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# UNIQUE LOCAL EXISTENCE
# OF SOLUTION IN LOW REGULARITY SPACE
# OF THE CAUCHY PROBLEM FOR THE MKDV EQUATION
# WITH PERIODIC BOUNDARY CONDITION

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### §1. Introduction and a main result

In the present note, we consider the local solvability of the Cauchy problem for the modified Korteweg-de Vries equation on the one-dimensional torus $\mathbf{T} = \mathbf{R}/(2\pi\mathbf{Z})$.

$$(1.1) \qquad \partial_t u + \partial_x^3 u + u^2 \partial_x u = 0, \qquad t \in [-T, T], \quad x \in \mathbf{T},$$

$$(1.2) \qquad u(0, x) = u_0(x), \qquad x \in \mathbf{T},$$

where $T$ is a positive constant and the unknown function $u$ is real-valued. If $u$ is a "nice" solution of (1.1)-(1.2), then we have the conservation law of $L^2$ norm, that is, $\|u(t)\|_{L^2} = \|u_0\|_{L^2}$. In that case, we change the spatial variable $x$ to $x + ct$ with $c = \|u_0\|_{L^2}^2$ to rewrite equation (1.1) as follows.

$$(1.3) \qquad \partial_t u + \partial_x^3 u + \left(u^2 - \frac{1}{2\pi} \int_{\mathbf{T}} u^2(t, x) \, dx\right)\partial_x u = 0, \quad t \in [-T, T], \quad x \in \mathbf{T}.$$

Hereafter, we consider equation (1.3) instead of (1.1), since (1.3) is better than (1.1) as far as the the time local well-posedness is concerned.

In [1] Bourgain introduced the Fourier restriction norm method to study the well-posedness of the Cauchy problem for nonlinear dispersive wave equations such as the (modified) Korteweg-de Vries and the nonlinear Schrödinger equations (for nonlinear wave equations, see, e.g., [10]). For simplicity, we refer to the (time) local well-posedness as (LWP) and we put $H^s \equiv H^s(\mathbf{T})$ and $L^p \equiv L^p(\mathbf{T})$. In [1], Bourgain proved (LWP) of (1.2)-(1.3) in $H^s$ for $s \geq 1/2$ (for the global existence of solution, see Colliander, Keel, Staffilani, Takaoka and Tao [3] and Kappeler and Topalov [6]). His proof in [1] is based on the following trilinear estimate.

$$(1.4) \qquad \left\|\left(u^2 - \frac{1}{2\pi} \int_{\mathbf{T}} u^2(t, x) \, dx\right)\partial_x u\right\|_{Y_{-1/2, s}} \leq C\|u\|_{Y_{1/2, s}}^3$$

for $s \geq 1/2$, where

$$\|u\|_{Y_{b,s}} = \Big( \sum_{k=-\infty}^{+\infty} \int_{-\infty}^{+\infty} \langle k \rangle^{2s} \langle \tau - k^3 \rangle^{2b} |\tilde{u}(\tau,k)|^2 \, d\tau \Big)^{1/2},$$

$$Y_{b,s} = \{u \in \mathcal{S}'(\mathbf{R} \times \mathbf{R}); \ u(t, x + 2\pi) = u(t,x), \ \|u\|_{Y_{b,s}} < +\infty\}$$

for $b, s \in \mathbf{R}$. Here, $\tilde{u}$ denotes the Fourier transform of $u$ with respect to $t$ and $x$ and $\langle a \rangle = (1 + |a|)$ for $a \in \mathbf{R}$. In [8] Kenig, Ponce and Vega showed that when $s < 1/2$, the trilinear estimate (1.4) breaks down. In [2] Bourgain also proved that the flow map corresponding to (1.2)-(1.3) is not in $C^3$ for $s < 1/2$ as a mapping from the initial data in $H^s$ to the solution of (1.2)-(1.3) in $C([-T, T]; H^s)$. But these negative results do not necessarily imply that (1.2)-(1.3) is ill-posed in $H^s$ for $s < 1/2$. In [4], Christ, Colliander and Tao showed that when $s < 1/2$, the uniform continuous dependence of solution on initial data breaks down (for the case of $\mathbf{R}$, see [9]). That is, when $s < 1/2$, the continuous dependence on initial data does not hold in the following sense.

(UCD) $\qquad \forall R > 0, \exists \eta(r) \in C([0, \infty)), \exists T > 0; \ \eta \geq 0, \eta(r) \to 0 \ (r \to +0),$

$$\|u_0\|_{H^s}, \|v_0\|_{H^s} \leq R \implies \text{the solutions } u, v \text{ exist on } [-T, T] \text{ and}$$

$$\|u - v\|_{C([-T,T];H^s)} \leq \eta(\|u_0 - v_0\|_{H^s}).$$

This formulation of the continuous dependence on initial data seems slightly too strong, though the proof using the contraction argument often yields (UCD). In fact, two of the authors, Takaoka and Tsutsumi [12] proved (LWP) in $H^s$ for $1/2 > s > 3/8$, though the dependence of solutions on initial data is not uniformly continuous. A new ingredient of the proof in [12] is the use of the modified Fourier restriction norm defined as follows. For $b, s \in \mathbf{R}$, we put

(1.5)

$$\|v\|_{Z_{b,s}} = \Big( \sum_{k=-\infty}^{+\infty} \int_{-\infty}^{+\infty} \langle k \rangle^{2s} \langle \tau - k^3 - k|\hat{u}_0(k)|^2 \rangle^{2b} |\tilde{v}(\tau,k)|^2 \, d\tau \Big)^{1/2},$$

$$Z_{b,s} = \{v \in \mathcal{S}'(\mathbf{R} \times \mathbf{R}); \ v(t, x + 2\pi) = v(t,x), \ \|v\|_{Z_{b,s}} < +\infty\},$$

where $u_0$ is the initial data given in (1.2) and $\hat{u}$ denotes the Fourier transform of $u$ with respect to $x$ only. The $Z_{b,s}$ norm takes into account the effect of the nonlinear term. In the present note, we study the local solvability of (1.2)-(1.3) in yet lower regularity sapces.

We have the following theorem.

**Theorem 1.1.** *Assume that $3/8 \geq s > 1/4$. For any $u_0 \in H^s$, there exists $T = T(\|u_0\|_{H^s}) > 0$ such that (1.2)-(1.3) has a solution on $[-T, T]$ satisfying*

(1.6) $\qquad u \in C([-T, T]; H^s),$

(1.7) $\qquad \varphi \displaystyle\sum_{k=-\infty}^{\infty} e^{i(kx - k \int_0^t |\hat{u}(\tau,k)|^2 \, d\tau)} \hat{u}(t, k) \ \in Y_{1/2,s},$

*where $\varphi$ is any cut-off function in $C^\infty(\mathbf{R})$ with supp $\varphi \subset [-T, T]$. Furthermore, if $s > 1/3$ or if $k|\hat{u}_0(k)|^2 \in \ell^\infty$, the uniqueness and the continuous dependence also hold.*

*Remark 1.1.* (i) The property (1.7) is equivalent to the following:

$$(1.8) \qquad \varphi \sum_{k=-\infty}^{\infty} e^{i(kx - k \int_{t_0}^{t} |\hat{u}(\tau,k)|^2 \, d\tau)} \hat{u}(t,k) \ \in Y_{1/2,s}, \qquad t_0 \in (-T,T),$$

where $\varphi$ is any cut-off function in $C^{\infty}(\mathbf{R})$ with supp $\varphi \subset [-T,T]$.

(ii) In [6], Kappeler and Topalov showed by the inverse scattering method that when $s \geq 0$, the Cauchy problem of the mKdV equation on $\mathbf{T}$ is globally well-posed in $H^s$. While their proof in [6] heavily depends on the complete integrability of the mKdV equation, our proof is applicable to the equation with nonlinearity $u^2 \partial_x u$ replaced by $(u + u^2)\partial_x u$.

### §2. Sketch of Proof of Theorem 1.1

We first begin with the following observation. We take the Fourier coefficients in the spatial varible of equation (1.3) to have

$$(2.1) \qquad \partial_t \hat{u}(t,k) - ik^3 \hat{u}(t,k) = -i \sum_{\substack{k_1+k_2+k_3=k \\ k_1+k_2 \neq 0}} \hat{u}(k_1)\hat{u}(k_2)k_3\hat{u}(k_3)$$

$$= -i \Big\{ \sum_{\substack{k_1+k_2+k_3=k \\ (k_1+k_2)(k_2+k_3)(k_3+k_1) \neq 0}} \hat{u}(k_1)\hat{u}(k_2)k_3\hat{u}(k_3)$$

$$+ \sum_{\substack{k_1+k_2+k_3=k \\ (k_1+k_2)(k_3+k_1) \neq 0 \\ k_2+k_3=0}} \hat{u}(k_1)\hat{u}(-k_3)k_3\hat{u}(k_3)$$

$$+ \sum_{\substack{k_1+k_2+k_3=k \\ (k_1+k_2)(k_2+k_3) \neq 0 \\ k_3+k_1=0}} \hat{u}(-k_3)\hat{u}(k_2)k_3\hat{u}(k_3)$$

$$+ \sum_{\substack{k_1+k_2+k_3=k \\ k_1+k_2 \neq 0 \\ k_2+k_3=k_3+k_1=0}} k_3\hat{u}(k_3)\hat{u}(-k_3)^2 \Big\}$$

$$= -i\frac{k}{3} \Big\{ \sum_{\substack{k_1+k_2+k_3=k \\ (k_1+k_2)(k_2+k_3)(k_3+k_1) \neq 0}} \hat{u}(k_1)\hat{u}(k_2)\hat{u}(k_3) \Big\}$$

$$+ ik|\hat{u}(k)|^2\hat{u}(k).$$

At the last equality of (2.1), we have used the facts that $\hat{v}(-k) = \overline{\hat{v}(k)}$ for a real-valued function $v$ and so the second and the third terms vanish on the right hand side of the second equlity of (2.1), since the summation is taken over all positive and negative $k_3$. The first term on the right hand side of (2.1) is a "good" term. Actually, it can be estimated in $H^s$, $s > 1/4$, which is pointed out in [1, Remarks (ii) (a) and (b) after Proposition 8.37 on page 228]. On the other hand, the second term on the right hand side of (2.1) is the worst. It is clear what effect this term has on the solution. This term gives rise to the rapid oscillation of solution, which breaks (UCD) in $H^s$, $s < 1/2$. In fact, let us try the estimate of this worst term in $H^s$. Then, we have

$$|k|^s \big| k|\hat{u}(k)|\hat{u}(k) \big| \leq (|k|^{1/2}|\hat{u}(k)|)^2 |k|^s |\hat{u}(k)|$$

$$\leq C\big( |k|^s |\hat{u}(k)| \big)^3, \qquad |k| \gg 1$$

as long as $s \geq 1/2$. The lower bound for Bourgain's trilinear estimate comes from this fact. If we consider, instead of the solution $u(t)$ itself,

$$w(t,k) = e^{i(kx+tk^3+k \int_0^t |\hat{u}(\tau,k)|^2 \ d\tau)} \hat{u}(t,k),$$

then we can formally eliminate the worst term to have

$$(2.2) \qquad \partial_t w = -\frac{i}{2\pi} \sum_{(k_1+k_2)(k_2+k_3)(k_3+k_1)\neq 0} e^{i\phi} w(t,k_1) w(t,k_2) k_3 w(t,k_3),$$

$$(2.3) \qquad \phi = t(k^3 - k_1^3 - k_2^3 - k_3^2) - ik \int_0^t |\hat{u}(\tau,k)|^2 \ d\tau$$

$$+ i \sum_{j=1}^3 k_j \int_0^t |\hat{u}(\tau,k_j)|^2 \ d\tau.$$

Now the problem is how to control $e^{i\phi}$. This is quite non-obvious, because $e^{i\phi}$ depends on the solution which we have to estimate. The following lemma plays a crucial role in the estimate of the right hand side of (2.2).

**Lemma 2.1.** *Let* $\phi \in C^2(\mathbf{R})$ *and let* $M$ *be a sufficiently large positive constant. Assume that*

$$|\phi'(t)| \geq CM, \qquad |\phi''(t)| \leq CM^{-1/2}|\phi'(t)|^2, \qquad t \in \mathbf{R},$$

*where* $C$ *does not depend on* $M$. *Then, we have*

$$\left\| e^{i\phi} f_1 f_2 f_3 \right\|_{H_t^{-1/2}} \leq C_0 M^{-1/2} \prod_{j=1}^3 \|f_j\|_{H_t^{1/2}},$$

*where* $C_0$ *is a positive constant independent of* $M$ *and* $H_t^s$ *denotes the Sobolev space with respect to the time variable* $t$.

Lemma 2.1 enables us to control $e^{i\phi}$, but it does not yield an estimate of the difference of two solutions. If we wish to prove the uniqueness and the continuous dependence of solution on initial data, we have to estimate the difference of two solutions. It is possible to do so in some sense if we additionally assume that $s > 1/3$ or $k|\hat{u}_0(k)|^2 \in \ell^\infty$. To be more specific, when $s > 1/3$, the norm $Z_{b,s}$ difined as in (1.5) helps us to derive a kind of smoothing property for the solution (see, e.g., [12, Corollary 2.6 on page 3019]). When $k|\hat{u}_0(k)|^2 \in \ell^\infty$, we can show that $k|\hat{u}(t,k)|^2 \in L^\infty(0,T;\ell^\infty)$ in the same way as in the proof of Lemma 2.5 of [12]. The details of the proof will appear elsewhere.

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