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ON THE ENERGY CRITICAL FOCUSING NON-LINEAR WAVE EQUATION

CARLOS E. KENIG AND FRANK MERLE

1. Introduction

In this note we consider the energy critical non-linear wave equation

$$\begin{cases} \partial_t^2 u - \Delta u = \pm |u|^{\frac{4}{N-2}} u & (x,t) \in \mathbb{R}^N \times \mathbb{R} \\ u\big|_{t=0} = u_0 \in \dot{H}^1(\mathbb{R}^N) \\ \partial_t u\big|_{t=0} = u_1 \in L^2(\mathbb{R}^N) \end{cases}$$

Here the $-$ sign corresponds to the defocusing problem, while the $+$ sign corresponds to the focusing problem. The theory of the local Cauchy problem (CP) for this equation was developed in many papers, see for instance [27], [9], [22], [30], [31], [32], [15] etc.

In particular, one can show that if $\|(u_0,u_1)\|_{\dot{H}^1\times L^2} \le \delta$, δ small, there exists a unique solution with $(u,\partial_t u) \in C\big(\mathbb{R}; \dot{H}^1(\mathbb{R}^N) \times L^2(\mathbb{R}^N)\big)$ with the norm

$$\|u\|_{L_{xt}^{\frac{2(N+1)}{N-2}}} < \infty$$

$\big($i.e., the solution scatters in $\dot{H}^1(\mathbb{R}^N) \times L^2(\mathbb{R}^N)\big)$.

In the defocusing case, Struwe [35] in the radial case, when $N=3$, Grillakis [11] in the general case when $N=3$, and then Grillakis [12], Shatah-Struwe [29], [30], [31] (and others [15]) in higher dimensions, proved that this also holds for any (u_0,u_1) with $\|(u_0,u_1)\|_{\dot{H}^1\times L^2} < \infty$ and that, (for $3 \le N \le 5$) for more regular (u_0,u_1) the solution preserves the smoothness for all time. This topic has been the subject of intense investigation. See the recent work of Tao [37] for a recent installment in it and further references.

In the focusing case, these results do not hold. In fact, the classical identity

$$\frac{d^2}{dt^2}\int |u(x,t)|^2 = 2\left[\int (\partial_t u)^2 + |\nabla u|^2 - |u(t)|^{\frac{2N}{N-2}}\right] \tag{1.1}$$

was used by Levine [21] to show that if $(u_0,u_1)\in H^1\times L^2$ is such that

$$E\big((u_0,u_1)\big)=\int \frac{1}{2}\left|\nabla u_0\right|^2+\frac{1}{2}\left|u_1\right|^2-\frac{(N-2)}{2N}\left|u_0\right|^{\frac{2N}{N-2}}<0,$$

the solution must break down in finite time. Moreover,

$$W(x)=W(x,t)=\frac{1}{\left(1+\frac{|x|^2}{N(N-2)}\right)^{\frac{(N-2)}{2}}}$$

is in $\dot{H}^1(\mathbb{R}^N)$ and solves the elliptic equation

$$\Delta W+|W|^{\frac{4}{N-2}}W=0,$$

so that scattering cannot always occur even for global (in time) solutions.

In the paper ([16]), we show the following sharp small energy data result:

Theorem 1.1. *Let $(u_0,u_1)\in\dot{H}^1\times L^2$, $3\le N\le 5$. Assume that $E\big((u_0,u_1)\big)<E\big((W,0)\big)$. Let u be the corresponding solution of the Cauchy problem, with maximal interval of existence $I=\big(-T_-(u_0,u_1),T_+(u_0,u_1)\big)$. Then:*

i) If $\int\left|\nabla u_0\right|^2<\int\left|\nabla W\right|^2$, then

$$I=(-\infty,+\infty)\ \text{and}\ \|u\|_{L_{xt}^{\frac{2(N+1)}{N-2}}}<\infty.$$

ii) If $\int\left|\nabla u_0\right|^2>\int\left|\nabla W\right|^2$, then

$$T_+(u_0,u_1)<+\infty,\ T_-(u_0,u_1)<+\infty.$$

Our proof follows the new point of view into these problems that we introduced in [17], where we obtained the corresponding result for the energy critical non-linear Schrödinger equation for radial data. We will consider only here the proof of part *i*), part *ii*) follows from more standart arguments.

2. Linear estimates and the Cauchy problem

In this section we will review the theory of the Cauchy problem

$$\text{(CP)}\qquad \begin{cases}\partial_t^2u-\Delta u=|u|^{\frac{4}{N-2}}u & (x,t)\in\mathbb{R}^N\times\mathbb{R}\\ u\big|_{t=0}=u_0\in\dot{H}^1(\mathbb{R}^N)\\ \partial_t u\big|_{t=0}=u_1\in L^2(\mathbb{R}^N)\end{cases}$$

i.e. the $\dot{H}^1$ critical, focusing Cauchy problem for NLW, and some of the associated linear theory. Consider thus

$$\text{(LCP)} \qquad \begin{cases} \partial_t^2 w - \Delta w = h & (x,t) \in \mathbb{R}^N \times \mathbb{R} \\ w\big|_{t=0} = w_0 \in \dot{H}^1(\mathbb{R}^N) \\ \partial_t w\big|_{t=0} = w_1 \in L^2(\mathbb{R}^N) \end{cases}$$

the associated linear problem. The solution operator to (LCP) is given by:

$$\begin{aligned} w(x,t) &= \cos(t\sqrt{-\Delta})w_0 + (-\Delta)^{1/2}\sin(t\sqrt{-\Delta})w_1 \\ &+ \int_0^t \frac{\sin\left((t-s)\sqrt{-\Delta}\right)}{\sqrt{-\Delta}} h(s)ds \\ &= S(t)(w_0,w_1) + \int_0^t \frac{\sin\left((t-s)\sqrt{-\Delta}\right)}{\sqrt{-\Delta}} h(s)ds. \end{aligned}$$

Let us now define the $S(I)$, $W(I)$ norm for an interval I by

$$\|v\|_{S(I)} = \|v\|_{L_I^{\frac{2(N+1)}{N-2}} L_x^{\frac{2(N+1)}{N-2}}} \text{ and } \|v\|_{W(I)} = \|v\|_{L_I^{\frac{2(N+1)}{N-1}} L_x^{\frac{2(N+1)}{N-1}}}.$$

Theorem 2.1 (See [27], [9], [30])**.** *Assume $(u_0,u_1) \in \dot{H}^1 \times L^2$, $0 \in I$ an interval and $\|(u_0,u_1)\|_{\dot{H}^1\times L^2} \leq A$. Then, (for $3 \leq N \leq 5$) there exists $\delta = \delta(A)$ such that if*

$$\|S(t)\left((u_0,u_1)\right)\|_{S(I)} < \delta,$$

there exists a unique solution u to (CP) in $\mathbb{R}^N \times I$, with $(u,\partial_t u) \in C(I;\dot{H}^1 \times L^2)$, $\left\|D_x^{1/2}u\right\|_{W(I)} + \left\|\partial_t D_x^{-1/2}u\right\|_{W(I)} < +\infty$, $\|u\|_{S(I)} \leq 2\delta$.

Remark 2.2. We recall that, since we are working in the focusing case, from the work of Levine ([21], [34]) we have that if $(u_0,u_1) \in H^1 \times L^2$ is such that $E((u_0,u_1)) < 0$, then the maximal interval of existence is finite.

3. Variational estimates

In section 3 we prove some elementary variational estimates which yield the necessary sharp coercivity for our arguments and which follows from arguments in [17].

Note that by invariances of the equation, for $\theta_0 \in [-\pi,\pi]$, $\lambda_0 > 0$, $x_0 \in \mathbb{R}^N$, $W_{\theta_0,x_0,\lambda_0}(x) = e^{i\theta_0}\lambda_0^{\frac{(N-2)}{2}} W(\lambda_0(x-x_0))$ is still a solution. By the work of Aubin [3], Talenti [36] we have the following characterization of W:

$$\forall u \in \dot{H}^1, \quad \|u\|_{L^{2^*}} \leq C_N \|\nabla u\|_{L^2}; \tag{3.1}$$

moreover,
(3.2)

If $\|u\|_{L^{2^*}} = C_N \|\nabla u\|_{L^2}$, $u \neq 0$, then $\exists(\theta_0, \lambda_0, x_0) : u = W_{\theta_0, x_0, \lambda_0}$,

where C_N is the best constant of the Sobolev inequality (3.1) in dimension N.

Theorem 3.1 (Energy trapping)**.** *Let u be a solution of (CP), with $(u, \partial_t u)\big|_{t=0} = (u_0, u_1) \in \dot{H}^1 \times L^2$ and maximal interval of existence I. Assume that, for $\delta_0 > 0$,*

$$E((u_0, u_1)) \leq (1 - \delta_0) E((W, 0)) \text{ and } \|\nabla u_0\|_{L^2}^2 < \|\nabla W\|_{L^2}^2 .$$

Then, there exists $\bar{\delta} = \bar{\delta}(\delta_0)$ such that, for $t \in I$, we have

$$\|\nabla_x u(t)\|_{L^2}^2 \leq (1 - \bar{\delta}) \|\nabla W\|_{L^2}^2 \tag{3.3}$$

$$\int |\nabla_x u(t)|^2 - |u(t)|^{2^*} \geq C_{\bar{\delta}} \int |\nabla_x u(t)|^2 \tag{3.4}$$

$$E((u(t), \partial_t u(t))) \geq 0. \tag{3.5}$$

Corollary 3.2. *Let u be as in Theorem 3.1. Then for all $t \in I$ we have $E((u(t), \partial_t u(t))) \simeq \|(u(t), \partial_t u(t))\|_{\dot{H}^1 \times L^2}^2 \simeq \|(u_0, u_1)\|_{\dot{H}^1 \times L^2}^2$, with comparability constants which depend only on δ_0.*

4. Existence and compactness of a critical element; further properties of critical elements

In section 4, using the work of Bahouri-Gerard [4] and the concentration compactness argument from [17] we produce a "critical element" for which scattering fails and which enjoys a compactness property because of its criticality. At this point, we show a crucial orthogonality property of "critical elements" related to a second conservation law in the energy space which exploits the finite speed of propagation for the wave equation and its Lorentz invariance. This is the extra ingredient that allows us to go beyond the radial case. Let us consider the statement:

(SC) For all $(u_0, u_1) \in \dot{H}^1 \times L^2$, with $\int |\nabla u_0|^2 < \int |\nabla W|^2$ and $E((u_0, u_1)) < E((W, 0))$, if u is the corresponding solution of (CP) with maximal interval of existence I then $I = (-\infty, +\infty)$ and $\|u\|_{S((-\infty,+\infty))} < \infty$.

In addition, for a fixed $(u_0, u_1) \in \dot{H}^1 \times L^2$, with $\int |\nabla u_0|^2 < \int |\nabla W|^2$ and $E((u_0, u_1)) < E((W, 0))$, we say that $(SC)((u_0, u_1))$ holds if, for u the corresponding solution of (CP), with maximal interval of existence I, we have $I = (-\infty, +\infty)$ and $\|u\|_{S(-\infty,+\infty)} < \infty$.

By linear arguments, there exists $\eta_0 > 0$ such that if (u_0, u_1) is as in (SC), and $E((u_0, u_1)) \leq \eta_0$, then $SC((u_0, u_1))$ holds. Moreover, for any (u_0, u_1) as in (SC), (3.5) shows that

$$E((u_0, u_1)) \geq 0.$$

Thus, there exists a number E_C, $\eta_0 \leq E_C \leq E((W, 0))$ such that, if (u_0, u_1) is as in (SC) and $E((u_0, u_1)) < E_C$, then $(SC)((u_0, u_1))$ holds and E_C is optimal with this property. For the rest of this section we will assume that $E_C < E((W, 0))$. Using concentration compactness ideas, we prove that there exists a critical element $(u_{0,C}, u_{1,C})$ at the critical level of energy E_C, so that $SC((u_{0,C}, u_{1,C}))$ does not hold and from the minimality, this element has a compactness property up to the symmetries of the equation (which will give rigidity in the problem).

We then use the finite speed of propagation and Lorentz transformations to establish support and orthogonality properties of critical elements, which are essential to treat the nonradial case.

Proposition 4.1. *There exists $(u_{0,C}, u_{1,C})$ in $\dot{H}^1 \times L^2$, with*

$$E((u_{0,C}, u_{1,C})) = E_C < E((W, 0)), \quad \int |\nabla u_{0,C}|^2 < \int |\nabla W|^2$$

such that if u_C is the solution of (CP) with data $(u_{0,C}, u_{1,C})$ and with maximal interval of existence I, $0 \in \overset{\circ}{I}$, then $\|u_C\|_{S(I)} = +\infty$.

Proposition 4.2. *Assume that u_C is as in Proposition 4.1 and that (say) $\|u_C\|_{S(I_+)} = +\infty$, where $I_+ = [0, \infty) \bigcap I$. Then there exists $x(t) \in \mathbb{R}^N$, $\lambda(t) \in \mathbb{R}^+$, for $t \in I_+$, such that $K = \{\vec{v}(x, t), t \in I_+\}$ has the property that $\overline{K}$ is compact in $\dot{H}^1 \times L^2$, where*

$$\vec{v}(x, t) = \left(\frac{1}{\lambda(t)^{\frac{(N-2)}{2}}} u_C \left(\frac{x - x(t)}{\lambda(t)}, t \right), \frac{1}{\lambda(t)^{\frac{N}{2}}} \partial_t u_C \left(\frac{x - x(t)}{\lambda(t)}, t \right) \right).$$

A corresponding conclusion is reached if $\|u_C\|_{S(I_-)} = +\infty$, where $I_- = (-\infty, 0) \bigcap I$.

We turn now to the next important property of u_C (at least in the nonradial situation): the second invariant of the equation in the energy space for u_C is zero.

Proposition 4.3. *Assume that u_C is as in Proposition 4.2 and I_+ is a finite interval. Then,*

$$\int \nabla u_{0,C} . u_{1,C} = 0.$$

This follows from the fact somehow that using the Lorentz transformation, you will still get at a lower level of energy, a solution with a space-time norm too large.

In sections 5 and 6 we prove a rigidity theorem (Theorem 4.4), which allows us to conclude the argument.

Theorem 4.4. *Assume that* $(u_0, u_1) \in \dot{H}^1 \times L^2$ *is such that*

$$E((u_0, u_1)) < E((W, 0)), \quad \int |\nabla u_0|^2 < \int |\nabla W|^2, \int \nabla u_0 . u_1 = 0.$$

Let u *be the solution of* (CP) *with* $(u(0), \partial_t u(0)) = (u_0, u_1)$, *with maximal interval of existence* $(-T_-(u_0, u_1), T_+(u_0, u_1))$. *Assume that there exist* $\lambda(t) > 0$, $x(t) \in \mathbb{R}^N$, *for* $t \in [0, T_+(u_0, u_1))$, *with the property that*

$$\begin{aligned} K \quad = \quad & \left\{ \vec{v}(x,t) = \left(\frac{1}{\lambda(t)^{\frac{(N-2)}{2}}} u\left(\frac{x - x(t)}{\lambda(t)}, t\right), \frac{1}{\lambda(t)^{\frac{N}{2}}} \partial_t u\left(\frac{x - x(t)}{\lambda(t)}, t\right) \right), \right. \\ & \left. t \in [0, T_+(u_0, u_1)) \right\} \end{aligned}$$

has the property that $\overline{K}$ *is compact in* $\dot{H}^1 \times L^2$.

Then, $T_+(u_0, u_1) < \infty$ *is impossible.*

Moreover, if $T_+(u_0, u_1) = +\infty$ *and we assume that* $\lambda(t) \geq A_0 > 0$, *for* $t \in [0, \infty)$, *we must have* $u \equiv 0$.

5. Rigidity Theorem. Part 1: Infinite time interval and self-similarity for finite time intervals

The first case of the rigidity theorem deals with infinite time of existence. This uses localized conservations laws and related ones, very much in the spirit of the corresponding localized virial identity used in [17].

We next turn to the proof of Theorem 4.4 in the case when

$$T_+(u_0, u_1) = +\infty, \ \lambda(t) \geq A_0.$$

The conclusion of the proof follows then from a reduction to a concentrated situation uniform in time given by free by the use of the compactness.

$$\lambda(t) \geq A_0 > 0.$$

We will set

$$r(R) = \int_{|x|\geq R} \frac{|u|^2}{|x|^2} + |u|^{2^*} + |\nabla u|^2 + |\partial_t u|^2 \, dx.$$

Lemma 5.1. *The following identities hold: for all* $t \geq 0$

i) $\partial_t \left(\int \frac{1}{2}(\partial_t u)^2 + \frac{1}{2} |\nabla_x u|^2 - \frac{1}{2^*} |u|^{2^*} \right) = 0$

ii) $\partial_t \int \nabla u . \partial_t u = 0$

iii) $\partial_t \left(\int \psi_R(x) . \nabla u \partial_t u \right) = -\frac{N}{2} \int (\partial_t u)^2 + \frac{(N-2)}{2} \left[\int |\nabla_x u|^2 - |u|^{2^*} \right] + O(r(R))$

iv) $\partial_t \left(\int \phi_R u u_t \right) = \int (\partial_t u)^2 - \int |\nabla u|^2 + \int |u|^{2^*} + O(r(R))$

v) $\partial_t \left(\int \psi_R \left\{ \frac{1}{2}(\partial_t u)^2 + \frac{1}{2} |\nabla_x u|^2 - \frac{1}{2^*} |u|^{2^*} \right\} \right) = -\int \nabla u \partial_t u + O(r(R)).$

we can assume $x(0) = 0$.

Lemma 5.2. *There exist* $\epsilon_1 > 0$*,* $C > 0$*, such that, if* $\epsilon \in (0, \epsilon_1)$*, there exists* $R_0(\epsilon)$ *so that if* $R > 2R_0(\epsilon)$*, then there exists* $t_0 = t_0(R, \epsilon)$*,* $0 \leq t_0 \leq CR$*, with the property that for all* $0 < t < t_0$ *we have* $\left| \frac{x(t)}{\lambda(t)} \right| < R - R_0(\epsilon)$ *and* $\left| \frac{x(t_0)}{\lambda(t_0)} \right| = R - R_0(\epsilon)$*.*

The proof use the following quantity

$$z_R(t) = \int \psi_R(x) . \nabla_x u u_t + \left(\frac{N}{2} - \alpha\right) \int \phi_R u u_t, \quad 0 < \alpha < 1.$$

Then,

$$z_R'(t) = -C_\alpha E + O(r(R))$$

which allows us to conclude.

Lemma 5.3. *There exist* $\epsilon_2 > 0$*,* $R_1(\epsilon) > 0$*,* $C_0 > 0$ *such that if* $R > R_1(\epsilon)$*,* $t_0 = t_0(R, \epsilon)$ *is as in Lemma 5.2, then for* $0 < \epsilon < \epsilon_2$*,*

$$t_0(R, \epsilon) \geq \frac{C_0 R}{\epsilon}.$$

The proof use the following quantity

$$y_R(t) = \int \psi_R(x) e(u)(x, t) dx$$

and the fact that $\int \nabla u_0 u_1 = 0$.

This concludes the proof of Theorem 4.4 in the case when $T_+(u_0, u_1) = +\infty$.

6. Rigidity Theorem. Part 2: Self-similar variables and conclusion of the proof of the rigidity theorem

The second case of the rigidity theorem deals with finite time of existence of the critical element. This case is dealt with in [17] through the use of the L^2 conservation law, which is absent for the wave equation. We proceed in two stages. First we show that the solution must have self-similar behavior. Then, in section 6, following Merle-Zaag ([24]) and earlier work on non-linear heat equations by Giga-Kohn ([8]), we introduce self-similar variables and the new resulting equation, which has a monotonic energy. We then show that there exists a non-trivial asymptotic solution w^*, which solves a (degenerate) elliptic non-linear equation. Finally, using the estimates we proved on w^* and the unique continuation principle, we show that w^* must be zero, a contradiction which gives our rigidity theorem and conclude the proof of the main theorem.

In the case when $T_+(u_0, u_1) = 1 < +\infty$, using again

$$z(t) = \int x\nabla u\partial_t u + \left(\frac{N}{2} - \alpha\right)\int u\partial_t u, \qquad 0 < \alpha < 1,$$

which is defined for $0 \le t < 1$ and the finite speed of propagation, we obtain

Proposition 6.1. *Assume that* (u_0, u_1) *is as in Theorem 4.4, with* $T_+(u_0, u_1) = 1$. *Then* $\operatorname{supp} \nabla u, \partial_t u \subset B(0, 1-t)$ *and*

$$\vec{K} = \left\{(1-t)^{\frac{N}{2}}\big(\nabla u((1-t)x, t),\, \partial_t u((1-t)x, t)\big)\right\}$$

has compact closure in $L^2(\mathbb{R}^N)^N \times L^2(\mathbb{R}^N)$.

We now set,

$$y = x/(1-t), \quad s = -\log(1-t), \quad 0 \le t < 1$$

and define

$$w(y, s, 0) = (1-t)^{\frac{N-2}{2}} u(x, t) = e^{-s\frac{(N-2)}{2}} u(e^{-s}y, 1 - e^{-s}). \tag{6.1}$$

Note that $w(y, s, 0)$ is defined for $0 \le s < +\infty$, and that $\operatorname{supp} w(-, s, 0) \subset \{|y| \le 1\}$. We also consider, for $\delta > 0$, small,

$$y = \frac{x}{1 + \delta - t}, \qquad s = -\log(1 + \delta - t),$$

$$w(y, s, \delta) = (1 + \delta - t)^{\frac{N-2}{2}} u(x, t) = e^{-s\frac{(N-2)}{2}} u(e^{-s}y, 1 + \delta - e^{-s}) \tag{6.2}$$

Note that $w(y,s,\delta)$ is defined for $0 \le s < -\log\delta$, and that

$$\operatorname{supp} w(-,\delta) \subset \left\{ |y| \le \frac{e^{-s}-\delta}{e^{-s}} = \frac{(1-t)}{(1+\delta-t)} \le 1-\delta \right\}.$$

The w solve, in their domain of definition, the equation (see [24]):

$$\begin{aligned} \partial_s^2 w =& \frac{1}{\rho}\operatorname{div}\big(\rho\nabla w - \rho(y.\nabla w)y\big) - \frac{N(N-2)}{4}w \\ & + |w|^{\frac{4}{N-2}} w - 2y\nabla\partial_s w - (N-1)\partial_s w, \end{aligned} \tag{6.3}$$

where $\rho = (1-|y|^2)^{-\frac{1}{2}}$.

For $w(y,s,\delta)$, $\delta > 0$ as above, we now define

$$\begin{aligned} \tilde{E}(w(s)) = \int_{B_1} \Big\{ & \frac{1}{2}\big[(\partial_s w)^2 + |\nabla w|^2 - (y.\nabla w)^2\big] \\ & + \frac{N(N-2)}{8}w^2 - \frac{(N-2)}{2N}|w|^{2^*} \Big\} \frac{dy}{(1-|y|^2)^{1/2}}. \end{aligned} \tag{6.4}$$

Proposition 6.2. *Let $w = w(y,s,\delta)$, $\delta > 0$ be as above. Then, for $0 < s_1 < s_2 < \log(\frac{1}{\delta})$, the following identities hold:*

i) $\tilde{E}(w(s_2)) - \tilde{E}(w(s_1)) = \int_{s_1}^{s_2}\int_{B_1} \frac{(\partial_s w)^2}{(1-|y|^2)^{3/2}} dy\, ds$

ii) $\lim\limits_{s\to\log(\frac{1}{\delta})} \tilde{E}(w(s)) \le E = E(u_0,u_1)$.

We are able to find a w^* such that we have the following

Proposition 6.3. $w^* \in H_0^1(B_1)$,
$\int_{B_1} \frac{|w^*(y)|^2}{(1-|y|^2)^2} < \infty$ *and w^* solves the (degenerate) elliptic equation*

$$\frac{1}{\rho}\operatorname{div}\big(\rho\nabla w^* - \rho(y.\nabla w^*)y\big) - \frac{N(N-2)}{4}w^* + |w^*|^{\frac{4}{N-2}} w^* = 0, \tag{6.5}$$

where $\rho(y) = (1-|y|^2)^{-1/2}$.
Moreover, $w^* \not\equiv 0$ and

$$\int \frac{|w^*(y)|^{2^*}}{(1-|y|^2)^{1/2}}dy + \int \frac{\big[|\nabla w^*(y)|^2 - (y.\nabla w^*(y))^2\big]}{(1-|y|^2)^{1/2}}dy < +\infty, \tag{6.6}$$

The contradiction which finishes the proof of Theorem 4.4 is then provided by the following elliptic result given by a well-known argument of Trudinger [40] and the classical unique continuation theorem of Aronszajn, Krzywicki and Szarski (see [2] and [13], Section 17.2)

Proposition 6.4. *Assume that $w \in H_0^1(B_1)$, is such that*

i) $\int \frac{|w(y)|^2}{(1-|y|^2)^2}dy < \infty$ *(a consequence of $w \in H_0^1(B_1)$)*

ii) $\int \frac{|w(y)|^{2^*}}{(1-|y|^2)^{1/2}} dy + \int \frac{|\nabla w(y)|^2 - (y.\nabla w(y))^2}{(1-|y|^2)^{1/2}} dy < \infty$

iii) w verifies the (degenerate) elliptic equation (6.5)*.*

Then, $w \equiv 0$.

Remark 6.5. For this part of the argument, no size or energy conditions are needed.

The results in this section yield the contradiction which completes the proof of Theorem 4.4.

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