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ANISOTROPIC INVERSE PROBLEMS AND CARLEMAN ESTIMATES

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Abstract. This note reports on recent results on the anisotropic Calderón problem obtained in a joint work with Carlos E. Kenig, Mikko Salo and Gunther Uhlmann [8]. The approach is based on the construction of complex geometrical optics solutions to the Schrödinger equation involving phases introduced in the work [12] of Kenig, Sjöstrand and Uhlmann in the isotropic setting. We characterize those manifolds where the construction is possible, and give applications to uniqueness for the corresponding anisotropic inverse problems in dimension $n \geq 3$.

Contents

1. Introduction

In a foundational paper of 1980 [6], A. Calderón asked whether it is possible to determine the electrical conductivity of a body by making current and voltage measurements at the boundary. This inverse problem, also known as Electrical Impedance Tomography (EIT), has in particular applications to medical imaging. The question has been extensively studied for isotropic conductivities, but the case where the conductivities depend on the direction is also of great interest. For instance muscle tissues may have different conductivities in the transverse and longitudinal directions.

The problem can be formulated in mathematical terms as follows. Let $\Omega \subset \mathbf{R}^n$ be a bounded open set with smooth boundary, the conductivity in the anisotropic case is represented by a smooth symmetric positive matrix $\gamma = (\gamma^{jk})_{1\leq j,k\leq n}$ depending smoothly on $x \in \overline{\Omega}$. If there are no sources or sinks of current in Ω, the potential u on Ω induced by a voltage potential $f \in H^{\frac{1}{2}}(\partial\Omega)$ on the boundary solves the Dirichlet problem

$$\begin{cases} \dfrac{\partial}{\partial x_j}\Big(\gamma^{jk}\dfrac{\partial u}{\partial x_k}\Big) = 0 & \text{in } \Omega, \\ \qquad\qquad\quad u = f & \text{on } \partial\Omega. \end{cases}$$

Here and throughout this note we are using Einstein's summation convention: repeated indices in lower and upper position are summed. The boundary measurements are given by the Dirichlet-to-Neumann map, defined by

$$\Lambda_\gamma f = \gamma^{jk}\frac{\partial u}{\partial x_j}\nu_k\Big|_{\partial\Omega}$$

where $\nu = (\nu_1, \dots, \nu_n)$ denotes the unit outer normal to $\partial\Omega$ and u is the solution of the Dirichlet problem. The inverse problem is whether one can determine γ by knowing Λ_γ. In the isotropic case, the inverse problem is restricted to conductivities which are multiple of the identity $\gamma^{jk} = c\,\delta^{jk}$.

Unfortunately, the Dirichlet-to-Neumann map Λ_γ doesn't determine γ uniquely, an observation due to L. Tartar. Indeed, if $\psi : \overline{\Omega} \to \overline{\Omega}$ is a diffeomorphism which is the identity on the boundary $\psi|_{\partial\Omega} = \mathrm{Id}$ then we have

$$\Lambda_{\tilde{\gamma}} = \Lambda_\gamma$$

where

$$\tilde{\gamma}\circ\psi = \frac{1}{|\det\psi'|}\,{}^t\psi'\,\gamma\,\psi'.$$

The question is whether this is the only obstruction to unique determination of the conductivity. In the 2-dimensional case, as was shown by Sylvester [30], the anisotropic conductivity problem can be reduced to the isotropic one by using isothermal coordinates. In this note, we will therefore be interested in the case $n \geq 3$. It is noteworthy that the 2 and higher dimensional problems are quite different in many aspects. Some evidence of this fact will be given in section 3.

Let us review some of the known results and begin by the isotropic case. In [6] A. Calderón studied the linearized problem and proved uniqueness for conductivities close to constants. Then R. Kohn and M. Vogelius [14] proved that the knowledge of the Dirichlet-to-Neumann map determines the Taylor expansion of the conductivity at the boundary, settling the case of analytic conductivities. It was J. Sylvester and

G. Uhlmann who finally solved Calderón's problem in [31] in dimension $n \geq 3$. Their approach is based on the observation that the conductivity problem can be reduced to a similar inverse problem on the Schrödinger operator

$$-\Delta + q \quad \text{with} \quad q = \frac{\Delta\sqrt{c}}{\sqrt{c}}$$

and on the construction of solutions

$$e^{\frac{1}{h}\langle x,\zeta\rangle}(1 + r(x,\zeta,h)), \qquad \zeta \in \mathbf{C}^n, \quad \zeta^2 = 0 \tag{1.1}$$

to the Schrödinger equation by means of complex geometrical optics. Finally in [20], A. Nachman proved uniqueness for the conductivity problem in dimension $n = 2$. His result was recently extended to L^∞ conductivities by K. Astala and L. Päivärinta [3] in the isotropic case and by K. Astala, M. Lassas and L. Päivärinta [2] in the anisotropic case. In dimension $n \geq 3$, the best regularity for which uniqueness has been obtained so far is $W^{\frac{3}{2},\infty}$, a result of L. Päivärinta, A. Panchenko and G. Uhlmann [22]. The problem is still open for less regular conductivities.

In dimension $n \geq 3$, the anisotropic conductivity problem is of geometrical nature. Let (M, g) be a smooth compact oriented manifold with boundary. Consider the Dirichlet problem

$$\begin{cases} \Delta_g u = 0 & \text{in } M, \\ u = f & \text{on } \partial M. \end{cases}$$

The Dirichlet-to-Neuman map Λ_g maps any function $f \in H^{\frac{1}{2}}(\partial M)$ on the boundary into the normal derivative of the corresponding solution of the Dirichlet problem

$$\Lambda_g f = \partial_\nu u\big|_{\partial M} = g^{jk}\nu_k \frac{\partial u}{\partial x_j}\Big|_{\partial M}.$$

The inverse problem is to recover g from Λ_g. There is a similar obstruction to uniqueness as in the Calderón problem since

$$\Lambda_{\psi^* g} = \Lambda_g \tag{1.2}$$

if $\psi : M \to M$ is a diffeomorphism which is the identity on the boundary. The relation between the conductivity problem and the geometrical problem is given by

$$g_{jk} = (\det \gamma)^{\frac{1}{n-2}} \gamma^{jk}.$$

In [17], J. Lee and G. Uhlmann proved the analogue of Kohn and Vogelius' results, i.e. that the knowledge of the Dirichlet-to-Neumann map determines the Taylor expansion of the metric at the boundary. It is usually referred to this result as boundary determination. Boundary determination implies that (1.2) is the only obstruction to the unique determination of the metric for real-analytic manifolds. This was first proved in dimension[1] $n \geq 3$ for strongly convex and simply connected manifolds. In [16], M. Lassas and G. Uhlmann removed the remaining topological assumptions on the manifold. Einstein manifolds are real-analytic in the interior and it was conjectured in [16] that Einstein manifolds are determined, up to isometry from the Dirichlet-to-Neumann map. This was recently proven by C. Guillarmou and A. Sa Barreto [9]. These results on the anisotropic Calderón problem are all based on the analyticity of the metric. The recovery of the metric in the interior of M proceeds by analytic continuation, using the knowledge of Taylor series of g at the boundary. Thus these results do not give information from the interior of the manifold.

On the other hand, in the isotropic case (where g is a conformal multiple of the Euclidean metric), many results are available even for non-smooth coefficients. These results are based on special complex geometrical optics solutions to elliptic equations (1.1) introduced in [31]. However, complex geometrical optics solutions have not been available in the anisotropic case, which has been a major difficulty in the study of that problem. The purpose of this work was to try to generalize this approach (described in more details in section 3) to the anisotropic case, and to find out when the construction is possible.

2. Uniqueness results

In this section we state the main uniqueness results obtained in [8] in collaboration with Carlos Kenig, Mikko Salo and Gunther Uhlmann. Let us first introduce the class of manifolds for which we can prove uniqueness results in inverse problems. For this we need the notion of simple manifolds [26].

Definition 2.1. *A manifold* (M, g) *with boundary is* simple *if* ∂M *is strictly convex, and for any point* $x \in M$ *the exponential map* $\exp_x$ *is a diffeomorphism from some closed neighborhood of* 0 *in* T_xM *onto* M*.*

Definition 2.2. *A compact manifold with boundary* (M, g)*, of dimension* $n \geq 3$*, is* admissible *if it is conformal to a submanifold with boundary of* $\mathbf{R} \times (M_0, g_0)$ *where* (M_0, g_0) *is a compact simple* $(n-1)$*-dimensional manifold.*

Examples of admissible manifolds include the following:

[1] See [8] for an account of the corresponding results in dimension 2.

1. Bounded domains in Euclidean space, in the sphere minus a point, or in hyperbolic space. In the last two cases, the manifold is conformal to a domain in Euclidean space via stereographic projection.
2. More generally, any domain in a locally conformally flat manifold is admissible, provided that the domain is appropriately small. Such manifolds include locally symmetric 3-dimensional spaces, which have parallel curvature tensor so their Cotton tensor vanishes.
3. Any bounded domain M in $\mathbf{R}^n$, endowed with a metric which in some coordinates has the form
$$g(x_1, x') = c(x) \begin{pmatrix} 1 & 0 \\ 0 & g_0(x') \end{pmatrix},$$
with $c > 0$ and g_0 simple, is admissible.
4. The class of admissible metrics is stable under C^2-small perturbations of g_0.

The first inverse problem involves the Schrödinger operator

$$\mathcal{L}_{g,q} = -\Delta_g + q,$$

where q is a smooth complex valued function on (M, g). We make the standing assumption that 0 is not a Dirichlet eigenvalue of $\mathcal{L}_{g,q}$ in M. Then the Dirichlet problem

$$\begin{cases} \mathcal{L}_{g,q} u = 0 & \text{in } M, \\ u = f & \text{on } \partial M \end{cases}$$

has a unique solution for any $f \in H^{1/2}(\partial M)$, and we may define the Dirichlet-to-Neumann map

$$\Lambda_{g,q} : f \mapsto \partial_\nu u|_{\partial M}.$$

Let us point out the gauge invariance of the Dirichlet-to-Neumann map under conformal change of metrics

$$\Lambda_{g,q} = \Lambda_{c^{-1}g, c(q-q_c)} \quad \text{where} \quad q_c = c^{\frac{n-2}{4}} \Delta_g \big(c^{-\frac{n-2}{4}}\big). \tag{2.1}$$

Given a fixed admissible metric, one can determine the potential q from boundary measurements.

Theorem 2.3. *Let (M, g) be admissible, and let q_1 and q_2 be two smooth functions on M. If $\Lambda_{g,q_1} = \Lambda_{g,q_2}$, then $q_1 = q_2$.*

This result was known previously in dimensions $n \geq 3$ for the Euclidean metric [31] and for the hyperbolic metric [11]. We remark that in the two dimensional case global uniqueness is not known even for the Euclidean metric. It is only known for potentials coming from conductivities [20] or for a generic class of potentials [28].

We obtain similar uniqueness results for the Schrödinger operator in the presence of a magnetic field. Let A be a smooth complex valued 1-form on M (the magnetic potential), and denote

$$\mathcal{L}_{g,A,q} = d_{\bar{A}}{}^* d_A + q,$$

where $d_A = d + iA\wedge\ : C^\infty(M) \to \Omega^1(M)$ and $d_A{}^*$ is the formal adjoint[2] of d_A. As before, we assume throughout that 0 is not a Dirichlet eigenvalue of $\mathcal{L}_{g,A,q}$ in M, and consider the Dirichlet problem

$$\begin{cases} \mathcal{L}_{g,A,q} u = 0 & \text{in } M, \\ u = f & \text{on } \partial M. \end{cases}$$

We define the Dirichlet-to-Neumann map as the magnetic normal derivative

$$\Lambda_{g,A,q} : f \mapsto d_A u(\nu)|_{\partial M}.$$

This map is invariant under gauge transformations of the magnetic potential:

$$\Lambda_{g,A+d\psi,q} = \Lambda_{g,A,q}$$

for any smooth function ψ which vanishes on the boundary. Thus, it is natural to expect to recover the magnetic field dA and the electric potential q from the map $\Lambda_{g,A,q}$.

Theorem 2.4. *Let (M, g) be admissible, let A_1, A_2 be two smooth* 1-*forms on M and let q_1, q_2 be two smooth functions on M. If $\Lambda_{g,A_1,q_1} = \Lambda_{g,A_2,q_2}$, then $dA_1 = dA_2$ and $q_1 = q_2$.*

This result was proved in [21] for the Euclidean metric. The proof in [8] is closer to [7] which considers partial boundary measurements. See [24] for further references on the inverse problem for the magnetic Schrödinger operator in the Euclidean case.

The next result considers the anisotropic Calderón problem. Under the additional condition that the metrics are in the same conformal class, one expects uniqueness since the only diffeomorphism that leaves a conformal class invariant is the identity. In dimensions $n \geq 3$ this was known earlier for metrics conformal to the Euclidean metric [31], conformal to the hyperbolic metric [11], and analytic metrics in the same conformal class [18] (based on [17]).

Theorem 2.5. *Let (M, g_1) and (M, g_2) be two admissible Riemannian manifolds in the same conformal class. If $\Lambda_{g_1} = \Lambda_{g_2}$, then $g_1 = g_2$.*

[2] For the sesquilinear inner product induced by the Hodge dual on the exterior form algebra.

Proof. We want to show that if (M, g) is admissible and c is smooth and positive and if $\Lambda_{cg} = \Lambda_g$ then $c = 1$. Boundary determination implies $c|_{\partial M} = 1$ and $\partial_\nu c|_{\partial M} = 0$, and then the assumption and (2.1) imply

$$\Lambda_{cg,0} = \Lambda_{g,0} = \Lambda_{cg,q},$$

where $q = -\Delta_g(c^{\frac{n-2}{4}})/c^{\frac{n+2}{4}}$. This is the analogue of Sylvester and Uhlmann's observation according to which the conductivity problem may be solved by considering the inverse problem on the Schrödinger equation. We conclude from Theorem 2.3 that $q = 0$, so $\Delta_g(c^{\frac{n-2}{4}}) = 0$ in M. Since $c^{\frac{n-2}{4}} = 1$ on ∂M, uniqueness of solutions for the Dirichlet problem shows that $c \equiv 1$. □

3. The isotropic Calderón problem

The purpose of this section is to describe the method developed in the isotropic case (and in dimension $n \geq 3$) to obtain uniqueness in the inverse problem for the Schrödinger equation.

The first step is to relate the information provided by the Dirichlet-to-Neumann map to the interior of the domain. This can be done by an integration by parts: if u_1 and u_2 are solutions of the equations $-\Delta u_j + q_j u_j = 0$ for $j = 1, 2$ then we have

$$\int_\Omega (q_1 - q_2) u_1 u_2 \, dx = \int_{\partial\Omega} (\Lambda_{q_1} - \Lambda_{q_2}) u_1 u_2 \, ds. \tag{3.1}$$

The aim is therefore to construct a family of solutions of the Schrödinger equation which is rich enough so that the cancellation

$$\int_\Omega (q_1 - q_2) u_1 u_2 \, dx = 0$$

implies $q_1 = q_2$. The use of solutions constructed by complex geometrical optics

$$u_j = e^{\frac{1}{h}\langle x, \zeta_j\rangle}\big(1 + h r_j(x, \zeta_j)\big), \quad \zeta_j \in \mathbf{C}^n \quad \zeta_j^2 = 0$$

with $\|r_j\|_{L^2} = \mathcal{O}(1)$ if $|\operatorname{Re}\zeta_j| = 1$, has proved to be successful.

Suppose that one is able to construct such solutions then one has

$$\int_\Omega (q_1 - q_2) e^{\frac{1}{h}\langle x, \zeta_1 + \zeta_2\rangle} \, dx = \mathcal{O}(h) \tag{3.2}$$

if $\operatorname{Re}(\zeta_1 + \zeta_2) = 0$. It seems natural after passing to the limit in (3.2) to try to obtain information on the Fourier transform of the function $1_\Omega(q_1 - q_2)$. This leads to the following choice for ζ_j

$$\zeta_1 = \eta + i\Big(\frac{h}{2}\xi + \tau\Big), \quad \zeta_2 = -\eta + i\Big(\frac{h}{2}\xi - \tau\Big),$$

so that taking the limit when h tends to 0 in (3.2) gives

$$\mathcal{F}\big(1_\Omega(q_1 - q_2)\big)(\xi) = 0$$

hence $q_1 = q_2$ on Ω. The constraints $\zeta_j^2 = 0$ and $|\operatorname{Re}\zeta_j| = 1$ give

$$\Big|\frac{h}{2}\xi \pm \tau\Big| = |\eta| = 1, \quad \frac{h}{2}\xi \pm \tau \perp \eta \tag{3.3}$$

which implies that ξ, η and τ are non-vanishing orthogonal vectors. This construction is therefore only possible when $n \geq 3$. Note that the choice of the spectral parameters ζ_1 and ζ_2 is done in such a way that the exponential growth of the product $u_1 u_2$ is canceled since

$$\operatorname{Re}(\langle x, \zeta_1\rangle + \langle x, \zeta_2\rangle) = 0. \tag{3.4}$$

To sum up, the method consists in

(1) Constructing solutions of the Schrödinger equation by means of complex geometrical optics, and obtain estimates on the correction r_j.
(2) Passing to the limit when h tends to 0 in the integration by parts formula (3.1), and use the injectivity of the Fourier transform.

The second part is somewhat flexible, and there are some alternative arguments using the Radon transform, or microlocal analysis (for instance in the case where the Dirichlet-to-Neumann map is only known on part of the boundary). In the anisotropic problem, we use an alternative argument (see final section).

We now proceed to the construction of the correction r_j. One way is to use Carleman estimates. We will use the following semi-classical Sobolev norm

$$\|u\|_{H^1_{\mathrm{scl}}} = \big(\|u\|^2_{L^2} + \|h\nabla u\|^2_{L^2}\big)^{\frac{1}{2}}.$$

Theorem 3.1. *Let q be a bounded function on Ω. There exist constants $C > 0$ and $h_0 \in (0,1)$ such that for all $\xi \in S^{n-1}$, $h \leq h_0$ and all $u \in C_0^\infty(\Omega)$ the following estimate holds*

$$\|e^{\frac{1}{h}\langle x,\xi\rangle} u\|_{H^1_{\mathrm{scl}}} \leq C h^{-1} \|e^{\frac{1}{h}\langle x,\xi\rangle} h^2(\Delta - q)u\|_{L^2}. \tag{3.5}$$

Proof. Without loss of generality we can assume $q = 0$, since a perturbation of the Carleman inequality by zero order terms yields an error which can be absorbed in the left-hand side of the inequality provided h is small enough. Consider the phase

$$\tilde{\varphi} = \langle x, \xi\rangle + \frac{h}{2}\langle x, \xi\rangle^2$$

and the conjugated operator

$$P_{\tilde{\varphi}} = -e^{\frac{\tilde{\varphi}}{h}} h^2 \Delta e^{-\frac{\tilde{\varphi}}{h}}.$$

It suffices to prove the *a priori* estimate

$$\|v\|_{H^1_{\mathrm{scl}}} \leq C_1 h^{-1} \|P_{\tilde{\varphi}} v\|_{L^2}$$

since on $\Omega \times S^{n-1}$

$$e^{\frac{1}{h}\langle x,\xi\rangle} \leq e^{\frac{1}{2}\langle x,\xi\rangle^2} e^{\frac{1}{h}\langle x,\xi\rangle} \leq C_2\, e^{\frac{1}{h}\langle x,\xi\rangle}.$$

One has

$$P_{\tilde{\varphi}} = \underbrace{h^2 D^2 - (1 + h\langle x,\xi\rangle)^2}_{=A} + \underbrace{2i(1 + h\langle x,\xi\rangle)\langle \xi, hD\rangle + h^2}_{=iB}$$

where A and B are two selfadjoint operators, and also

$$\|P_{\tilde{\varphi}} u\|^2_{L^2} = \|Au\|^2_{L^2} + \|Bu\|^2_{L^2} + i([A,B]u,u)_{L^2}.$$

A computation gives

$$i[A,B] = 4h^2\big((1 + h\langle x,\xi\rangle)^2 + \langle \xi, hD\rangle^2\big)$$

thus we get

$$\begin{aligned} i([A,B]u,u)_{L^2} &= 4h^2 \|(1 + h\langle x,\xi\rangle)u\|^2_{L^2} + 4h^2 \|\langle \xi, hD\rangle u\|^2_{L^2} \\ &\geq 2h^2 \|u\|^2_{L^2} \end{aligned}$$

if h is small enough. Therefore we get $\|P_{\tilde{\varphi}} u\|^2_{L^2} \geq 2h^2 \|u\|^2_{L^2}$.

We can furthermore control the gradient of u since we have

$$\begin{aligned} \|h\nabla u\|^2_{L^2} &= (Au,u)_{L^2} + \|(1 + h\langle x,\xi\rangle)u\|^2_{L^2} \\ &\leq \|Au\|^2_{L^2} + 3\|u\|^2_{L^2} \leq \|P_{\tilde{\varphi}} u\|^2_{L^2} + 3\|u\|^2_{L^2} \end{aligned}$$

if h is small enough. Finally we obtain

$$\begin{aligned} \|h\nabla u\|^2_{L^2} + \|u\|^2_{L^2} &\leq (1 + h^{-2})\|P_{\tilde{\varphi}} u\|^2_{L^2} + 3\|u\|^2_{L^2} \\ &\leq 4h^{-2} \|P_{\tilde{\varphi}} u\|^2_{L^2}. \end{aligned}$$

This completes the proof. □

The Carleman estimate can easily be modified into

$$\|e^{\frac{1}{h}\langle x,\xi\rangle} u\|_{L^2} \leq C h^{-1} \|e^{\frac{1}{h}\langle x,\xi\rangle} h^2(\Delta - q)u\|_{H^{-1}_{\mathrm{scl}}} \tag{3.6}$$

and classical arguments involving the Hahn-Banach theorem show that the equation

$$e^{-\frac{1}{h}\langle x,\xi\rangle} h^2(\Delta - q)\big(e^{\frac{1}{h}\langle x,\xi\rangle} u\big) = w \tag{3.7}$$

has a solution $u \in H^1(\Omega)$ satisfying $\|u\|_{H^1_{\mathrm{scl}}(\Omega)} \leq C h^{-1} \|w\|_{L^2(\Omega)}$. Now if we take $w = h^2 e^{i\langle x,\eta\rangle/h} q$ and denote $r = h^{-1} e^{-i\langle x,\eta\rangle/h} u$, where u solves (3.7), then we have

$$h^2(\Delta - q)\Big(e^{\frac{1}{h}\langle x,\xi + i\eta\rangle}(1 + hr)\Big) = 0$$

and $\|r\|_{H^1_{\mathrm{scl}}(\Omega)} \leq C\|q\|_{L^2(\Omega)}$.

This ends the proof of the uniqueness in the inverse problem on the Schrödinger equation. Note that only $q \in L^\infty(\Omega)$ is needed in the method. The question is to what extent this method can be generalized to the anisotropic setting.

4. Limiting Carleman weights

Let $h \in (0,1]$ be a small parameter, consider the semiclassical Laplace-Beltrami operator $P = -h^2\Delta_g$. If φ is a smooth real-valued function on M, consider the conjugated operator

$$P_\varphi = e^{\varphi/h} P e^{-\varphi/h} \tag{4.1}$$

and denote p_φ its semiclassical principal symbol.

We want to construct complex geometrical solutions

$$u = e^{-\frac{1}{h}(\varphi + i\psi)}(a + hr) \tag{4.2}$$

of the Schrödinger equation $(-\Delta_g + q)u = 0$. Given a function φ, the construction amounts to looking for solutions of the conjugated equation

$$P_\varphi v + h^2 q v = 0$$

of the form $v = e^{-\frac{i}{h}\psi}(a + hr)$ and then applying the usual WKB method. This includes solving the eikonal equation

$$p_\varphi(x, d\psi) = 0$$

and a transport equation on a. Note that P_φ is not a self-adjoint operator and that the symbol p_φ is complex valued. The existence of a solution ψ to the eikonal equation implies

$$\{\overline{p_\varphi}, p_\varphi\}(x, d\psi) = 0. \tag{4.3}$$

It seems natural to ask for the conjugated operator P_φ to be locally solvable in the semiclassical sense, in order to find the correction term r and go from an approximate solution to an exact solution. This means that the principal symbol p_φ of the conjugated operator needs to satisfy Hörmander's local solvability condition

$$\frac{1}{i}\{\overline{p_\varphi}, p_\varphi\} \leq 0 \text{ when } \; p_\varphi = 0.$$

Since applications of the complex geometrical optics construction to inverse problems require to construct solutions with both exponential weights $e^{\varphi/h}$ and $e^{-\varphi/h}$ in order to cancel possible exponential behaviour in the product of two solutions (see for instance the remark before (3.4) in section 3) and since $p_{-\varphi} = \overline{p_\varphi}$, it seems natural to impose the bracket condition

$$\{\overline{p_\varphi}, p_\varphi\} = 0 \quad \text{when} \quad p_\varphi = 0.$$

Working with weights φ satisfying this condition is a way to ensure that both the integrability condition (4.3) for the eikonal equation and the solvability condition on $P_{\pm\varphi}$ are fulfilled.

We define such weights as being limiting Carleman weights. Here it is natural to work with open manifolds.

Definition 4.1. *A real-valued smooth function φ in an open manifold (M, g) is said to be a* limiting Carleman weight *if it has non-vanishing differential, and if it satisfies on T^*M the Poisson bracket condition*

$$\{\overline{p_\varphi}, p_\varphi\} = 0 \quad \textit{when} \quad p_\varphi = 0, \tag{4.4}$$

where p_φ is the semiclassical principal symbol of the conjugated Laplace-Beltrami operator (4.1)*.*

This notion was introduced in [12], where the limiting Carleman weight $\varphi = \log|x|$ was used in the isotropic context to prove that the knowledge of the Dirichlet-to-Neumann map, measured on possibly small subsets of the boundary, determines $q \in L^\infty(\Omega)$.

The existence of limiting Carleman weights is a property which only depends on conformal classes of geometries. The following result is a characterization of those Riemannian manifolds which admit limiting Carleman weights.

Theorem 4.2. *If (M, g) is an open manifold having a limiting Carleman weight, then some conformal multiple of the metric g admits a parallel unit vector field. For simply connected manifolds, the converse is also true.*

Locally, a manifold admits a parallel unit vector field if and only if it is isometric to the product of an Euclidean interval and another Riemannian manifold. This is an instance of the de Rham decomposition theorem or is easy to prove directly. Thus, if (M, g) has a limiting weight φ, one can choose local coordinates in such a way that $\varphi(x) = x_1$ and

$$g(x_1, x') = c(x)\begin{pmatrix} 1 & 0 \\ 0 & g_0(x') \end{pmatrix},$$

where c is a positive conformal factor. Conversely, any metric of this form admits $\varphi(x) = x_1$ as a limiting weight. Note that in the case $n = 2$, limiting Carleman weights in (M, g) are exactly the harmonic functions with non-vanishing differential.

The following lemma gives some geometrical properties of limiting Carleman weights. The Levi-Civita connection on (M, g) is denoted by D and $D^2\varphi$ is the Hessian of the function φ.

Lemma 4.3. *A function φ with non-vanishing differential is a limiting Carleman weight if and only if $|\nabla\varphi|^{-2}\nabla\varphi$ is a conformal Killing field. In particular, if φ is a limiting Carleman weight then the level sets of φ in (M, g) are totally umbilical submanifolds with normal $|\nabla\varphi|^{-1}\nabla\varphi$, with principal curvatures equal to*

$$\mu = -|\nabla\varphi|^{-3} D^2\varphi(\nabla\varphi, \nabla\varphi).$$

Umbilical hypersurfaces are known for the Euclidean space: they are parts of either hyperplanes or hyperspheres. Using this information, it is possible to determine all the limiting Carleman weights for the Euclidean metric.

Theorem 4.4. *Let Ω be an open subset of $\mathbf{R}^n$, $n \geq 3$, and let e be the Euclidean metric. The limiting Carleman weights in (Ω, e) are locally of the form*

$$\varphi(x) = a\varphi_0(x - x_0) + b$$

where $a \in \mathbf{R} \setminus \{0\}$ and φ_0 is one of the following functions:

$$\langle x, \xi\rangle, \quad \arg\langle x, \omega_1 + i\omega_2\rangle,$$
$$\log|x|, \quad \frac{\langle x, \xi\rangle}{|x|^2}, \quad \arg\big(e^{i\theta}(x + i\xi)^2\big), \quad \log\frac{|x+\xi|^2}{|x-\xi|^2}$$

with ω_1, ω_2 orthogonal unit vectors, $\theta \in [0, 2\pi)$ and $\xi \in \mathbf{R}^n \setminus \{0\}$.

Limiting Carleman weights can be used to prove Carleman estimates.

Theorem 4.5. *Let (U, g) be an open Riemannian manifold and (M, g) be a smooth compact Riemannian submanifold with boundary such that $M \Subset U$. Suppose that φ is a limiting Carleman weight on (U, g). Let q be a smooth function on M. There exist two constants $C > 0$ and $h_0 \in (0, 1]$ such that for all functions $u \in C_0^\infty(\operatorname{int} M)$ and all $0 < h \leq h_0$, one has the inequality*

$$\|e^{\frac{\varphi}{h}} u\|_{H^1_{\mathrm{scl}}(M)} \leq Ch\|e^{\frac{\varphi}{h}}(\Delta - q)u\|_{L^2(M)}. \tag{4.5}$$

A direct application of the commutator method will not be enough to get an *a priori* estimate assuming the bracket condition (4.4); one needs to use convexification. This classical argument consists in taking a modified weight $f \circ \varphi$ where f is a convex function chosen so that the bracket in (4.4) becomes positive. The proof is similar to the one given in the isotropic setting.

We would like to express our deepest thanks to Johannes Sjöstrand who made substantial contributions to the paper [8]. His unpublished notes on characterizing limiting Carleman weights in the Euclidean case are the basis for the study described in this section. In particular he proved that the level sets of limiting Carleman weights in the Euclidean case are either hyperspheres or hyperplanes.

5. Sketch of proof for the uniqueness

5.1. **Complex geometrical solutions.** Let (M,g) be an admissible Riemannian manifold. Following the program described in section 3 in the isotropic setting, we first want to construct the complex geometrical optics solutions (4.2). Suppose that φ is a limiting Carleman weight on an open Riemannian manifold (U,g) containing (M,g), the eikonal equation reads

$$|\nabla\psi|^2 = |\nabla\varphi|^2, \quad \langle\nabla\varphi,\nabla\psi\rangle = 0. \tag{5.1}$$

and the transport equation

$$2(\nabla\varphi + i\nabla\psi)a + \Delta(\varphi+i\psi)a = 0. \tag{5.2}$$

We will work in special coordinates to solve both equations.

We know that (M,g) is conformally embedded in $\mathbf{R}\times(M_0,g_0)$ for some compact simple $(n-1)$-dimensional (M_0,g_0). Assume, after replacing M_0 with a slightly larger simple manifold if necessary, that for some simple $(U,g_0)\Subset(\operatorname{int} M_0,g_0)$ one has

$$(M,g)\Subset(\mathbf{R}\times\operatorname{int} U,g)\Subset(\mathbf{R}\times\operatorname{int} M_0,g).$$

On the manifold $\mathbf{R}\times M_0$ the metric g has the form

$$g(x) = c(x)\begin{pmatrix} 1 & 0 \\ 0 & g_0(x') \end{pmatrix},$$

where $c>0$ and g_0 is simple. We choose the limiting Carleman weight to be $\varphi(x)=x_1$.

In these coordinates one has

$$\nabla\varphi = \frac{1}{c}\frac{\partial}{\partial x_1} \quad \text{and} \quad |\nabla\varphi| = \frac{1}{c}$$

thus the eikonal equation now reads

$$|\nabla\psi| = \frac{1}{c}, \quad \partial_{x_1}\psi = 0.$$

Under the given assumptions on (M,g), there is an explicit construction for ψ. Let $\omega\in U$ be a point such that $(x_1,\omega)\notin M$ for all x_1. Take (r,θ) to be polar normal coordinates in (U,g_0) with center ω, that is

$x' = \exp_\omega^U(r\theta)$ where $r > 0$ and $\theta \in S^{n-2}$. In these coordinates (which depend on the choice of ω) the metric has the form

$$g(x_1, r, \theta) = c(x_1, r, \theta) \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & m(r,\theta) \end{pmatrix},$$

where m is a smooth positive definite matrix. A solution of the eikonal equation is $\psi(x) = r = d_U(x', \omega)$.

With this choice of ψ and in these coordinates (x_1, r, θ) one has

$$\varphi + i\psi = x_1 + ir \quad \text{and} \quad \nabla\varphi + i\nabla\psi = \frac{2}{c}\overline{\partial}$$

with

$$\overline{\partial} = \frac{1}{2}\Big(\frac{\partial}{\partial x_1} + i\frac{\partial}{\partial r}\Big).$$

The transport equation now reads

$$4\overline{\partial} a + \Big(\overline{\partial} \log \frac{|g|}{c^2}\Big) a = 0.$$

The function

$$a = |g|^{-1/4} c^{1/2} a_0(x_1, r) b(\theta)$$

is a solution provided $\overline{\partial} a_0 = 0$.

As in the isotropic case, the Carleman estimates ensures the solvability of the conjugated equation and the construction of the correction term r. This completes the construction of complex geometrical optics solutions of the Schrödinger equation.

5.2. **The geodesical ray transform.** As in the isotropic setting, an integration by parts gives

$$\int_M (q_1 - q_2) u_1 u_2 \, dv = \int_{\partial M} (\Lambda_{q_1} - \Lambda_{q_2}) u_1 u_2 \, ds = 0 \tag{5.3}$$

for any solution $u_j \in H^1(M)$ of $-\Delta_g u_j + q_j u_j = 0$. Following the first part of this section, we take two solutions of the form

$$u_1 = e^{-\frac{1}{h}(x_1+ir)}\big(|g|^{-1/4} c^{1/2} e^{i\lambda(x_1+ir)} b(\theta) + h r_1\big),$$
$$u_2 = e^{\frac{1}{h}(x_1+ir)}\big(|g|^{-1/4} c^{1/2} + h r_2\big),$$

where λ is a real number and $\|r_j\|_{H^1_{\mathrm{scl}}(M)} = \mathcal{O}(1)$. Note that contrary to the isotropic case, we choose to kill the oscillations $e^{i\psi/h}$ in the product $u_1 u_2$; instead we will use the freedom in the choice of the amplitudes. Using these solutions in (5.3) and letting h tend to 0 gives

$$\int_{\mathbf{R}} \iint_{M_{x_1}} e^{i\lambda(x_1+ir)} (q_1 - q_2) c(x_1, r, \theta) b(\theta) \, dr \, d\theta \, dx_1 = 0,$$

with $M_{x_1} = \{(r,\theta)\,;\,(x_1,r,\theta) \in M\}$. The functions q_j can be extended to U by boundary determination. Since the former integral is zero for any smooth function b this implies

$$\int e^{-\lambda r}\Big(\underbrace{\int_{-\infty}^{\infty} e^{i\lambda x_1}(q_1-q_2)c(x_1,r,\theta)\,dx_1}_{=Q_\lambda(r,\theta)}\Big)\,dr = 0.$$

By analiticity of the Fourier transform if we can prove that $Q_\lambda = 0$ for λ small enough then we are done. But we have

$$\int_\gamma e^{-\lambda r} Q_\lambda(\gamma(r))\,dr = 0$$

for all the geodesics γ in U issued from the point ω. Varying ω we obtain that the attenuated geodesic ray transform of Q_λ

$$I^\lambda Q_\lambda(x',\xi') = \int_0^{\tau(x',\xi')} e^{-\lambda t} Q_\lambda(\gamma_{x',\xi'}(t))\,dt$$

vanishes identically. Here (x',ξ') is an inward pointing vector of M_0

$$(x',\xi') \in TM_0, \quad x' \in \partial M_0, \quad |\xi'| = 1, \quad \langle \xi', \nu_0(x')\rangle < 0,$$

(ν_0 is the outer unit normal vector to ∂M_0), $\gamma_{x',\xi'}$ denotes the geodesic starting from $\gamma(0) = x'$ with speed $\dot\gamma(0) = \xi'$ and $\tau(x',\xi')$ is the time when $\gamma_{x',\xi'}$ exits M_0. To finish the proof it suffices therefore to prove the injectivity of I^λ for λ small.

We denote

$$\partial_+ SM_0 = \{(x',\xi') \in TM_0,\, x' \in \partial M_0,\, |\xi'| = 1,\, \langle \xi', \nu_0(x')\rangle < 0\}$$

the set of inward pointing vectors. The injectivity of the geodesic ray transform is given in the following theorem.

Theorem 5.1. *Let (M_0, g_0) be a compact simple manifold with smooth boundary. There exists $\varepsilon > 0$ such that for all λ satisfying $|\lambda| \leq \varepsilon$ if*

$$I^\lambda f(x',\xi') = 0$$

for all $(x',\xi') \in \partial_+ SM_0$, then $f = 0$.

This result is proved by a perturbation argument (which explains why the parameter λ has to be taken small) from the well known [1], [19], [26] injectivity of the geodesic ray transform $I = I^0$. In the inverse problem for the magnetic Schrödinger equation, we need a similar result for the geodesic transform acting on 1-forms.

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