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Lifshitz tails for some non monotonous random models

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Abstract

In this talk, we describe some recent results on the Lifshitz behavior of the density of states for non monotonous random models. Non monotonous means that the random operator is not a monotonous function of the random variables. The models we consider will mainly be of alloy type but in some cases we also can apply our methods to random displacement models.

RÉSUMÉ. Cet exposé décrit des résultats récents sur le comportement de Lifshitz de la densité d'états de certains modèles aléatoires non monotones. Ici, non monotone signifie que l'opérateur aléatoire n'est pas une fonction monotone des variables aléatoires. L'essentiel des résultats sont obtenus pour des modèles d'Anderson continus ; néanmoins, certains résultats s'appliquent aussi aux modèles de déplacements aléatoires.

1 The basic model

Consider the continuous alloy type (or Anderson) random Schrödinger operator:

$$H_\omega = -\Delta + V_\omega \text{ where } V_\omega(x) = \sum_{\gamma\in\mathbb{Z}^d} \omega_\gamma V(x-\gamma) \tag{1.1}$$

on $\mathbb{R}^d$, $d \geq 1$, where V is the site potential, and $(\omega_\gamma)_{\gamma\in\mathbb{Z}^d}$ are the random coupling constants. We assume

(H1) (1) $V:\ \mathbb{R}^d \to \mathbb{R}$ is continuous, non identically vanishing and supported in $(-1/2, 1/2)^d$;

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(2) $(\omega_\gamma)_\gamma$ are independent identically distributed (i.i.d.) random variables distributed in $[a, b]$ $(a < b)$ with essential infimum a and essential supremum b.

Let Σ be the almost sure spectrum of H_ω and $E_- = \inf \Sigma$. When V has a fixed sign, it is well-known that the $E_- = \inf(\sigma(-\Delta + V_{\overline{b}}))$ if $V \leq 0$ and $E_- = \inf(\sigma(-\Delta + V_{\overline{a}}))$ if $V \geq 0$. Here, $\overline{x}$ is the constant vector $\overline{x} = (x)_{\gamma \in \mathbb{Z}^d}$. Moreover, in this case, it is well-known that the integrated density of states of the Hamiltonian (see (2.1)) admit a Lifshitz tail near E_-, i.e., that the integrated density of states at energy E decays exponentially fast as E goes to E_- from above. We refer to [9, 8, 16, 15, 7, 6, 11] for precise statements.

The case we address is is that of V assuming both signs, i.e., there may exist $x_+ \neq x_-$ such that

$$V(x_-) \cdot V(x_+) < 0.$$

The basic difficulty this property introduces is that the variations of the potential V_ω as a function of the random variable ω_γ is not monotonous. In the monotonous case, to get the minimum, one can simply minimize with respect to each of the random variables individually. In the non monotonous case, this uncoupling between the different random variables may fail. We obtain results for reflection symmetric potential since, as we will see, for these potentials we also have analogous decoupling between the different random variables. Thus we make the following symmetry assumption on V:

(H2) V is reflection symmetric i.e. for any $\sigma = (\sigma_1, \dots, \sigma_d) \in \{0, 1\}^d$ and any $x = (x_1, \dots, x_d) \in \mathbb{R}^d$,

$$V(x_1, \dots, x_d) = V((-1)^{\sigma_1} x_1, \dots, (-1)^{\sigma_d} x_d).$$

We now consider the operator $H^N_\lambda = -\Delta + \lambda V$ with Neumann boundary conditions on the cube $[-1/2, 1/2]^d$. Its spectrum is discrete, and we let $E_-(\lambda)$ be its ground state energy. It is a simple eigenvalue and $\lambda \mapsto E_-(\lambda)$ is a real analytic concave function defined on $\mathbb{R}$. We first observe:

Proposition 1.1 ([13]). *Under the above assumptions (H1) and (H2), $E_- = \inf(E_-(a), E_-(b))$.*

For a and b sufficiently small, this result was proved in [14] without the assumption (H2) but with an additional assumption on the sign of $\displaystyle\int_{\mathbb{R}^d} V(x)dx$. The method used by Najar relies on a small coupling constant expansion for the infimum of Σ. These ideas were first used in [4] to treat other non monotonous perturbations, in this case magnetic ones, of the Laplace operator. In [1], the authors study the minimum of the almost sure spectrum for a random displacement model i.e. the random potential is defined as $V_\omega(x) = \sum_{\gamma \in \mathbb{Z}^d} V(x - \gamma - \xi_\gamma)$ where $(\xi_\gamma)_\gamma$ are i.i.d. random variables supported in a sufficiently small compact.

2 Lifshitz tails

We now turn to the results on Lifshitz tails. We denote by $N(E)$ the integrated density of states of H_ω, i.e., it is defined by the limit

$$N(E) = \lim_{L\to+\infty} \frac{\#\{\text{eigenvalues of } H^N_{\omega,L} \ \le E\}}{(2L)^d} \tag{2.1}$$

where $H^N_{\omega,L}$ is the operator H_ω restricted to the cube $[-L-1/2, L+1/2]^d$ with Neumann boundary conditions. This quantity has been the source of a lot of studies and we refer to [15, 17] for extensive review.
The behavior of the density of states will heavily depend on whether $E_-(a) = E_-(b)$ or not.

2.1 When $E_-(a) \neq E_-(b)$

This is the generic case i.e. it holds for generic V satisfying (H1) and (H2) once a and b are fixed and for given V and a (or b) for all b (or a) except one value. It is also the simplest case and the one when the results are the most similar to those obtained in the monotonous case.
We prove

Theorem 2.1 ([13])**.** *Suppose Assumptions (H1) and (H2) and $E_-(a) \neq E_-(b)$. Then*

$$-\frac{d}{2} - \alpha_- \le \liminf_{E\to E_-^+} \frac{\log|\log N(E)|}{\log(E-E_-)} \le \limsup_{E\to E_-^+} \frac{\log|\log N(E)|}{\log(E-E_-)} \le -\frac{d}{2} - \alpha_+$$

where $c = a$ if $E_-(a) < E_-(b)$ and $c = b$ if $E_-(a) > E_-(b)$ and

$$\alpha_- = -\liminf_{\varepsilon\to0} \frac{\log|\log \mathbb{P}(\{|c-\omega_0| \le \varepsilon\})|}{\log\varepsilon} \ge 0,$$
$$\alpha_+ = -\frac{1}{2}\liminf_{\varepsilon\to0} \frac{\log|\log \mathbb{P}(\{|c-\omega_0| \le \varepsilon\})|}{\log\varepsilon} \ge 0.$$

In particular if the tails of the random variables $(\omega_\gamma)_\gamma$ are not exponential i.e. if $\alpha_- = \alpha_+ = 0$, we obtain standard Lifshitz tails.
Combining Theorem 2.1 with the Wegner estimates obtained in [10, 5] and the multiscale analysis as developed in [3], we learn

Theorem 2.2. *Assume (H1) and (H2) hold and that that the common distribution of the random variables admits an absolutely continuous density. Then, the bottom edge of the spectrum of H_ω exhibits complete localization in the sense of [3].*

2.2 When $E_-(a) = E_-(b)$

In this case, the distribution of the random variables plays a crucial role for the existence of Lifshitz tails. The condition $E_-(a) = E_-(b)$ can be interpreted as a resonance condition.

Theorem 2.3 ([12]). *Suppose Assumptions (H1) and (H2) and $E_- := E_-(a) = E_-(b)$. Then,*

(1) If the random variables $(\omega_\gamma)_\gamma$ are not Bernoulli distributed i.e. if $\mathbb{P}(\omega_0 = a) + \mathbb{P}(\omega_0 = b) < 1$ then

$$-\frac{d}{2} - \alpha_- \leq \liminf_{E \to E_-^+} \frac{\log|\log N(E)|}{\log(E - E_-)} \leq \limsup_{E \to E_-^+} \frac{\log|\log N(E)|}{\log(E - E_-)} \leq -\frac{1}{2} - \alpha_+.$$

(2) If $\mathbb{P}(\omega_0 = a) + \mathbb{P}(\omega_0 = b) = 1$, there exists potentials V satisfying assumption (H1) and (H2) such that $E_-(a) = E_-(b)$ and, there exists $C > 0$ such that, for $E \geq E_-$,

$$\frac{1}{C}(E - E_-)^{d/2} \leq N(E) \leq C(E - E_-)^{d/2}. \tag{2.2}$$

So we see that, in some cases, when the random variables are Bernoulli distributed, the integrated density of states exhibits van Hove singularities at the bottom of the spectrum. On the other hand, when the distributions is not Bernoulli, the density of states always exhibits Lifshitz tails but the Lifshitz exponent may be 1/2 even in d dimensions (assume to simplify $\alpha_- = \alpha_+ = 0$). Actually the exponent 1/2 is the best one can get in general i.e. without further assumptions on V. One can indeed in some cases prove a lower bound of the same type as the upper bound given in Theorem 2.3.

Whether this happens or not depends on a another resonance condition which can be expressed as follows. Assume $E_-(a) = E_-(b)$. Let e_j be a vector of the canonical basis of $\mathbb{R}^d$. We say that H_a^N and H_b^N (see section 1) match in the direction e_j if $E_-(a) = E_-(b)$ is also the ground state energy of the operator $-\Delta + aV(\cdot) + bV(\cdot - e_j)$ on the parallelepiped $[-1/2, 1/2]^d \cup (e_j + [-1/2, 1/2]^d)$ with Neumann boundary conditions.

One then can show that, if H_a^N and H_b^N match in all directions then one is in case (2) of Theorem 2.3. If they don't match at least in one direction, one is in case (1). If they don't match in exactly one direction, then one can show that the Lifshitz exponent is 1/2 (under the assumption $\alpha_- = \alpha_+ = 0$).

In view of these results (and their proofs), it is natural to conjecture that, for H_ω defined by (1.1), if $\mathbb{P}(\omega_0 = a) + \mathbb{P}(\omega_0 = b) \neq 1$, then

$$\limsup_{E \to E_-^+} \frac{\log|\log N(E)|}{\log(E - E_-)} < 0$$

even without the assumptions (H1) or (H2).

2.3 A random displacement model

These techniques also enable to treat a Bernoulli displacement model. Consider

$$H_\omega = -\Delta + V_\omega \text{ where } V_\omega(x) = \sum_{\gamma\in\mathbb{Z}^d} V(x-\gamma-\xi_\gamma). \tag{2.3}$$

where

(H1')
- $V : \mathbb{R}^d \to \mathbb{R}$ is continuous, non identically vanishing and supported in $(-r,r)^d$, $0<r<1/2$ and satisfies (H2);
- $(\xi_\gamma)_\gamma$ are independent identically distributed (i.i.d.) random variables distributed in $\{-1/2+r, 1/2-r\}^d$ such that all these points have a positive probability.

By [1], the configurations that minimize the ground state energy are given by a symmetric "clusterization". For example, if one restricts H_ω to a cube $[-L-1/2, L+1/2]^d$, a minimizer is then given by $(\xi_\gamma)_j = (-1)^{\gamma_j}(1/2-r)$ for $\gamma \in [-L-1/2, L+1/2]^d \cap \mathbb{Z}^d$ where $(\xi_\gamma)_j$ is the j-th component of the vector ξ_γ. Moreover, this minimizing configuration is unique up to the symmetries of the problem: if one consider the global minimizer $(\xi_\gamma)_{\gamma\in\mathbb{Z}^d}$ defined above, all other minimizers in the cube $[-L-1/2, L+1/2]^d$ are obtained by translations of the global minimizer restricted to the cube ([2]). On Fig 1, we represent a few cells of the lattice $\mathbb{Z}^2$ with the location of the support of the minimizing potential represented as disks. Let us call E_- the minimizing ground state energy.

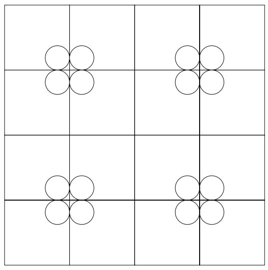

Figure 1: The minimizing configuration in dimension 2

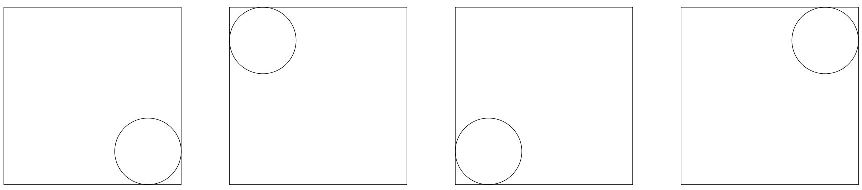

Figure 2: The 4 configurations in dimension 2

For $\xi \in \{-1/2+r, 1/2-r\}^d$, define the operator

$$H_\xi = -\Delta + V(x-\xi)$$

acting on $[-1/2,1/2]^d$ with Neumann boundary conditions. On Fig 2, we show the 4 configuration for the supports of V obtained in dimension 2.

Due to the symmetry assumption on V, all the H_ξ have the same ground state energy.

Theorem 2.4. *Consider the model H_ω defined by* (2.3) *and assume (H1') is satisfied. Let $N(E)$ denote the integrated density of states of H_ω and E_- be the infimum of the almost sure spectrum of H_ω. Then,*

(1) if, at least, two of the $(H_\xi)_{\xi\in\{-1/2+r,1/2-r\}^d}$ do not match in, at least, one direction (for a definition of matching, see the comments following Theorem 2.3). Then, one has

$$\limsup_{E\to E_-^+}\frac{\log|\log N(E)|}{\log(E-E_-)}\leq-\frac{1}{2}. \tag{2.4}$$

(2) if all the $(H_\xi)_{\xi\in\{-1/2+r,1/2-r\}^d}$ match in all directions, then

$$N(E)\geq\frac{1}{C}(E-E_-)^{d/2}. \tag{2.5}$$

Note that (2.5) may not always be optimal; in some cases, $N(E)$ may be much larger ([2]).

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