Centre de Mathématiques Laurent Schwartz
ECOLE POLYTECHNIQUE

SEMINAIRE

Equations aux Dérivées Partielles

2007-2008

Yvan Martel and Frank Merle

On the collision of two solitons for the generalized KdV equation in the nonintegrable case

Séminaire É. D. P. (2007-2008), Exposé n° XII, 10 p.

<http://sedp.cedram.org/item?id=SEDP_2007-2008____A12_0>

U.M.R. 7640 du C.N.R.S.
F-91128 PALAISEAU CEDEX
Fax : 33 (0)1 69 33 49 49
Tél : 33 (0)1 69 33 49 99

Séminaire *Équations aux dérivées partielles*
Centre de mathématiques Laurent Schwartz, 2007-2008
Exposé n° XII, 1-10

On the collision of two solitons for the generalized KdV equation in the nonintegrable case

Yvan Martel(1) and Frank Merle(2)

(1) Université de Versailles Saint-Quentin-en-Yvelines, Mathématiques,
45, av. des Etats-Unis, 78035 Versailles cedex, France
martel@math.uvsq.fr
(2) Université de Cergy-Pontoise, IHES and CNRS, Mathématiques
2, av. Adolphe Chauvin, 95302 Cergy-Pontoise cedex, France
Frank.Merle@math.u-cergy.fr

In this talk, we are concerned with the generalized Korteweg-de Vries equations (gKdV)

$$\partial_t u + \partial_x(\partial_x^2 u + f(u)) = 0, \quad t, x \in \mathbb{R}. \tag{1}$$

We call **soliton** a solution of (1) of the form $u(t,x) = Q_c(x - ct)$, for $c > 0$.

We study the following general questions about the collision of two solitons: let $u(t)$ be a solution of (1) such that

$$u(t,x) - [Q_{c_1}(x - c_1 t) + Q_{c_2}(x - c_2 t)] \to 0 \text{ in } H^1(\mathbb{R}) \text{ as } t \to -\infty, \tag{2}$$

where $Q_{c_1}(x - c_1 t)$, $Q_{c_2}(x - c_2 t)$ are two solitons ($0 < c_2 < c_1$). Formally, the two solitons have to collide.

- What is the behavior of $u(t)$ during and after the collision ?
- Do the two solitons survive the collision at the main orders ? In other words, for large positive time, do we recover a two soliton structure

$$u(t,x) - \Big[Q_{c_1^+}(x - c_1^+ t - \delta_1) + Q_{c_2^+}(x - c_2^+ t - \delta_2) + \eta(t,x)\Big] \to 0 \text{ in } H^1(\mathbb{R}) \text{ as } t \to +\infty, \tag{3}$$

  where $0 < c_2^+ < c_1^+$, δ_1, $\delta_2 \in \mathbb{R}$, and $\eta(t)$ is a residue, small compared to Q_{c_1}, Q_{c_2} ?
- If the two soliton structure is preserved, are the velocities modified (i.e. $c_j^+ \neq c_j$), and the trajectories changed (nonzero shift $\delta_j \neq 0$)?
- Is the collision elastic or inelastic ? We say that the collision is elastic if the residue is zero, i.e. $\eta(t) = 0$ after the collision.

We recall some previous works concerning the collision of solitons for (1):

- Integrable case ($f(u) = u^2$ or u^3). In this case, there exist explicit multi-soliton solutions describing the interaction of solitons. For these solutions, we have (2), (3) with

$c_j = c_j^+$, $\eta = 0$ and the shifts δ_j are explicit, nonzero. In particular, **the collision is elastic in the integrable case**.

See Fermi, Pasta and Ulam [5], Zabusky and Kruskal [24], Lax [8], Hirota [6], the survey by Miura [19]. There are many other references concerning the integrable case, see for example references in [19].

- Numerical predictions for nonintegrable models and experiments. In the case of nonintegrable models, from numerical studies, **the collision seems inelastic but almost elastic** (small but nonzero dispersive residue $\eta(t)$).

  Recall that a typical example of nonintegrable model similar to KdV is the BBM equation. See for example the following references on the BBM equation: Eilbeck and McGuire [4], Bona et al. [2], [7]. See Shih [20] for some nonintegrable gKdV equations.

  We also refer to Craig et al. [3] and references therein for a comparison between numerics for Euler with free surface, explicit multi-solitons of KdV and experiments on water tanks.

In this talk, I report on recent works in collaboration with Frank Merle ([13, 14, 15, 16]) describing the collision of two solitons Q_{c_1}, Q_{c_2} for nonintegrable gKdV equations (1), in the case of two solitons of different scale, i.e. under the following assumption:

$$0 < c_2 \ll c_1.$$

Note that under standard assumptions, Q_{c_1} is globally stable in H^1 (see below) and thus it will survive the collision, up to a perturbation of order $\|Q_{c_2}\|_{H^1}$. However, in the nonintegrable situation, i.e. without exceptional algebraic structure, it is not clear whether the small soliton Q_{c_2} survives the collision. We introduce a new framework to understand this kind of questions.

1 General setting and previous stability results

First, we present the assumptions on f used throughout these notes.

Assumption on f. For $p = 2, 3, 4$

$$f(u) = u^p + f_1(u), \quad \lim_{u\to 0} \left|\frac{f_1(u)}{u^p}\right| = 0.$$

This assumption means that the nonlinearity f is **subcritical** in a neighborhood of 0. Recall that criticality means $f(u) = u^5$, for which all solitons have the same L^2 norm.

Second, we recall known results on existence and stability of solitons. From Berestycki and Lions [1] and Weinstein [23], there exists $c_* > 0$ such that for all $c \in (0, c_*)$, there exists a (even) solution $Q_c \in H^1$ of

$$Q_c'' + f(Q_c) = c\,Q_c$$

such that $R_c(t,x) = Q_c(x - ct)$, which is a soliton of (1), is **stable and asymptotically stable** in the energy space H^1 in the following sense.

Orbital stability (Weinstein [23]). *Let $0 < c < c_*$. Let $u(t)$ be a solution of* (1).

$$\|u(0) - Q_c\|_{H^1} = \alpha_0 \textit{ small} \;\Rightarrow\; \sup_t \|u(t) - Q_c(. - \rho(t))\|_{H^1} \le K\alpha_0,$$

for some function $\rho(t)$ such that $|\rho'(t) - c| \leq K\alpha_0$.

Recall that the proof of this result is based on the two conservation laws for (1)

$$\int u^2(t) = \int u^2(0), \quad E(u(t)) = \frac{1}{2}\int u_x^2(t) - \int F(u(t)) = E(u(0)),$$

where $F(u) = \int_0^u f(s)ds$. Note that for $f(u) = u^p$, with $p = 2$, 3 or 4, we have $c_* = +\infty$.

Asymptotic stability (Martel and Merle [11, 12, 10, 13]). *Under the same assumptions, there exists $c^+ > 0$ ($|c^+ - c| \leq K\alpha_0$) such that*

$$u(t) - Q_{c^+}(. - \rho(t)) \to 0 \textit{ in } H^1(x > \tfrac{c}{10}t) \textit{ and } \rho'(t) \to c^+ \text{ as } t \to +\infty.$$

This result means that the solution converges in the energy space to a limiting soliton Q_{c^+} of speed c^+ close to c, locally around the center of mass $\rho(t)$ of the soliton, and on the right of it (see the original papers for more comments on this result).

In the rest of this paper, we consider $0 < c_2 < c_1 < c_*$.

Existence of asymptotic multi-solitons (Martel [9]). *There exists a unique H^1 solution $U(t)$ of (1) such that*

$$\lim_{t\to-\infty} \left\|U(t) - [Q_{c_1}(. - c_1t) + Q_{c_2}(. - c_2t)]\right\|_{H^1} \to 0.$$

From [9], the behavior of $U(t)$ as $t \to +\infty$ is not known, except that if the soliton Q_{c_2} is small, then Q_{c_1} is stable, up to a perturbation of order $\|Q_{c_2}\|_{H^1} \sim Kc_2^{\frac{1}{p-1}-\frac{1}{4}}$.

Stability of multi-solitons in H^1 (Martel, Merle and Tsai [17]). *Let $u(t)$ be a solution of (1). For $\alpha_0 > 0$ small, $T > 0$ large:*

$$\begin{aligned}&\left\|u(T) - [Q_{c_1}(. - c_1T) + Q_{c_2}(. - c_2T)]\right\|_{H^1} \leq \alpha_0 \quad \Rightarrow \\ &\sup_{t\geq T}\left\|u(t) - [Q_{c_1}(. - \rho_1(t)) + Q_{c_2}(. - \rho_2(t))]\right\|_{H^1} \leq K(\alpha_0 + e^{-\gamma T}).\end{aligned}$$

Moreover, there exists c_1 close to c_1, c_2^+ close to c_2 such that

$$u(t) - \left[Q_{c_1^+}(. - \rho_1(t)) + Q_{c_2^+}(. - \rho_2(t))\right] \to 0 \text{ in } H^1(x > \tfrac{c_2}{10}t) \text{ and } \rho_j(t) \to c_j^+ \text{ as } t \to +\infty.$$

This result means that the two soliton structure is asymptotically stable when the two solitons are sufficiently decoupled ($T \gg 1$). Note that this result implies the global in time stability of the multi-soliton solutions in the integrable case (see Corollary 1 in [17]). See references in [17] for previous results on this question.

We also refer to a recent article of Tao [22] where these results are reviewed in a larger perspective.

2 Stability of two soliton collision (general nonlinearity)

Our first main result concerns the case of a general nonlinearity.

Theorem 1 ([16]). *Assume $0 < c_2 < c_0(c_1) \ll c_1 < c_*$, where $c_0(c_1)$ is small enough. Let $U(t)$ be the solution of (1) such that*

$$\lim_{t\to-\infty} \left\| U(t) - [Q_{c_1}(.-c_1 t) + Q_{c_2}(.-c_2 t)] \right\|_{H^1(\mathbb{R})} \to 0.$$

There exist $c_1^+ \underset{c_2\sim 0}{\sim} c_1$, $c_2^+ \underset{c_2\sim 0}{\sim} c_2$, such that

$$c_1^+ \geq c_1, \quad c_2^+ \leq c_2,$$

$$w^+(t,x) = U(t,x) - \left[Q_{c_1^+}(x-\rho_1(t)) + Q_{c_2^+}(x-\rho_2(t)) \right],$$

satisfies

$$\lim_{t\to+\infty} \|w^+(t)\|_{H^1(x\geq \frac{c_2}{10}t)} = 0, \quad \sup_{t\in\mathbb{R}} \|w^+(t)\|_{H^1} \leq K c_2^{\frac{1}{p-1}}.$$

Comments on Theorem 1.

1. The two solitons are preserved through the collision. Indeed, the size of the perturbative term $w^+(t)$ satisfies

$$\sup_t \|w^+(t)\|_{H^1} \leq K c_2^{\frac{1}{p-1}} \quad \text{whereas} \quad \|Q_{c_2}\|_{H^1} \sim K c_2^{\frac{1}{p-1}-\frac{1}{4}}.$$

2. Speed change is related to a dispersive residue

$$\|w^+(t)\|_{H^1} \not\to 0 \text{ as } t\to+\infty \quad \textbf{if and only if} \quad c_1^+ > c_1 \text{ and } c_2^+ < c_2.$$

3. The behavior of $U(t)$ is globally stable in time in H^1 (see Theorem 3 in [16]). Presice upper bounds on $c_1^+ - c_1$ and $c_2 - c_2^+$ are available, see Theorem 1 in [16]. Nevertheless, under general assumptions on $f(u)$, we do not know whether the residue is zero (elastic or inelastic collision).

3 Detailed description for the quartic KdV equation

In the case of the quartic gKdV equation

$$\partial_t u + \partial_x(\partial_x^2 u + u^4) = 0 \quad t, x \in \mathbb{R}, \tag{4}$$

which is nonintegrable, we refine the information given by Theorem 1. In particular, we prove the residue is not zero, thus the collision is close to elastic but inelastic.

By the scaling invariance of the equation

$$\text{if } u(t,x) \text{ is solution then } \forall \lambda > 0,\ \lambda^{\frac{2}{3}} u(\lambda^3 t, \lambda x) \text{ is solution},$$

we are reduced to the case

$$c_1 = 1,\ Q = Q_1, \qquad 0 < c_2 = c \ll 1.$$

Theorem 2 ([15]). *Under the assumptions of Theorem 1, for equation* (4)*,*

$$c_1^+ - 1 \geq K\, c^{\frac{17}{6}}, \quad 1 - \frac{c_2^+}{c} \geq K\, c^{\frac{8}{3}},$$

$$0 < \mathbf{K}\, \mathbf{c}^{\frac{17}{12}} \leq \|w_x^+(t)\|_{L^2} + c^{\frac{1}{2}} \|w^+(t)\|_{L^2} \leq K' c^{\frac{11}{12}}, \quad \text{for } t \text{ large}$$

Comments on Theorem 2.

1. Theorem 2 proves the nonexistence of a pure 2-soliton solution in this regime, since $\liminf_{t\to+\infty} \|w^+(t)\|_{H^1} > 0$.

2. Theorem 2 is the first rigorous result describing an inelastic collision between two nonlinear objects for the gKdV equations or similar models.

3. The collision is almost elastic. Indeed,

$$\|w^+(t)\|_{L^2} \leq K \|Q_c\|_{L^2}^7.$$

Finally, we state the existence of special symmetric solutions of (4) which give a precise description of the collision of two solitons.

Theorem 3 ([15]). *Assume* $0 < c \ll 1$*. There exists a solution* $\varphi(t)$ *of* (4) *such that*

$$\varphi(-t, -x) = \varphi(t, x),$$

$$w^-(t, x) = \varphi(t, x) - \left[Q(x - t + \tfrac{1}{2}\Delta) + Q_c(x - ct + \tfrac{1}{2}\Delta_c)\right],$$

$$w^+(t, x) = \varphi(t, x) - \left[Q(x - t - \tfrac{1}{2}\Delta) + Q_c(x - ct - \tfrac{1}{2}\Delta_c)\right],$$

$$\lim_{-\infty} \|w^-(t)\|_{H^1(x \leq \frac{1}{10}ct)} = 0, \quad \lim_{+\infty} \|w^+(t)\|_{H^1(x \geq \frac{1}{10}ct)} = 0,$$

$$K\, c^{\frac{17}{12}} \leq \|w_x^\pm(t)\|_{L^2} + c^{\frac{1}{2}} \|w^\pm(t)\|_{L^2} \leq K' c^{\frac{17}{12}}, \quad \text{for } \pm t \text{ large},$$

$$\Delta \sim -\frac{K_1}{c^{1/6}} < 0, \quad \Delta_c \sim -K_2 < 0.$$

Comments on Theorem 3.

1. The solution $\varphi(t)$ is a generalization of the notion of multi-solitons in the nonintegrable situation. We can obtain $\varphi(t)$ at any order of c.

2. We have constructed a solution with symmetry. The velocities at $t = \pm\infty$ are thus identical. The quantities Δ, Δ_c represent the shift on the trajectories of each soliton. Note that the shift Δ on Q becomes negative infinite as $c \to 0$. The shift Δ_c on Q_c is negative and of size 1.

3. Such a solution $\varphi(t)$ is not unique but the lower bound on the defect is universal.

4. From critical Cauchy theory for (4) due to Tao [21], we conjecture that $w^+(t)$ disperses at $t \to \pm\infty$.

5. Theorem 3 extends to general nonlinearities $f(u)$, except the lower bound on $w^+(t)$. In general $\Delta \sim K_1 \int Q_c$, for some constant K_1.

4 Sketch of the proofs (quartic case)

The proofs are based both on algebraic computations (during the interaction) and on asymptotic analysis. These different arguments are structured as follows. Define $T_c = c^{-\frac{1}{2}-\frac{1}{100}}$.

1. Asymptotic arguments for $|t| > T_c$. For $|t| > T_c$, we expect the solitons to be decoupled. We apply refinements of previous stability and asymptotic stability arguments ([14]). The arguments are based on monotonicity properties of localized L^2 quantities and on Virial type identities.

2. Construction of an explicit approximate solution. This is the main step of the proof where we perform algebraic computations relevant in the collision region $|t|<T_c$.

3. Justification of the algebra on $[-T_c, T_c]$. We estimate the difference between approximate and exact solutions by using long time stability arguments (note that $T_c \to +\infty$ as $c \to 0$). For this, we introduce a modified Hamiltonian structure (refinement of [23]).

In the rest of these notes, we concentrate first on the construction of an approximate solution (step 2 above) and then we sketch a proof of nonexistence (weak version of Theorem 2).

4.1 Approximate solution at all order for $|t| < T_c$

Let

$$y_c = x + (1-c)t, \quad y = x - \alpha(y_c),$$

$$v(t,x) = Q(y) + Q_c(y_c) + W(t,x),$$

$$\alpha'(s) = \sum_{\substack{1\leq k\leq k_0\\ 0\leq \ell\leq \ell_0}} a_{k,\ell}\, c^\ell Q_c^k(s),$$

$$W(t,x) = \sum_{\substack{1\leq k\leq k_0\\ 0\leq \ell\leq \ell_0}} c^\ell \left(Q_c^k(y_c) A_{k,\ell}(y) + (Q_c^k)'(y_c) B_{k,\ell}(y)\right),$$

where $(a_{k,\ell}, A_{k,\ell}, B_{k,\ell})$ are to be determined so that

$$\|\partial_t v + \partial_x(\partial_x^2 v - v - v^4)\|_{L^2} \leq K c^{N(k_0,\ell_0)},$$

and $N(k_0,\ell_0) \to +\infty$ as $k_0, \ell_0 \to +\infty$. Note that the introduction of parameters $(a_{k,\ell})$ is related to the shift of Q.

For each (k,ℓ), we obtain the following typical system

$$(\Omega_{k,\ell}) \quad \begin{cases} (\mathcal{L}A_{k,\ell})' + a_{k,\ell}(3Q - 2Q^4)' = F_{k,\ell} \\ (\mathcal{L}B_{k,\ell})' + a_{k,\ell}(3Q'') - 3A''_{k,\ell} - 4Q^3 A_{k,\ell} = G_{k,\ell} \end{cases}$$

where $F_{k,\ell}$ and $G_{k,\ell}$ are given in terms of $(a_{k',\ell'}, A_{k',\ell'}, B_{k',\ell'})$, for $k' \leq k$, $\ell' \leq \ell$, with either $k' < k$ or $\ell' < \ell$ and where $\mathcal{L}A = -A'' + A - 4Q^3 A$ is the linearized operator around Q.

The system $(\Omega_{k,\ell})$ can be solved when $F_{k,\ell}$ and $G_{k,\ell}$ have certain parity properties (there is no uniqueness, two free parameters appear). Note that the parameter $a_{k,\ell}$ is necessary in solving the system.

For all k, ℓ, we obtain functions $A_{k,\ell}$, $B_{k,\ell}$ with the following structure: localized functions (L^2) plus a polynomial function (with an explicit control on the degrees of the polynomial).

Then, the next step is the recomposition of the approximate solution at $t = \pm T_c$ and the identification of a nonzero defect.

Formally, we find the defect at the rank $k = 2$, $\ell = 0$. Indeed, if we concentrate on the main terms ($k = 1, 2$ and $\ell = 0$), we obtain the following properties

$$A_{1,0},\ A_{2,0} \in L^2,$$

$$B_{1,0}(x) = -b_{1,0}\frac{Q'(x)}{Q(x)} + \tilde{B}_{1,0}(x), \quad \tilde{B}_{1,0} \in L^2, \quad b_{1,0} \neq 0,$$

$$B_{2,0}(x) = -b_{2,0}\frac{Q'(x)}{Q(x)} + \tilde{B}_{2,0}(x), \quad \tilde{B}_{2,0} \in L^2, \quad b_{2,0} \neq 0.$$

Observe that $\frac{Q'}{Q}$ is a bounded function such that $\lim_{\pm+\infty} -\frac{Q'}{Q} = \pm 1$. Moreover, $A_{1,0}$, $A_{2,0}$, $\tilde{B}_{1,0}$ and $\tilde{B}_{2,0}$ have exponential decay properties.

For $t = +T_c$, we have $y_c << y$ which means that the two solitons are decoupled so that the two terms $(Q_c^k)'(y_c)\tilde{B}_{k,0}(y)$ for $k = 1, 2$ in the definition of v are negligeable. Thus,

$$\begin{aligned} v(T_c, x) &\sim Q(y) + Q_c(y_c) - b_{1,0}Q_c'(y_c) - b_{2,0}(Q_c^2)'(y_c) + \dots \\ &\sim Q(y) + Q_c(y_c - b_{1,0}) - b_{2,0}(Q_c^2)'(y_c - b_{1,0}) + \dots \end{aligned}$$

The term $-b_{1,0}Q_c'(y_c)$ is interpreted as a translation of the soliton Q_c. In contrast, the term $-b_{2,0}(Q_c^2)'$ **is a defect of size** $\|(Q_c^2)'\|_{L^2} = Kc^{\frac{11}{12}}$, in the sense that it cannot be combined with Q_c to form a pure soliton.

Thus, we cannot recompose $v(T_c, x)$ as the sum of two solitons at this order, with a defect of order $c^{\frac{11}{12}}$.

4.2 Nonexistence of a pure 2-soliton (quartic case)

We combine the approximate solution constructed above with some analysis arguments.

- From the algebra, there exists a nonsymmetric approximate solution $\tilde{v}$ of (4) such that

$$\left\|\tilde{v}(-T_c) - Q(.+\tfrac{1}{2}\Delta) - Q_c(.-(1-c)T_c+\tfrac{1}{2}\Delta_c)\right\|_{H^1} \leq Kc,$$

$$\left\|\tilde{v}(T_c) - Q(.-\tfrac{1}{2}\Delta) - Q_c(.+(1-c)T_c-\tfrac{1}{2}\Delta_c) + \mathbf{2b_{2,0}(Q_c^2)'}\right\|_{H^1} \leq Kc.$$

This approximate solution is constructed with the structure described in section 4.1. Note that at the main orders, Δ and Δ_c are explicit (see Theorem 3).

- By contradiction, assume that there exists a pure 2-soliton $U(t)$:

$$\left\|U(t) - Q(.-t-x_{1,\pm}) - Q_c(.-ct-x_{2,\pm})\right\|_{H^1} \to 0 \quad \text{as } t \to \pm\infty.$$

By stability, after time and space translations, there exist $T_+ > T_c$, δ_+, such that

$$\|U(-T_c, .-T_c) - Q(.+\tfrac{1}{2}\Delta) - Q_c(.-(1-c)T_c+\tfrac{1}{2}\Delta_c)\|_{H^1} \leq Kc,$$

$$\|U(T_+, .-\delta_+) - Q(.-\tfrac{1}{2}\Delta) - Q_c(.+(1-c)T_c-\tfrac{1}{2}\Delta_c)\|_{H^1} \leq Kc.$$

A priori there is no relation between T_+ and T_c.

By stability analysis on $[-T_c, T_c]$ applied to $\tilde{v}$ and $U(t)$, there exists δ such that

$$\left\|U(T_c) - \tilde{v}(T_c, . - \delta)\right\|_{H^1} \leq Kc.$$

• We have $T_+ \sim T_c$ by using the large time stability of the two soliton structure. Indeed, $\|(Q_c^2)'\|_{H^1} \sim Kc^{\frac{11}{12}}$ and

$$\left\|U(T_c) - Q(. - \rho_1) - Q_c(. - \rho_2)\right\|_{H^1} \leq Kc^{\frac{11}{12}}, \quad \rho_1 - \rho_2 \sim T_c,$$

$$\Rightarrow \quad \forall t > T_c, \ \left\|U(t) - Q(. - \rho_1(t)) - Q_c(. - \rho_2(t))\right\|_{H^1} \leq Kc^{\frac{5}{12}},$$

with $\rho_1(t) - \rho_2(t) \sim (1-c)t$.

• A contradiction then follows from

$$\|U(T_c) - Q(. - \tilde{\rho}_1) - Q_c(. - \tilde{\rho}_2)\|_{H^1} \leq Kc,$$

$$\left\|U(T_c) - Q(. - \rho_1) - Q_c(. - \rho_2) + \mathbf{2b_{2,0}(Q_c^2)'(. - \rho_2)}\right\|_{H^1} \leq Kc$$

since $\|(Q_c^2)'\|_{H^1} \sim Kc^{\frac{11}{12}}$.

We refer to the original paper [15] for a qualitative proof, with precise lower and upper bound estimates on the residue as $t \to +\infty$.

5 Case of the BBM equation

For the BBM equation, we study the collision of a soliton of speed $c_0 > 1$ with a small soliton of speed $c > 1$ close to 1. This is a joint work with T. Mizumachi [18].

After renormalization, it is equivalent to study the collision of Q by a small soliton $R_\sigma \sim Q_\sigma$, $\sigma = c - 1 > 0$ small, for the equation

$$(1 - \lambda\partial_x^2)\partial_t u + \partial_x(\partial_x^2 u - u + u^2) = 0, \quad \lambda = \tfrac{c_0 - 1}{c_0} \in (0, 1).$$

A similar analysis proves the existence of a small but nonzero residue (the collision is inelastic but almost elastic), confirming various numerical predictions on the collision of solitons of the BBM equation.

References

[1] H. Berestycki and P.-L. Lions, Nonlinear scalar field equations. I. Existence of a ground state, Arch. Rational Mech. Anal. **82**, (1983) 313–345.

[2] J.L. Bona, W.G. Pritchard and L.R. Scott, Solitary-wave interaction, Phys. Fluids 23, **438**, (1980).

[3] W. Craig, P. Guyenne, J. Hammack, D. Henderson and C. Sulem, Solitary water wave interactions. Phys. Fluids **18**, (2006), 057106.

[4] J. C. Eilbeck and G. R. McGuire, Numerical study of the regularized long-wave equation. II. Interaction of solitary waves, J. Computational Phys. **23** (1977), 63–73.

[5] E. Fermi, J. Pasta and S. Ulam, Studies of nonlinear problems, I, Los Alamos Report LA1940 (1955); reproduced in Nonlinear Wave Motion, A.C. Newell, ed., American Mathematical Society, Providence, R. I., 1974, pp. 143–156.

[6] R. Hirota, Exact solution of the Korteweg-de Vries equation for multiple collisions of solitons, Phys. Rev. Lett., **27** (1971), 1192–1194.

[7] H. Kalisch and J.L. Bona, Models for internal waves in deep water, Discrete and Continuous Dynamical Systems, **6** (2000), 1–20.

[8] P. D. Lax, Integrals of nonlinear equations of evolution and solitary waves, Comm. Pure Appl. Math. **21**, (1968) 467–490.

[9] Y. Martel, Asymptotic N–soliton–like solutions of the subcritical and critical generalized Korteweg–de Vries equations, Amer. J. Math. **127** (2005), 1103–1140.

[10] Y. Martel, Linear problems related to asymptotic stability of solitons of the generalized KdV equations, SIAM J. Math. Anal. **38** (2006), 759–781.

[11] Y. Martel and F. Merle, Asymptotic stability of solitons for subcritical generalized KdV equations, Arch. Ration. Mech. Anal. **157** (2001) 219–254.

[12] Y. Martel and F. Merle, Asymptotic stability of solitons of the subcritical gKdV equations revisited. Nonlinearity **18** (2005), no. 1, 55–80.

[13] Y. Martel and F. Merle, Asymptotic stability of solitons of the gKdV equations with general nonlinearity. To appear in Math. Annalen. arXiv:0706.1174v2

[14] Y. Martel and F. Merle, Refined asymptotics around solitons for gKdV equations. To appear in Discrete and Continuous Dynamical Systems. Series A. arXiv:0706.1178v2

[15] Y. Martel and F. Merle, Description of two soliton collision for the quartic gKdV equations. arXiv:0709.2672v1

[16] Y. Martel and F. Merle, Stability of two soliton collision for nonintegrable gKdV equations. arXiv:0709.2677v1

[17] Y. Martel, F. Merle and Tai-Peng Tsai, Stability and asymptotic stability in the energy space of the sum of N solitons for the subcritical gKdV equations, Commun. Math. Phys. **231**, (2002) 347–373.

[18] Y. Martel, F. Merle and T. Mizumachi, preprint.

[19] R.M. Miura, The Korteweg–de Vries equation: a survey of results, SIAM Review **18**, (1976) 412–459.

[20] L.Y. Shih, Soliton–like interaction governed by the generalized Korteweg-de Vries equation, Wave motion **2** (1980), 197–206.

[21] T. Tao, Scattering for the quartic generalised Korteweg-de Vries equation, J. Diff. Eq. 232 (2007), 623651.

[22] T. Tao, Why solitons are stable ? arXiv:0802.2408v2

[23] M.I. Weinstein, Lyapunov stability of ground states of nonlinear dispersive evolution equations, Comm. Pure Appl. Math. **39**, (1986) 51–68.

[24] N.J. Zabusky and M.D. Kruskal, Interaction of "solitons" in a collisionless plasma and recurrence of initial states, Phys. Rev. Lett. **15** (1965), 240–243.