,1}^{\text{TF}} \quad (44)$$

Sketch of proof:

1. *Upper bound.* For slow to rapid rotation (cases (i)-(ii)) we use a trial function of the form

$$\Psi(\mathbf{x}_1, \dots, \mathbf{x}_N) = \prod_{i=1}^N \varphi^{\text{GP}}(\mathbf{x}_i) F(\mathbf{x}_1, \dots, \mathbf{x}_N) \quad (45)$$

with a *real valued* function F and a GP minimizer φ^{GP} . Then

$$\begin{aligned} \frac{\langle \Psi, H\Psi \rangle}{\langle \Psi, \Psi \rangle} &= NE^{\text{GP}} + 4\pi g N \int |\varphi_i^{\text{GP}}|^4 \\ &+ \langle \Psi, \Psi \rangle^{-1} \int \left(\sum_i |\nabla_i F|^2 - 8\pi g |\varphi_i^{\text{GP}}|^2 |F|^2 + \sum_{i<j} v_{ij} |F|^2 \right) \prod_i |\varphi_i^{\text{GP}}|^2. \end{aligned} \quad (46)$$

For this computation the GP equation (36) for φ^{GP} has been used.

As a trial function for the last integral in (46) we can take a ‘Dyson wave function’ [3] of the form

$$F(\mathbf{x}_1, \dots, \mathbf{x}_N) = F_1(\mathbf{x}_1) F_2(\mathbf{x}_1, \mathbf{x}_2) \cdots F_N(\mathbf{x}_1, \dots, \mathbf{x}_N). \quad (47)$$

Here $F_i(\mathbf{x}_1, \dots, \mathbf{x}_i) = f(\min_{j<i} |\mathbf{x}_i - \mathbf{x}_j|)$ and f is essentially the zero energy scattering solution for the interaction potential v (assumed to be nonnegative, radially symmetric and of short range.) Note: F is not symmetric, but this is allowed because the quadratic form in F defined by the integral in (46) is real valued [7]. The bound for this term is obtained in a similar way as for the nonrotating case in [5]. The proof is then completed by bounding E^{GP} in terms of E^{TF} and also the GP density in terms of the TF density.

For ultrarapid rotations, $\omega \rightarrow \infty$, the trial function has to be modified. One can take

$$\Psi(\mathbf{x}_1, \dots, \mathbf{x}_N) = \prod_{i=1}^N \varphi(\mathbf{x}_i) F(\mathbf{x}_1, \dots, \mathbf{x}_N) \quad (48)$$

with

$$\varphi(\mathbf{x}) = \sqrt{\rho_\epsilon^{\text{TF}}(\mathbf{x})} \exp(iS(\mathbf{x})) \quad (49)$$

where $\rho_\epsilon^{\text{TF}}$ is a regularized TF density and the phase factor S corresponds to a ‘giant vortex’ centered at the origin:

$$S(\mathbf{x}) = \left[\frac{1}{2} r_\Omega^2 \Omega \right] \vartheta \quad (50)$$

where $[\cdot]$ denotes the integer part and r_Ω is the radius of the set \mathcal{M}_Ω where $V - \Omega^2 r^2 / r$ is minimal. (This set is always a subset of a cylinder.)

2. *Lower bound.* In the case $\omega < \infty$ one first uses the diamagnetic inequality

$$|(i\nabla + \mathbf{A})\varphi|^2 \geq |\nabla\varphi|^2. \quad (51)$$

Then one writes

$$V - \frac{\omega^2}{4} r^2 = V - \frac{\omega^2}{4} r^2 - \mu + \mu \geq -2\rho^{\text{TF}} + \mu \quad (52)$$

with the TF chemical potential $\mu = \mu_{1,\omega}^{\text{TF}}$ that satisfies

$$\mu = E^{\text{TF}} + \int (\rho^{\text{TF}})^2. \quad (53)$$

We then have to bound the Hamiltonian

$$\tilde{H} = \sum_i (-\Delta_i - 2g' \rho_i^{\text{TF}}) + \sum_{i < j} v'_{ij} \quad (54)$$

where the primes indicate suitable scalings of the coupling constant and the interaction potential. A lower bound on (54) is achieved by introducing Neumann boxes inside which ρ^{TF} is approximately constant and using the basic bound for the energy of n particles in a box of side length ℓ [4]:

$$E^{\text{QM}}(n, \ell) \geq 4\pi a(n/\ell^3)(1 - o(1)) \quad (55)$$

where $o(1) \rightarrow 0$ if $an/\ell^3 \rightarrow 0$ and $n \rightarrow \infty$. The result is a lower bound

$$-g' \int (\rho^{\text{TF}})^2 (1 + o(1)) \quad (56)$$

for \tilde{H} and altogether the bound

$$NE^{\text{TF}}(1 - o(1)) \quad (57)$$

for the energy.

The case of ultrarapid rotations, $\omega \rightarrow \infty$, is simpler than the case of finite ω since in the lower bound for the energy we can ignore the (positive) interaction altogether. Namely, with Ψ_N a normalized ground state, we can write

$$\Omega^{-\frac{2s}{s-2}} N^{-1} \langle \Psi_N, H \Psi_N \rangle = C_{\Psi_N} + \inf_{\mathbb{R}^3} W \quad (58)$$

with $W(\mathbf{x}) = V(\mathbf{x}) - r^2/4$ and

$$C_{\Psi_N} = \Omega^{-\frac{2s}{s-2}} \|[i\nabla + A]\Psi_N\|_2^2 + \int_{\mathbb{R}^3} \hat{\rho}^{\text{TF}}(\vec{x}) \left(W(\mathbf{x}) - \inf_{\mathbb{R}^3} W \right) d\mathbf{x} + \Omega^{-\frac{2s}{s-2}} N^{-1} \sum_{1 \leq i < j \leq N} \langle \Psi_N, v(|\mathbf{x}_i - \mathbf{x}_j|) \Psi_N \rangle. \quad (59)$$

Since the interaction potential v is by assumption nonnegative the same holds for C_{Ψ_N} , and $\inf W = E_{0,1}^{\text{TF}}$.

What about the subleading terms beyond the TF term? This is question is highly nontrivial already in the GP context, i.e., ignoring the many-body aspects. The following has been proved in [11] for a two-dimensional GP theory in a ‘flat’ trap (corresponding to $s = \infty$), or equivalently, in a 3D cylinder (‘beer can’) with Neumann boundary conditions:

THEOREM (Energy to subleading order)

If $\log g \ll \Omega \ll g^{1/2}$, then

$$E^{\text{GP}} = E^{\text{TF}} + (\Omega/2) \log(\Omega/g)(1 + o(1)).$$

If $g^{1/2} \lesssim \Omega \ll g/|\log g|$ then

$$E^{\text{GP}} = E^{\text{TF}} + (\Omega/2) \log(g^{1/2})(1 + o(1)).$$

7 Conclusions

For rapidly rotating, dilute Bose gases in anharmonic traps the many-body leading energy and density asymptotics as g and/or $\Omega \rightarrow \infty$ can be calculated exactly from a simple density functional. The subleading order has been calculated within GP theory. The following (difficult) problems are open:

- Derive the subleading order of the energy from the many-body Hamiltonian.
- Prove BEC into GP minimizers when g and Ω are not fixed.

References

- [1] E.H. Lieb, R. Seiringer, J.P. Solovej, J. Yngvason, *The Mathematics of the Bose Gas and its Condensation*, (Birkhäuser, Basel, 2005).
- [2] E.H. Lieb, J. Yngvason, The Ground State Energy of a Dilute Two-dimensional Bose Gas, *J. Stat. Phys.* **103** (2001), 509–526.
- [3] F.J. Dyson, Ground-State Energy of a Hard-Sphere Gas, *Phys. Rev.* **106** (1957), 20–26.
- [4] E.H. Lieb, J. Yngvason, Ground State Energy of the Low Density Bose Gas, *Phys. Rev. Lett.* **80** (1998), 2504–2507.

- [5] E.H. Lieb, R. Seiringer, J. Yngvason, Bosons in a Trap: A Rigorous Derivation of the Gross-Pitaevskii Energy Functional, *Phys. Rev. A* **61** (2000), 043602.
- [6] E.H. Lieb, R. Seiringer, Derivation of the Gross-Pitaevskii Equation for Rotating Bose Gases, *Commun. Math. Phys.* **264** (2006), 505-537.
- [7] R. Seiringer, Ground State Asymptotics of a Dilute, Rotating Gas, *J. Phys. A: Math. Gen.* **36** (2003), 9755-9778.
- [8] E. H. Lieb, R. Seiringer, J. Yngvason, The Yrast Line of a Rapidly Rotating Bose Gas: Gross-Pitaevskii Regime, *Phys. Rev. A* **79** (2009), 063626
- [9] M. Lewin, R. Seiringer, *Strongly correlated phases in rapidly rotating Bose gases*, arXiv:0906.0741
- [10] J.-B. Bru, M. Correggi, P. Pickl, J. Yngvason, The TF Limit for Rapidly Rotating Bose Gases in Anharmonic Traps, *Commun. Math. Phys.* **280** (2008), 517-544.
- [11] M. Correggi, J. Yngvason, Energy and Vorticity in Fast Rotating Bose-Einstein Condensates, arXiv:0806.3191, *J.Phys. A* **41** (2008), 445002