Centre de Mathématiques Laurent Schwartz
ECOLE POLYTECHNIQUE

SEMINAIRE

Equations aux Dérivées Partielles

2006-2007

François Castella

Time averaging for the strongly confined nonlinear Schrödinger equation

Séminaire É. D. P. (2006-2007), Exposé n° II, 22 p.

<http://sedp.cedram.org/item?id=SEDP_2006-2007____A2_0>

U.M.R. 7640 du C.N.R.S.
F-91128 PALAISEAU CEDEX
Fax : 33 (0)1 69 33 49 49
Tél : 33 (0)1 69 33 49 99

Time averaging for the strongly confined nonlinear Schrödinger equation.

François Castella

This text is a summary of a joint work by N. Ben Abdallah[1], F. Castella[2], F. Méhats[1].

1 Introduction

We study the asymptotic behavior of a nonlinear gas of quantum particles, evolving in the three dimensional space $(x, z) \in \mathbb{R}^3$ ($x \in \mathbb{R}^2$, $z \in \mathbb{R}$), yet strongly confined along the vertical z direction. The dynamics of the gas essentially occurs along the remaining, horizontal x plane, and our goal is to recover the limiting dynamics along x, by performing the relevant averaging procedure.

Such nonlinear and strongly confined gases are typically encountered in the study of Bose condensation. In this context, an atomic gas is confined in a given region of space, and an appropriate cooling procedure makes it possible to set all atoms in the same quantum state, described by the same wave function Ψ. This somehow "macroscopic" wave function satisfies a Schrödinger equation. The fact that the underlying gas is made up of many atoms which interact pairwise is usually taken into account using a mean-field model, and the appropriate equation is nonlinear.

Mathematically speaking, this situation is described by a nonlinear Schrödinger equation in the presence of a small parameter. The confining Hamiltonian in the z direction, called H_z in the sequel, carries a weight $1/\varepsilon$ which, as $\varepsilon \to 0$, enhances the time oscillations of Ψ, of the form $\exp(-itH_z/\varepsilon)$ (roughly), and the difficulty is to average out these oscillations.

We show that the strong confinement allows to develop an averaged model over the discrete eigenspaces of H_z. This model describes the limiting dynamics along the x plane. The point is, we are able to completely develop the averaging procedure over *all* the eigenspaces *at once*. The limiting model is an infinite system of coupled, nonlinear, Schrödinger equations, describing the averaged evolution of Ψ over each eigenspace. In particular, all energy levels are coupled through the averaged nonlinearity. This contrasts with the previous study performed in [BMSW] (see also [BM] in the similar spirit), where only the ground state, *i.e.* the eigenspace associated with the lowest eigenenergy of H_z, is treated, and the limiting model is a single, scalar, nonlinear Schrödinger equation, describing the averaged evolution of Ψ over this single eigenspace. This also contrasts with the Born-Oppenheimer situation, where the emphasis is more on the separation between two distinguished eigenspaces, but the spectrum is not necessarily discrete.

The key observation in the present study, that makes it possible to perform a clean averaging procedure, relies on the fact that the operator $\exp(-itH_z/\varepsilon)$ is *almost periodic* in time. In other words, it carries a discrete, possibly infinite, number of independent time-oscillations. This observation allows to average $\exp(-itH_z/\varepsilon)$ in time without having

[1]Institut de Mathématiques de Toulouse - Université Paul Sabatier - 118 rte de Narbonne - 31062 Toulouse Cedex

[2]IRMAR, Université de Rennes 1 - Campus de Beaulieu - 35042 Rennes Cedex, France

to deal with the difficulty of *small denominators* (see [BCD] in the context of laser-matter interaction). It also allows to formulate our limiting model in a "good" functional framework, without having to project it over all the eigenspace of H_z, a difficult if not impossible task, that is the very reason why the text [BMSW] restricts to a situation where only the ground state is occupied. Obviously, the counterpart is that our error terms are bounded by nothing better than $o(1)$: a simpler, periodic framework (i.e. only one time-oscillation, as in [BMSW]) certainly allows to obtain improved convergence rates, yet it has to be stressed that such a simplified framework is *not* generic. In the course of the analysis, we are also led to identifying the Sobolev scale associated with the operator $H_z = -\Delta_z + V_c(z)$ (see below for the notations), *i.e.* the domain of the successive powers $(-\Delta_z + V_c(z))^m$ $(m \geq 0)$. This turns out to be an important and difficult step of our analysis, which leads us to use an appropriate pseudo-differential calculus, based on the Weyl-Hörmander calculus and the associated Sobolev spaces developed by Bony and Chemin in [BC], later used by Hellfer and Nier in [HN].

1.1 The model

Let (x, z) be the variable in $\mathbb{R}^3 = \mathbb{R}^2 \times \mathbb{R}$, where $z \in \mathbb{R}$ lies in the vertical direction (say), and $x \in \mathbb{R}^2$ belongs to the horizontal plane[3]. Accordingly, take two Hamiltonians

$$H_x = -\Delta_x + V(x), \qquad \text{and} \quad H_z = -\frac{\partial^2}{\partial z^2} + V_c(z), \tag{1.1}$$

where both potentials $V(x)$ and $V_c(z)$ are assumed smooth, C^∞, real valued, and bounded from below. Without loss of generality, we may assume that both potentials are bounded away from zero, i.e.

$$V(x) \geq 1 \text{ and } V_c(z) \geq 1.$$

Other, more specific, assumptions on the potentials $V_c(z)$ and $V(x)$ are needed, which are detailed now.

A key assumption of this paper is that V_c is *confining*, i.e.

$$V_c(z) \underset{|z|\to\infty}{\longrightarrow} +\infty. \tag{1.2}$$

As is well known [RS], this ensures that the spectrum of $H_z = -\Delta_z + V_c(z)$ is discrete, when considered as a linear, unbounded operator over $L^2(\mathbb{R})$, with domain

$$D(H_z) = \{\Psi(z) \in L^2 \text{ s.t. } \partial_z^2 \Psi \in L^2 \text{ and } V_c(z)\, \Psi \in L^2\}.$$

Throughout this paper, the eigenelements of H_z will be denoted by the collection of eigenenergies $E_p \geq 0$ and eigenfunctions $\chi_p(z)$, as p runs in $\mathbb{N}$. They satisfy, for any index p,

$$H_z \chi_p(z) = E_p\, \chi_p(z). \tag{1.3}$$

Also, it is well known that $E_p \to +\infty$ as $p \to \infty$, and the χ_p's may be chosen so as to form an orthonormal basis of $L^2(\mathbb{R})$ (we will thus assume this orthonormality property from now on).

[3]Needless to say, our techniques are immediately adapted in any dimension $\mathbb{R}^d = \mathbb{R}^{d-p} \times \mathbb{R}^p$

For later functional analytic purposes, we shall actually assume a *reinforced version of confinement* in the z direction. This is a more technical point. Our study requires the following three conditions

$$\forall \alpha \in \mathbb{N}, \qquad \frac{\partial^\alpha V_c}{\partial z^\alpha}(z) = O\Big(V_c(z)\Big) \quad \text{as} \quad |z| \to \infty, \tag{1.4}$$

$$\exists M_z > 0, \qquad V_c(z) = O\left(|z|^{M_z}\right) \quad \text{as} \quad |z| \to \infty, \tag{1.5}$$

$$\exists M_z' > 0, \qquad \frac{|\nabla_z V_c(z)|}{V_c(z)} = O\left(|z|^{-M_z'}\right) \quad \text{as} \quad |z| \to \infty. \tag{1.6}$$

In other words, $V_c(z)$ should roughly behave like a symbol at infinity in z (this is the meaning of assumptions (1.4) and (1.6)), and V should have at most polynomial growth at infinity in z (this is assumption (1.5)). These assumptions typically exclude potentials behaving like $\exp(z)$ at infinity or so, for which the analysis we present in this text probably becomes false anyhow. Obviously, assumptions (1.2) and (1.4) are met in the prototype case where H_z simply is the harmonic oscillator $-\Delta_z + z^2$, which is the example we keep in mind throughout the paper, relevant in the context of Bose condensation.

Concerning the potential $V(x)$ in the x direction, the present study may be carried either when $V(x)$ is confining or when it is uniformly bounded. For definiteness, and because the physical situation we have in mind is again Bose condensation, we shall assume $V(x)$ is confining as is $V_c(z)$, namely

$$V(x) \underset{|x|\to\infty}{\longrightarrow} +\infty, \tag{1.7}$$

$$\forall \alpha \in \mathbb{N}, \quad \frac{\partial^\alpha V}{\partial x^\alpha}(x) = O\Big(V(x)\Big) \quad \text{as} \quad |x| \to \infty, \tag{1.8}$$

$$\exists M_x > 0, V(x) = O\left(|x|^{M_x}\right) \quad \text{as} \quad |z| \to \infty, \tag{1.9}$$

$$\exists M_x' > 0, \frac{|\nabla_x V(x)|}{V(x)} = O\left(|x|^{-M_x'}\right) \quad \text{as} \quad |x| \to \infty. \tag{1.10}$$

We stress that these assumptions are *not* essential in our analysis, and the alternative situation where $V(x) \in C_b^\infty(\mathbb{R}^2)$ (C^∞ and bounded functions) could be handled as well (the analysis is actually simpler then). Again, the prototype potential we have in mind is the harmonic oscillator $-\Delta_x + x^2$.

Now, let $\varepsilon > 0$ be the small parameter that measures the relative strength of the confinement in the z direction, relative to that in the x plane. Take a smooth nonlinearity $F : \mathbb{R} \mapsto \mathbb{R}$, $F \in C^\infty(\mathbb{R})$. The definite example in the context of Bose condensation is $F(u) = \pm u$. Our goal is to study the following nonlinear Schrödinger equation, written in dimensionless form, along the limit $\varepsilon \to 0$:

$$i\partial_t \Psi^\varepsilon(t,x,z) = H_x\, \Psi^\varepsilon + \frac{1}{\varepsilon} H_z\, \Psi^\varepsilon + F(|\Psi^\varepsilon|^2)\, \Psi^\varepsilon. \tag{1.11}$$

Here $H_x = -\Delta_x + V(x)$, and $H_z = -\Delta_z + V_c(z)$, as before (see (1.1)). In other words, we study the idealized limit where confinement in z is infinite, and the quantum particles are essentially confined in the horizontal plane $\mathbb{R}^2$.

Needless to say, an initial datum is also prescribed for (1.11), namely

$$\Psi^\varepsilon(0,x,z) = \Psi_0(x,z) \in L^2(\mathbb{R}^2\times\mathbb{R}). \tag{1.12}$$

In order to have "good" uniform bounds on Ψ^ε, and on the nonlinear term $F(|\Psi^\varepsilon|^2)$, we shall additionally assume that Ψ_0 possesses a "good" regularity in the Sobolev scale induced by the nonnegative, self-adjoint operators H_x and H_z. This is a delicate point of our analysis, which we now briefly discuss.

Namely, we shall suppose the following:

There exists an $m > 3/2$ such that

$$\Psi_0 \in B_m := \big\{u \in L^2(\mathbb{R}^3) \text{ s.t. } H_x^{m/2} u \in L^2(\mathbb{R}^3), \text{ and } H_z^{m/2} u \in L^2(\mathbb{R}^3)\big\}. \tag{1.13}$$

As we show later, it turns out the spaces B_ℓ ($\ell \geq 0$) form a scale of Hilbert spaces, and they may be endowed with either the norm

$$\|u\|_{B_\ell}^2 := \|u\|_{L^2(\mathbb{R}^3)}^2 + \|H_x^{\ell/2}\, u\|_{L^2(\mathbb{R}^3)}^2 + \|H_z^{\ell/2}\, u\|_{L^2(\mathbb{R}^3)}^2, \tag{1.14}$$

or the equivalent norm (we use the same notation for simplicity)

$$\|u\|_{B_\ell}^2 := \|u\|_{H^\ell(\mathbb{R}^3)}^2 + \|V(x)^{\ell/2}\, u\|_{L^2(\mathbb{R}^3)}^2 + \|V_{\rm c}(z)^{\ell/2}\, u\|_{L^2(\mathbb{R}^3)}^2, \tag{1.15}$$

where $H^\ell(\mathbb{R}^3)$ denotes the usual Sobolev space.

The reason for the present assumption is the following. First, the condition $m > 3/2$ in (1.13) makes B_m an algebra, and the nonlinear application $\Psi^\varepsilon \mapsto F(|\Psi^\varepsilon|^2)\,\Psi^\varepsilon$ is seen to be locally Lipschitz in B_m. Second, and more importantly, the fact that the operators $H_x^{m/2}$ and $H_z^{m/2}$ commute with $H_x + H_z/\varepsilon$ in (1.11), allows to prove Ψ^ε is uniformly bounded in B_m, despite the singular term H_z/ε. Note that the crucial point according to which both norms (1.14) and (1.15) are equivalent is not an obvious point, and the proof of this actually is an important and difficult step of our analysis, see below.

At this point of the discussion, we are in position to try to characterize the limit of Ψ^ε in B_m. This is where *almost-periodicity* enters, which is the key observation of the present text.

1.2 Heuristic approach to the strong confinement limit

The probably most natural approach is to first *project* the Schrödinger equation (1.11) over the orthonormal basis $(\chi_p)_{p\in\mathbb{N}}$. Admitting for the moment there exists a time $T_0 > 0$ such that Ψ^ε is bounded in $C^0([0,T_0];B_m)$, uniformly with respect to ε, we may write the orthogonal decomposition

$$\Psi^\varepsilon(t,x,z) = \sum_{p\geq 0} \psi_p^\varepsilon(t,x)\,\chi_p(z) \qquad \text{with} \quad \psi_p^\varepsilon(t,x) := \langle \Psi^\varepsilon(t,x,z), \chi_p(z)\rangle,$$

and it may be assumed that the ψ_p^ε's possess nice uniform bounds in the space $C^0([0,T_0];l^2(\mathbb{N};L^2(\mathbb{R}^2)))$. (The l^2 norm may be improved into a weighted l^2 norm, using the E_p's). Here and throughout the paper, we use the notations

$$\langle u, v\rangle := \int_{\mathbb{R}} u\,\overline{v}\,dz. \tag{1.16}$$

Using this, the Schrödinger equation (1.11) may be decomposed into

$$i\partial_t\psi_p^\varepsilon(t,x) = H_x\,\psi_p^\varepsilon + \frac{E_p}{\varepsilon}\,\psi_p^\varepsilon + \sum_{r\geq 0}\psi_r^\varepsilon \times \Big\langle F\Big(\Big|\sum_{q\geq 0}\psi_q^\varepsilon(t,x)\,\chi_q(z)\Big|^2\Big)\,,\,\overline{\chi_r(z)}\,\chi_p(z)\Big\rangle, \qquad (1.17)$$

an infinite system of coupled, nonlinear, Schrödinger equations, on the $\psi_p^\varepsilon(t,x)$'s ($p \in \mathbb{N}$, $x \in \mathbb{R}^2$).

In view of (1.17), $\partial_t\psi_p^\varepsilon$ clearly has size $O(1/\varepsilon)$. For this reason, it is now natural to *filter out* the oscillations $\exp(-itE_p/\varepsilon)$ of ψ_p^ε induced by H_z, in the spirit of Schochet and Grenier's works [Sc], [Gr]. Hence, we define, for each $p \geq 0$, the new unknown

$$\phi_p^\varepsilon(t,x) := \psi_p^\varepsilon(t,x)\ \exp\left(+itE_p/\varepsilon\right). \qquad (1.18)$$

The ϕ_p^ε's naturally satisfy the *filtered system*

$$i\partial_t\phi_p^\varepsilon(t,x) = H_x\,\phi_p^\varepsilon + \sum_{r\geq 0}\phi_r^\varepsilon \times \mathrm{e}^{-it\frac{E_r-E_p}{\varepsilon}}\ \Big\langle F\Big(\Big|\sum_{q\geq 0}\phi_q^\varepsilon(t,x)\,\chi_q(z)\,\mathrm{e}^{-it\frac{E_q}{\varepsilon}}\Big|^2\Big)\,,\,\overline{\chi_r}\,\chi_p\Big\rangle. \qquad (1.19)$$

Clearly, $\partial_t\phi_p^\varepsilon$ has become an $O(1)$ quantity. Even more, the system (1.19) is an infinite dimensional, nonlinear and coupled differential system on the ϕ_p^ε's ($p \in \mathbb{N}$), of the form

$$\partial_t u^\varepsilon = Au^\varepsilon + B(t/\varepsilon, u^\varepsilon), \qquad (1.20)$$

and the nonlinearity B showing up on the right-hand-side of (1.19) clearly possesses some "periodicity" in time, due to the oscillatory factors $\exp(itE_p/\varepsilon)$ etc. (To be more precise, the time dependence of the nonlinearity at hand turns out to be *almost-periodic*, as we discuss later in the text).

At this level, it now becomes quite tempting to *average* in time the system (1.19), or, equivalently, the toy model (1.20). This is actually the key ingredient in Schochet's work [Sc]. Indeed, it is well known that, provided the function $B(\tau,u)$ entering (1.20) possesses some *ergodicity* property in time, the reference system (1.20) converges towards

$$\partial_t u = Au + B_{\mathrm{av}}(u), \qquad \text{where } B_{\mathrm{av}}(u) := \lim_{T\to\infty}\frac{1}{T}\int_0^T B(\tau,u)\,d\tau. \qquad (1.21)$$

We refer to [SV] and [LM] for statements of this form in the context of ODE's. We also refer to [BCD], [BCDG], or more recently [CDG1], [CDG2] for this kind of averaging procedure in the context of laser-matter interaction, yet for infinite dimensional systems. We also refer to the deep paper [MS] in the context of fluid mechanics, for the use of similar averaging tools in infinite dimensional systems (here, very fine *resonance* questions are considered). In any circumstance, a prototype "ergodicity" assumption on the time-behavior of $B(\tau,u)$ is that B is periodic in time. A more general assumption is that $B(\tau,u)$ is quasi-periodic in time, which means $B(\tau,u) \equiv \mathcal{B}(\omega_1\tau,\ldots,\omega_N\tau,u)$, where $\mathcal{B}$ is 1-periodic in its first N arguments, and the ω_i's are rationally independent frequencies. An even more general assumption is that $B(\tau,u)$ is almost-periodic in time, which somehow corresponds to the quasi-periodic framework with $N = +\infty$ independent frequencies, and we refer to the sequel on that point.

For this reason, and despite the differential system satisfied by the ϕ_p^ε's is infinite dimensional, it is reasonable to expect that the ϕ_p^ε's in (1.19) converge *at least formally* towards the ϕ_p's, solution to the *averaged system*

$$
\begin{aligned}
&i\partial_t\phi_p(t,x) = H_x\,\phi_p(t,x) + \sum_{r\geq 0}\phi_r(t,x) \qquad (1.22)\\
&\times \lim_{T\to\infty}\frac{1}{T}\int_0^T\left[\left\langle F\left(\Big|\sum_{q\geq 0}\phi_q(t,x)\,\chi_q(z)\,\mathrm{e}^{-i\tau E_q}\Big|^2\right)\,,\,\overline{\chi_r(z)}\,\chi_p(z)\right\rangle\,\mathrm{e}^{-i\tau(E_r-E_p)}\right]d\tau.
\end{aligned}
$$

All these steps require some care yet, before becoming rigorous statements. In some sense, the goal of this paper is to rigorously prove the convergence towards (1.22), and even more to exhibit a functional framework that is well adapted to this infinite dimensional problem.

1.3 Rigorous results

The difficulty in making the above statements correct is twofolds. Firstly, the above procedure requires to actually decompose Ψ^ε over the χ_p's, hence to write down series expansions of the form $\sum_{r\geq 0}\cdots$ as in (1.22). However, it turns out to be extremely difficult to control the *convergence* of these series expansions, despite the fact that we have nice $l^2(L^2)$ bounds on the ϕ_p^ε's. This is essentially due to the lack of information on the behavior of the coefficient $\langle F(|\cdots|^2)\,,\,\overline{\chi_r}\,\chi_p\rangle$ appearing above, for large values of r and p. Indeed, no orthogonality property is at hand to estimate this coefficient, except in the very special case where $\chi_p(z)=\exp(ipz)$, corresponding to periodic boundary conditions on z (we may yet refer to W.-M. Wang's delicate analysis [W1], [W2], of factors of the form $\displaystyle\int_{\mathbb{R}}\chi_p(x)\,\chi_q(x)\,\chi_r(x)\,\chi_s(x)\,dx$ - $p,q,r,s\in\mathbb{N}$ - in the case when the χ_p's are the eigenfunctions of the harmonic oscillator). Secondly, there is actually a deeper difficulty. Indeed, in order to quantitatively prove the convergence of systems of the form (1.20) towards (1.21), one usually needs *small denominator estimates.* They turn out to be extremely difficult to recover in the present context, and in truth they very probably are false. For instance, in the reference situation where $F(u)=u$, equation (1.19) takes the simpler form

$$
i\partial_t\phi_p^\varepsilon(t,x) = H_x\,\phi_p^\varepsilon + \sum_{r,s,q\geq 0}\phi_r^\varepsilon(t,x)\,\phi_q^\varepsilon(t,x)\,\overline{\phi_s^\varepsilon}(t,x)\,\mathrm{e}^{-it(E_q-E_s+E_r-E_p)/\varepsilon}\,\langle\chi_q\,\chi_r\,,\,\chi_s\,\chi_p\rangle.
$$

As a consequence, the averaged system on the ϕ_p's is the same, up to the fact that the sum $\sum_{r,s,q\geq 0}\cdots$ eventually needs to be replaced by $\sum_{r,s,q\geq 0}\mathbf{1}[E_q-E_s+E_r-E_p=0]$. Yet rigorously proving the associated convergence result requires to have has some lower bound on

$$
\frac{\mathbf{1}[E_q-E_s+E_r-E_p\neq 0]}{E_q-E_s+E_r-E_p},
$$

usually Diophantine estimates or like. However, except in the very special case where H_z is the harmonic oscillator for which the E_p's are known and have the value $E_p=2p+1$, such estimates are generally not at hand.

These two difficulties make it necessary to find an alternative route.

A *first* solution is to choose an initial datum which lies in a *definite* energy level. More precisely, one may take an initial datum that lies in the fundamental energy level,

$$\Psi^\varepsilon(0,x,z) = \Psi_0(x,z) = \psi_0(x)\chi_0(z).$$

This is a non-generic situation. In this case, it has been proved in [BMSW] that, for later times, the solution $\Psi^\varepsilon(t,x,z)$ to (1.11) remains of the form

$$\Psi^\varepsilon(t,x,z) = \psi_0^\varepsilon(t,x)\chi_0(z) + \text{small remainder}.$$

As a consequence, the sums entering (1.17), (1.19), and (1.22) actually contain *one single term.* This is the key point. It allows to circumvent all the above mentioned difficulties, and the limiting model is, in that case, a single, nonlinear, Schrödinger equation, of the form

$$i\partial_t\phi_0(t,x) = H_x\phi_0 + \widetilde{F_{\rm av}}(|\phi_0|^2)\,\phi_0.$$

Here, the new, averaged[4] nonlinearity $\widetilde{F_{\rm av}}$ is given, after the averaging procedure, by

$$\widetilde{F_{\rm av}}(u) := \left\langle F\left(u\,|\chi_0(z)|^2\right)\,,\,|\chi_0(z)|^2\right\rangle.$$

This gives a rigorous statement that fully justifies the heuristic limit (1.22) *in that particular case.*

Another possible route is the one we take in this text. It allows to treat the general case.

Here, we definitely want to place ourselves in a situation where $\Psi^\varepsilon(t,x,z)$ contains many energy levels, a generic situation. As we said, explicitly decomposing Ψ^ε over the χ_p's leads to hard small denominators difficulties, and the convergence of the sums entering the expected limiting system (1.22) is far from obvious. For this reason, we adopt the following point of view.

Instead of filtering out the oscillations in (1.11) *after* the projection over the χ_p's, which leads to (1.19), we rather do it *without* projecting. For that reason, we define the new unknown

$$\Phi^\varepsilon(t,x,z) := \exp(+itH_z/\varepsilon)\,\Psi^\varepsilon(t,x,z), \tag{1.23}$$

in analogy with (1.18). It satisfies

$$i\partial_t\Phi^\varepsilon(t,x,z) = H_x\Phi^\varepsilon + \mathrm{e}^{+itH_z/\varepsilon}\,F\left(\left|\mathrm{e}^{+itH_z/\varepsilon}\,\Phi^\varepsilon\right|^2\right)\,\mathrm{e}^{+itH_z/\varepsilon}\,\Phi^\varepsilon. \tag{1.24}$$

In other words, introducing the function

$$\tau \mapsto F(\tau,u) := \mathrm{e}^{+i\tau H_z}\,F\left(\left|\mathrm{e}^{-i\tau H_z}\,u\right|^2\right)\mathrm{e}^{-i\tau H_z}\,u, \tag{1.25}$$

equation (1.24) reads

$$i\partial_t\Phi^\varepsilon(t,x,z) = H_x\Phi^\varepsilon + F\left(\frac{t}{\varepsilon},\Phi^\varepsilon(t)\right). \tag{1.26}$$

[4]the reason for the tilda above $\widetilde{F_{\rm av}}$ is clear below.

This is an infinite dimensional ODE, which is still of the form (1.20).

The key point lies in the observation that for any given function $u(x,z)$ having reasonable Sobolev-like regularity (namely $u \in B_m$ for some $m > 3/2$, see (1.13)), the to-be-averaged function $F(\tau, u)$ is *almost-periodic in time, with values in the Sobolev space* B_m.

The proof of this fact is *not* obvious. The almost-periodicity of $F(\tau, u)$ roughly means that $F(\tau, u)$ has *countably* many frequencies in τ, which in turn translates the fact that the spectrum of H_z is discrete as well: in view of definition (1.25) indeed, the oscillations of $F(\tau, u)$ are only created by those of the propagator $\mathrm{e}^{\pm i\tau H_z}$ (the latter are discrete), appropriately combined with the nonlinearity $F(|u|^2)\,u$ (and almost periodicity usually is stable upon composition with nonlinearities).

The interesting fact about almost-periodic functions is that they do possess a well defined *long time average*, and the formula

$$F_{\mathrm{av}}(u) := \lim_{T\to\infty} \frac{1}{T} \int_0^T F(\tau, u)\, d\tau \tag{1.27}$$

makes sense in B_m. Of course, the convergence rate in (1.27) is $o(1)$ only, contrary to true *periodic* functions, for which the associated remainder term is $O(1/T)$: the point is, the long time average *exists*, *beyond* any "small denominator" consideration or like.

In any circumstance, the limiting equation for $\Phi = \lim \Phi^\varepsilon$ now naturally reads

$$i\partial_t \Phi(t,x,z) = H_x \Phi + F_{\mathrm{av}}(\Phi). \tag{1.28}$$

Equation (1.28) gives a rigorous statement corresponding to the heuristic limit (1.22) discussed before. Note that the observation according to which we are here dealing with almost-periodic functions (hence the possibility to average in time), with values in a good Sobolev space (hence the possibility to do nonlinear analysis), are the two crucial ingredients in the present study.

To summarize, our main result is the following

Main Theorem

Take $m > 3/2$. Take a function $\Psi_0(x,z)$ having the Sobolev-like regularity,

$$\Psi_0(x,z) \in B_m := \left\{ u \in L^2(\mathbb{R}^3),\ \text{s.t.}\ H_x^{m/2} u \in L^2(\mathbb{R}^3)\ \text{and}\ H_z^{m/2} u \in L^2(\mathbb{R}^3) \right\}.$$

Define $\Psi^\varepsilon(t,x,z)$ as the solution to

$$i\partial_t \Psi^\varepsilon = H_x \Psi^\varepsilon + \frac{1}{\varepsilon} H_z \Psi^\varepsilon + F\left(|\Psi^\varepsilon|^2\right)\, \Psi^\varepsilon, \quad \Psi^\varepsilon(0,x,z) = \Psi_0(x,z).$$

Equivalently, define the filtered function $\Phi^\varepsilon(t,x,z) = \exp(+itH_z/\varepsilon)\, \Psi^\varepsilon$ *as the solution to*

$$i\partial_t \Phi^\varepsilon = F\left(\frac{t}{\varepsilon}, \Phi^\varepsilon\right), \quad \Phi^\varepsilon(0,x,z) = \Psi_0(x,z),$$

where $F(\tau, u) = \mathrm{e}^{+i\tau H_z}\, F\left(\left|\mathrm{e}^{-i\tau H_z}\right|^2\right)\, \mathrm{e}^{-i\tau H_z} u$. *Last, define $\Phi(t,x,z)$ as the solution to the averaged equation*

$$i\partial_t \Phi = H_x \Phi + F_{\mathrm{av}}(\Phi), \quad \Phi(0,x,z) = \Psi_0(x,z),$$

where $F_{\mathrm{av}}(u) = \lim_{T\to\infty} (1/T) \int_0^T F(\tau, u) d\tau$ *in* B_m. *Then, the following holds*

(i) There is a $T_0 > 0$, *depending only on* $\|\Psi_0\|_{B_m}$ *and on the nonlinear function* F, *such that* $\Psi^\varepsilon(t)$, $\Phi^\varepsilon(t)$, *and* $\Phi(t)$ *exist and possess the smoothness* $C^0([0,T_0]; B_m)$, *independently of* ε. *Incidentally,* B_m *is a Hilbert space and an algebra, when endowed with either of the norms (1.14) or (1.15).*

(ii) The following convergence holds

$$\|\Phi^\varepsilon(t) - \Phi(t)\|_{C^0([0,T_0];B_m)} \underset{\varepsilon\to 0}{\longrightarrow} 0.$$

(iii) The solution $\Phi(t)$ *to the averaged system has the following conserved quantities*

$$\|\Phi(t)\|_{L^2(\mathbb{R}^3)} = \mathrm{const}, \quad \langle \Phi(t), H_z \Phi(t) \rangle_{L^2(\mathbb{R}^3)} = \mathrm{const},$$

$$\left\langle H_x^{1/2}\,\Phi(t)\,,\, H_x^{1/2}\Phi(t)\right\rangle^2_{L^2(\mathbb{R}^3)} + \int_{\mathbb{R}^3} G_{\mathrm{av}}(\Phi(t))\, dx\, dz = \mathrm{const},$$

where $G_{\mathrm{av}}(\Psi)$ *is defined, for any* $\Psi \in B_m$, *as*

$$G_{\mathrm{av}}(\Psi) := \lim_{T\to\infty} \frac{1}{T} \int_0^T G\left(\left|\mathrm{e}^{-i\tau H_z}\,\Psi\right|^2\right) d\tau, \qquad \textit{and, } G(u) := \int_0^u F(v)\, dv.$$

Remarks on the Main Theorem:

• Obviously, upon projecting Φ on the χ_p's, system (1.26) may be seen as an infinite system of coupled nonlinear Schrödinger equations, involving the quantities $\phi_p(t,x) := \langle \Phi, \chi_p \rangle$. The underlying system coincides with the formally expected system (1.22).

• Needless to say, our main Theorem gives, as a particular case, the results obtained in [BMSW] when Ψ_0 is parallel with χ_0. Yet the (not to be improved) $o(1)$ convergence rate of our Theorem does not allow to recover the better convergence rates obtained in [BMSW] in this special situation.

• Note also that the above Theorem completely describes the asymptotic behavior of Ψ^ε, namely $\Psi^\varepsilon(t,x,z) \sim \exp(-itH_z/\varepsilon)\,\Phi(t,x,z)$ as $\varepsilon \to 0$.

• The reader's attention is drawn to the fact that the averaged system $i\partial_t \Phi = H_x \Phi + F_{\mathrm{av}}(\Phi)$ still is posed in the three dimensional space $\mathbb{R}^3$. It however entails a trivial dynamics in the vertical, z direction, which only plays the role of a parameter. Technically, factorizing out this z dependence is done upon projecting the averaged system over the basis of the χ_p's.

• Point (iii) of the Theorem gives conservation of mass, and conservation of the energy H_z in z, a natural conservation since the dynamics of Φ eventually occurs along the x direction only. It also gives the conservation of *total* energy in x. The latter may be used when the nonlinearity F has definite *sign* properties, so as to transform the above local-in-time convergence results, into global-in-time ones.

2 Sketch of proofs

2.1 Sobolev scale adapted to H_x and H_z

A first key ingredient of the present study lies in identifying the Sobolev scale adapted to H_x and H_z. Specifically, given any real number $\ell \geq 0$, we need to completely identify the norm

$$\|u\|_{B_\ell}^2 := \|u\|_{L^2(\mathbb{R}^3)}^2 + \|H_x^{\ell/2} u\|_{L^2(\mathbb{R}^3)}^2 + \|H_z^{\ell/2} u\|_{L^2(\mathbb{R}^3)}^2,$$
$$:= \|u\|_{L^2(\mathbb{R}^3)}^2 + \left\| (-\Delta_x + V(x))^{\ell/2} \; u \right\|_{L^2(\mathbb{R}^3)}^2 + \left\| (-\Delta_z + V_{\mathrm{c}}(z))^{\ell/2} \; u \right\|_{L^2(\mathbb{R}^3)}^2,$$

whenever u is smooth enough.

Our main result in this section is the following

Proposition 1 *Let $\ell \geq 0$ be a real number. Recall $H_x = -\Delta_x + V(x)$ and $H_z = -\Delta_z + V_{\mathrm{c}}(z)$. Then, the following equivalence holds,*

$$\|u\|_{B_\ell}^2 \sim \|u\|_{L^2(\mathbb{R}^3)}^2 + \| \, (-\Delta_x)^{\ell/2} \; u\|_{L^2(\mathbb{R}^3)}^2 + \| \, (-\Delta_z)^{\ell/2} \; u\|_{L^2(\mathbb{R}^3)}^2$$
$$+ \|V(x)^{\ell/2} \, u\|_{L^2(\mathbb{R}^3)}^2 + \|V_{\mathrm{c}}(z)^{\ell/2} \, u\|_{L^2(\mathbb{R}^3)}^2, \tag{2.1}$$

where, the symbol $\sim$ means that there are constants $c_0 > 0$ and $c_1 > 0$ such that $c_0 \times$ (r.h.s. of (2.1)) $\leq$ (l.h.s. of (2.1)) $\leq c_1 \times$ (r.h.s. of (2.1)), independently of u.

Remarks on Proposition 1:

• The identification of $\|u\|_{B_\ell}$ is a technically delicate, yet absolutely crucial step in the present paper. Indeed, the only uniform bound at hand on Ψ^ε, solution to (1.11), reads

$$\|\Psi^\varepsilon(t,x,z)\|_{L^2(\mathbb{R}^3)} + \|H_x^{m/2}\Psi^\varepsilon(t,x,z)\|_{L^2(\mathbb{R}^3)} + \|H_z^{m/2}\Psi^\varepsilon(t,x,z)\|_{L^2(\mathbb{R}^3)} = O(1),$$

on some non-trivial time interval $t \in [0, T_0]$, whenever the initial datum Ψ_0 belongs to B_m ($m > 3/2$). All other energy estimates give rise to commutators, hence diverging factors of the order $O(1/\varepsilon)$, due to the fast factor H_z/ε in (1.11): they only give access to bounds of the size $O(1/\varepsilon)$ as well, a useless information.

• The proof of Proposition 1 is *not* direct. Our proof uses Weyl-Hörmander's calculus, *i.e.* an appropriate pseudo-differential calculus adapted to the symbol $\xi^2 + \zeta^2 + V(x) + V_{\mathrm{c}}(z)$, see Bony and Chemin's work [BC]. This is also the route chosen by B. Helffer in the earlier work [He]: in this paper, B. Helffer completely identifies the Sobolev scale associated with the harmonic oscillator $-\Delta_z + z^2$, and the analogous of Proposition 1 is proved there in this case. We stress that even the identification of the norm $\|(1 - \Delta_z + z^2)^\ell \, u\|_{L^2}$ with the obvious $\|u\|_{L^2} + \left\|(-\Delta_z)^\ell \; u\right\|_{L^2} + \|z^{2\ell} \, u\|_{L^2}$ is *not* an easy result: it readily requires developping a pseudo-differential calculus that is adapted to the symbol $1 + \zeta^2 + z^2$.

• A pedestrian proof of Proposition 1, directly using commutators of both operators $-\Delta_z$ and $V_{\mathrm{c}}(z)$, and similarly in the x variable, probably is *out of reach*, even for integer values of ℓ. Indeed, such an analysis anyhow fails when dealing with factors of the form

$$\left\|(-\Delta_z)^{(\ell-k)/2} \; V_{\mathrm{c}}(z)^{k/2} \, u\right\|_{L^2(\mathbb{R}^3)} \quad \text{or} \quad \left\|V_{\mathrm{c}}(z)^{k/2} \; (-\Delta_z)^{(\ell-k)/2} \; u\right\|_{L^2(\mathbb{R}^3)},$$

whenever $0 \leq k \leq \ell$, and when it comes to trying to control such terms with the help of the mere term

$$\|u\|_{L^2(\mathbb{R}^3)} + \left\|V_{\mathrm{c}}(z)^{\ell/2}\right\|_{L^2(\mathbb{R}^3)} + \left\|(-\Delta_z)^{\ell/2} \; u\right\|_{L^2(\mathbb{R}^3)}.$$

• Note in passing that our identification of $\|u\|_{B_\ell}$ uses the fact that $V_{\rm c}(z)$ is confining, see (1.2). Even more, a crucial role is played by the reinforced assumptions (1.4) through (1.6), according to which $V_{\rm c}(z)$ behaves like a symbol at infinity in z, whose growth is at most polynomial. The similar assumptions (1.7) as well as (1.8) through (1.10) are used in the x direction. Note however that, would $V_{\rm c}$ be uniformly bounded together with all its derivatives instead of being confining, the results below would hold just the same, the proofs being actually simpler.

Sketch of proof of Proposition 1.

First step: Preliminary reduction and Weyl-Hörmander calculus

The key tool we use to prove the equivalence (2.1) is the Weyl-Hörmander calculus, see *e.g.* [BC]. Let us comment on that point, keeping the discussion at a rather informal level for the time being.

In terms of *symbols* (in the sense of pseudo-differential calculus, for *some* pseudo-differential calculus to be precised below), assertion (2.1) is fairly natural. Indeed, the principal symbol of $1 + H_x^\ell + H_z^\ell$ is

$$\sigma\left(1 + H_x^\ell + H_z^\ell\right)(x, z, \xi, \zeta) \equiv 1 + \left[\xi^2 + V(x)\right]^\ell + \left[\zeta^2 + V_{\rm c}(z)\right]^\ell,$$

where ξ and ζ are the Fourier variables associated with x resp. z, while the principal symbol of $1 + (-\Delta_x)^\ell + (-\Delta_z)^\ell + V(x)^\ell + V_{\rm c}(z)^\ell$ is

$$\sigma\left(1 + D_x^{2\ell} + D_z^{2\ell} + V(x)^\ell + V_{\rm c}(z)^\ell\right)(x, z, \xi, \zeta) \equiv 1 + \xi^{2\ell} + \zeta^{2\ell} + V(x)^\ell + V_{\rm c}(z)^\ell.$$

Formally exploiting the identification of the operators with their associated principal symbols, the whole equivalence (2.1) eventually (and informally) reduces to the existence of positive, universal constants c_0 and c_1 such that

$$c_0 \le \frac{1 + [\xi^2 + V(x)]^\ell + [\zeta^2 + V_{\rm c}(z)]^\ell}{1 + \xi^{2\ell} + \zeta^{2\ell} + V(x)^\ell + V_{\rm c}(z)^\ell} \le c_1, \tag{2.2}$$

independently of $(x, z, \xi, \zeta) \in \mathbb{R}^3 \times \mathbb{R}^3$. The point is, passing from the equivalence between *symbols* (2.2) to the equivalence between *norms* (2.1), one needs to have a proper quantization of symbols, hence a proper pseudodifferential calculus. In other words, one needs appropriate *weights* together with appropriate *metrics* to deduce (2.1) from (2.2) using a pseudo-differential machinery.

Now, the whole difficulty lies in the fact that the standard pseudodifferential calculus, based on the standard metrics

$$dx^2 + dz^2 + \frac{d\xi^2 + d\zeta^2}{1 + \xi^2 + \zeta^2}$$

can only give access to usual Sobolev-like norms, where only powers of $-\Delta_x$, $-\Delta_z$ are kept track of, or equivalently, one only takes into account powers of ξ^2 and ζ^2 as $|\xi|$ and/or $|\zeta|$ go to infinity: however, going from (2.2) to (2.1) requires not only counting powers of $-\Delta_x$, $-\Delta_z$ (*i.e.* powers of ξ^2 and ζ^2), but also powers of $V(x)$ and $V_{\rm c}(z)$ as $|x|$ and $|z|$ go to infinity; recall indeed that $V_{\rm c}$ and V are assumed confining, a key difficulty in the present perspective.

This is the reason why we need to consider an appropriate metric that keeps track of both aspects, and eventually develop the associated pseudo-differential machinery, based on the Weyl-Hörmander calculus.

Our proof of the equivalence (2.1 actually follows ideas developed by Bony and Chemin in [BC], and more recently ideas by Helffer and Nier [HN]. The whole point lies in defining the weight

$$M(x,z,\xi,\zeta) := \sqrt{1+\xi^2+\zeta^2+V(x)+V_{\rm c}(z)}, \tag{2.3}$$

and the associated metric

$$g(x,z,\xi,\zeta) := dx^2+dz^2+\frac{d\xi^2+d\zeta^2}{M^2(x,z,\xi,\zeta)}, \tag{2.4}$$

meaning that for any $(x',z',\xi',\zeta')\in\mathbb{R}^3\times\mathbb{R}^3$, we set $g(x,z,\xi,\zeta)(x',z',\xi',\zeta')=(x')^2+(z')^2+[(\xi')^2+(\zeta')^2]/M^2(x,z,\xi,\zeta)$. Choosing to work within the metric g equivalently means that for any given $\ell\in\mathbb{R}$, we shall deal with the class $S(M^\ell,g)$ of symbols $a(x,z,\xi,\zeta)\in C^\infty(\mathbb{R}^3\times\mathbb{R}^3)$ such that

$$\begin{aligned}&\forall\alpha,\beta\in\mathbb{N}^3,\quad \exists C_{\alpha,\beta}>0,\quad \forall(x,z,\xi,\zeta)\in\mathbb{R}^3\times\mathbb{R}^3,\\ &\qquad\qquad \left|\partial^\alpha_{x,z}\,\partial^\beta_{\xi,\zeta}a(x,z,\xi,\zeta)\right|\le C_{\alpha,\beta}\,M(x,z,\xi,\zeta)^{\ell-|\beta|}.\end{aligned} \tag{2.5}$$

The idea of using this class of symbols, *i.e.* this weight function and this metric, is actually borrowed from [HN].

It turns out that assumptions (1.4) through (1.6), as well as (1.8) through (1.10), ensure the metric g is *slow*, *temperate*, and it satisfies the *uncertainty principle*. These assumptions also ensure that for any $\ell\in\mathbb{R}$, the weight M^ℓ is *admissible* for the metric g. Lastly, we stress that the value of the *gain* in the present calculus is, following Hörmander [Ho], $M(x,z,\xi,\zeta)$

Now, given the metric g and the weight M^ℓ, to any symbol a in the class $S(M^\ell,g)$, Weyl-Hörmander calculus associates the operator

$$\begin{aligned}&u\in\mathcal{S}(\mathbb{R}^3)\mapsto a^{\rm w}\,u\in\mathcal{S}(\mathbb{R}^3)\text{ defined as}\\ &\left(a^{\rm w}\,u\right)(x)=\int_{\mathbb{R}^6}{\rm e}^{i(x-x')\cdot\xi+i(z-z')\cdot\zeta}\,a\left(\frac{x+x'}{2},\frac{z+z'}{2},\xi,\zeta\right)u(x',z')\,dx'\,dz'.\end{aligned} \tag{2.6}$$

and we write $a^{\rm w}\in{\rm Op}(S(M^\ell,g))$. In this language, we have

$$1-\Delta_x-\Delta_z+V(x)+V_{\rm c}(z)=\big(1+\xi^2+\zeta^2+V(x)+V_{\rm c}(z)\big)^{\rm w}\in{\rm Op}\,S(M^2,g).$$

Proving (2.1) now roughly reduces to proving the equivalence, whenever $\ell\in\mathbb{R}$,

$$\left\|\left(M^{\rm w}\right)^\ell u\right\|_{L^2(\mathbb{R}^3)}\sim\left\|\left(M^\ell\right)^{\rm w}u\right\|_{L^2(\mathbb{R}^3)} \tag{2.7}$$

Second step: functional calculus and Sobolev spaces based on the metric g

Now, the proof of (2.7) basically reformulates as: given the function $f(u) \equiv u^\ell$, and given the operator M^{w}, can one identify the operator $f(M^{\mathrm{w}})$ and even more specifically its symbol?

The answer to this question essentially requires two tools.

Firstly, we need to identify the operator M^{w} as a *self-adjoint* operator *on reasonably well defined functional spaces.* At this level of the analysis, we need to use spaces introduced by Bony and Chemin in [BC]. Let $\det(g(X))$ denotes the determinant of the quadratic form $g(X) = g(x, z, \xi, \zeta)$. Whenever $\ell \geq 0$, the work by Bony and Chemin allows to define the Sobolev space associated with the weight $M^\ell(X)$, and denoted by $H(M^\ell, g)$, as the set of functions $u = u(x, z)$ such that

$$\|u\|^2_{H(M^\ell,g)} \equiv \int_{\mathbb{R}^6} M^{2\ell}(X)\ \|\phi_X^{\mathrm{w}}\, u\|^2_{L^2(\mathbb{R}^3)}\ |\det(g(X))|^{1/2}\ dX < \infty, \tag{2.8}$$

where the operator ϕ_X^{w} conveniently microlocalizes u around the point X of phase-space, while the collection of functions ϕ_X provide a partition of unity. The set $H(M^\ell, g)$ clearly extends the usual definition of the standard Sobolev spaces $H^s(\mathbb{R}^3)$ ($s \in \mathbb{R}$). The natural orthogonality property ensures that definition (2.8) does not depend on the chosen partition of unity ϕ_X. With the above notation, we have

$$\forall \ell \leq \ell', \quad \mathcal{S}(\mathbb{R}^3) \subset H(M^\ell, g) \subset H(M^{\ell'}, g) \subset \mathcal{S}'(\mathbb{R}^3),$$

and the key point is the following result:

$$\text{The operator } \left(M^\ell\right)^{\mathrm{w}} \text{ with domain } H(M^\ell, g) \text{ is self-adjoint on } L^2. \tag{2.9}$$

Secondly, one needs to identify the operator $f(M^{\mathrm{w}}) = (M^{\mathrm{w}})^\ell$. Once the above self-adjointess is at hand, the idea lies in using the Helffer-Sjöstrand formula, according to which

$$f(M^{\mathrm{w}}) = \frac{1}{2\pi} \int_{\mathbb{C}} \frac{\partial \widetilde{f}}{\partial \overline{\lambda}}(\lambda)\ [M^{\mathrm{w}} - \lambda]^{-1}\ d\lambda \wedge d\overline{\lambda},$$

where $\lambda \in \mathbb{C}$, the measure $d\lambda \wedge d\overline{\lambda}$ is the standard 2-dimensional volume in $\mathbb{C}$, and $\widetilde{f}(\lambda)$ denotes an almost-analytic extension of f over $\mathbb{C}$. Therefore, studying $f(M^{\mathrm{w}})$ reduces to studying the resolvent of M^{w} (the similar argument is used in the functional calculus by Helffer and Robert [HR] in the context of the standard metrics $dx^2 + d\xi^2/(1+\xi^2)$). Now the key point in that direction is the folowing result, borrowed from [HN]:

$$\text{For any } \lambda \geq 0 \text{ the operator } \left[\left(M^2\right)^{\mathrm{w}} + \lambda\right]^{-1} \text{ belongs to } \operatorname{Op} S(M^{-2}, g). \tag{2.10}$$

This information, in conjunction with (a variant of) the Helffer-Sjöstrand formula, eventually allows to completely identify $f(M^{\mathrm{w}})$, and to complete the proof of Proposition 1. $\square$

2.2 Uniform bounds for Ψ^ε

The following Proposition now comes as an immediate consequence of Proposition 1.

Proposition 2 *Take a real number $\ell > 3/2$. Define the Sobolev space B_ℓ as the completion of the set of smooth functions $u(x,z)$ under the norm*

$$\|u\|_{B_\ell}^2 := \|u\|_{H^\ell(\mathbb{R}^3)}^2 + \|V(x)^{\ell/2}\, u\|_{L^2(\mathbb{R}^3)}^2 + \|V_{\rm c}(z)^{\ell/2}\, u\|_{L^2(\mathbb{R}^3)}^2.$$

Then, B_ℓ is a Hilbert space and $B_\ell \subset L^\infty(\mathbb{R}^3)$ continuously. Moreover, the following property holds true. Take any C^∞ nonlinear function f satisfying $f(0) = 0$. Then, the mapping $u \in B_\ell \mapsto f(u) \in B_\ell$ is well-defined and locally Lipschitz.

As an immediate corollary, we also have the following non-trivial uniform existence result.

Corollary 3 *Take a real number $m > 3/2$. Take an initial datum $\Psi_0(x,z)$ in (1.12) such that $\Psi_0(x,z) \in B_m$. Then, there is a $T_0 > 0$, independent of ε and only depending on $\|\Psi_0\|_{B_m}$ and the nonlinear function F, such that the nonlinear Schrödinger equation (1.11) with initial datum Ψ_0 possesses a unique solution $\Psi^\varepsilon(t,x,z)$ with the smoothness $\Psi^\varepsilon(t,x,z) \in C^0([0,T_0], B_m)$.*

2.3 Almost periodic functions with values in B_m

The key fact about almost-periodic functions $\Theta(t)$ (with values in $\mathbb{R}$, say), is the existence of their long time average $\Theta_{\rm av} := \lim_{T\to\infty} \frac{1}{T}\int_0^T \Theta(\tau)\, d\tau$, and the point is that no small-divisors estimate (or like) is needed to define such averages. In some sense, the small divisor estimates are encoded in the very definition of almost-periodic functions. This is the very reason for our introduction of the following (standard) definition of almost periodic functions with values in B_m.

Definition and Proposition 4 *Let $\ell \geq 0$. A function $\Theta(\tau) : \mathbb{R} \mapsto B_\ell$, $\Theta(\tau) \in C^0(\mathbb{R}; B_\ell)$, is called* almost-periodic*, and we note $\Theta(\tau) \in \mathrm{AP}(\mathbb{R}, B_\ell)$, whenever the set of translates*

$$\{t \mapsto \Theta(\tau + h),\ h \in \mathbb{R}\}$$

has compact closure in the norm $L^\infty(\mathbb{R}, B_\ell)$. Equivalently, $\Theta(\tau) \in \mathrm{AP}(\mathbb{R}, B_\ell)$ if and only if *$\Theta(\tau)$ is the strong limit of trigonometric polynomials, i.e. for any $\delta > 0$, there exists a trigonometric polynomial*

$$\Theta^\delta(\tau) = \sum_{n=1}^{N_\delta} \theta_{n,\delta} \exp(i\lambda_{n,\delta}\tau), \quad \text{such that} \quad \sup_{\tau\in\mathbb{R}} \|\Theta(\tau) - \Theta^\delta(\tau)\|_{B_\ell} \leq \delta,$$

where the $\theta_{n,\delta}$'s belong to B_ℓ, the $\lambda_{n,\delta}$'s belong to $\mathbb{R}$, and N_δ is some finite integer.

Remark. The above definition, namely the precompactness criterion, is usually called Bochner's criterion for almost-periodicity. The equivalence with being the uniform limit of trigonometric polynomials is a standard (and crucial) fact about almost-periodic functions. It is proved, e.g., in [LZ], and in any textbook about almost-periodic functions. □

With this definition, it turns out that one may do *Fourier analysis* on almost periodic functions. In particular, the long-time average of $\Theta(\tau)$ (which plays the role of the mean mode in standard Fourier analysis), is well defined, as shown by the

Proposition 5 *Let $\ell \geq 0$ and take $\Theta(\tau) \in \mathrm{AP}(\mathbb{R}, B_\ell)$. Then, the following strong limit exists in B_ℓ,*

$$\Theta_{\mathrm{av}} := \lim_{T\to\infty} \frac{1}{T} \int_0^T \Theta(\tau)\, d\tau.$$

More generally, for any $\lambda \in \mathbb{R}$, the Fourier-like coefficient

$$\widehat{\Theta}(\lambda) := \lim_{T\to\infty} \frac{1}{T} \int_0^T \Theta(\tau)\, \exp(-i\lambda\tau)\, d\tau,$$

is well-defined in B_ℓ, and $\widehat{\Theta}(\lambda)$ is non-zero for at most countably many values of λ.

We now turn to drawing the consequences of Proposition 5 that are of interest in our context. Our first result in that direction is the

Proposition 6 *Take $\ell > 3/2$ and take $\Theta(x, z) \in B_\ell$. Under these circumstances, the function*

$$F(\tau, \Theta)\ :\ \tau \mapsto \mathrm{e}^{+i\tau H_z}\, F\left(\left|\mathrm{e}^{-i\tau H_z}\,\Theta\right|^2\right)\, \mathrm{e}^{-i\tau H_z}\,\Theta$$

belongs to $\mathrm{AP}(\mathbb{R}; B_\ell)$. Hence, one may define the long time average as the limit in B_ℓ,

$$F_{\mathrm{av}}(\Theta) := \lim_{T\to\infty} \frac{1}{T} \int_0^T F(\tau, \Theta)\, d\tau.$$

Besides the function $\Theta \mapsto F_{\mathrm{av}}(\Theta)$ is locally Lipschitz in B_ℓ.

Next, we have all the necessary tools that allow to perform the natural nonlinear analysis of the averaged model $i\partial_t\Phi = H_x\Phi + F_{\mathrm{av}}(\Phi)$. Recall that the latter is to be derived from the oscillatory equation $i\partial_t\Psi^\varepsilon = H_x\Psi^\varepsilon + \varepsilon^{-1}H_z\Psi^\varepsilon + F(|\Psi^\varepsilon|^2)\,\Psi^\varepsilon$ in the present paper.

Proposition 7 *Take $m > 3/2$ and $\Psi_0 \in B_m$. Then, there is a $T_0 > 0$, only depending on $\|\Psi_0\|_{B_m}$ and the nonlinear function F, such that the solution Φ to the averaged equation*

$$i\partial_t\Phi = H_x\Phi + F_{\mathrm{av}}(\Phi), \qquad \Phi(0, x, z) = \Psi_0(x, z),$$

exists and is unique in $C^0([0, T_0]; B_m)$. Even more, it satisfies the conservation laws announced in the statement of the Main Theorem.

2.4 Sketch of proof of the Main Theorem

The key point is to perform the averaging procedure in time, i.e. to prove point (ii) of the main Theorem. As a very preliminary step, we mention that an easy regularising procedure allows to only prove a reduced version of the result, for initial data Ψ_0 that possess the improved regularity $\Psi_0 \in B_{m+2}$ (instead of B_m as in our main Theorem).

First step: reduction of the proof

We follow the strategy developed in [SV] for finite-dimensional ODE's (see [BCD] for an adaptation in the infinite-dimensional situation). The filtered function Φ^ε satisfies

$$i\partial_t \Phi^\varepsilon = H_x\,\Phi^\varepsilon + F(t/\varepsilon, \Phi^\varepsilon), \qquad \Phi^\varepsilon(0) = \Psi_0,$$
$$\text{where} \quad F(t,\Psi) := \mathrm{e}^{+itH_z}\,F(|\mathrm{e}^{-itH_z}\,\Psi|^2)\,\mathrm{e}^{-itH_z}\,\Psi \text{ is almost periodic.} \tag{2.11}$$

We wish to estimate the difference with the averaged system

$$i\partial_t \Phi = H_x\,\Phi + F_{\mathrm{av}}(\Phi), \qquad \Phi(0) = \Psi_0,$$
$$\text{where} \quad F_0(\Psi) := \lim_{T\to\infty} \frac{1}{T}\int_0^T F(\tau,\Psi)\,d\tau. \tag{2.12}$$

In order to do so, we choose a (large) time $T(\varepsilon)$ such that $T(\varepsilon) = o(1/\varepsilon)$ as $\varepsilon \to 0$. The "good" choice for $T(\varepsilon)$ is made precise below - see (2.21). Associated with $T(\varepsilon)$, we introduce the auxiliary solution $\widetilde{\Phi}^\varepsilon$ to

$$i\partial_t \widetilde{\Phi}^\varepsilon = H_x\,\widetilde{\Phi}^\varepsilon + \widetilde{F}_\varepsilon(t/\varepsilon, \widetilde{\Phi}^\varepsilon), \qquad \widetilde{\Phi}^\varepsilon(0) = \Psi_0,$$
$$\text{where} \quad \widetilde{F}_\varepsilon(t,\Psi) := \frac{1}{T(\varepsilon)}\int_t^{t+T(\varepsilon)} F(s,\Psi)\,ds. \tag{2.13}$$

Our strategy is to successively prove that the two terms $\Phi^\varepsilon - \widetilde{\Phi}^\varepsilon$ and $\widetilde{\Phi}^\varepsilon - \Phi$ go to zero in $C^0([0,T_0];B_m)$. As we shall see, each term requires specific arguments.

In any circumstance, it is easily proved that there exists a T_0, independent of ε, such that the solution $\widetilde{\Phi}^\varepsilon$ to (2.13) exists, is unique, and has the regularity $C^0([0,T_0],B_{m+2})$. Even more, there exists a common upper-bound $M > 0$ such that

$$\sup_{0<\varepsilon<1} \Big[\big\|\Phi^\varepsilon\big\|_{C^0([0,T_0];B_{m+2})} + \big\|\widetilde{\Phi}^\varepsilon\big\|_{C^0([0,T_0];B_{m+2})} + \big\|\Phi\big\|_{C^0([0,T_0];B_{m+2})} \Big] \le M. \tag{2.14}$$

Second step: estimating $\widetilde{\Phi}^\varepsilon - \Phi$

For any $u \in B_{m+2}$, we introduce the convergence rate

$$\delta(\varepsilon,u) := \sup_{0\le t\le 2T_0/\varepsilon} \Big\| \frac{\varepsilon}{2T_0}\int_0^t \Big[F(s,u) - F_{\mathrm{av}}(u)\Big]\,ds \Big\|_{B_m}. \tag{2.15}$$

Note that δ measures a convergence rate *with loss* of smoothness (loss of "two derivatives"). On top of that, take M is as in (2.14), and introduce the uniform convergence rate

$$\delta_M(\varepsilon) := \sup_{\|u\|_{B_{m+2}}\le M} \delta(\varepsilon, v). \tag{2.16}$$

Then, we can easily prove $\delta_M(\varepsilon) \underset{\varepsilon\to 0}{\longrightarrow} 0$. Besides, for any $0 \le t \le T_0$, we may establish

$$\sup_{\|u\|_{B_{m+2}}\le M} \Big\|\widetilde{F}_\varepsilon(t/\varepsilon,u) - F_{\mathrm{av}}(u)\Big\|_{B_m} \le 2\,T_0\,\frac{\delta_M(\varepsilon)}{\varepsilon\,T(\varepsilon)}. \tag{2.17}$$

Now, the following estimate is easily deduced for $t \in [0, T_0]$, where $C > 0$ only depends on T_0, M, F.

$$\forall 0 \leq t \leq T_0, \qquad \|\widetilde{\Phi}^\varepsilon(t) - \Phi(t)\|_{B_m} \leq C \, \frac{\delta_M(\varepsilon)}{\varepsilon \, T(\varepsilon)}. \tag{2.18}$$

Third step: estimating $\Phi^\varepsilon - \widetilde{\Phi}^\varepsilon$

This estimate is more delicate than the previous one. Introducing the difference $\Delta^\varepsilon(t) := \Phi^\varepsilon(t) - \widetilde{\Phi}^\varepsilon(t)$, we readily have for $0 \leq t \leq T_0$,

$$\begin{aligned}
\|\Delta^\varepsilon(t)\|_{B_m} &\leq \left\| \int_0^t \mathrm{e}^{i(t-s)H_x} \left[F(s/\varepsilon, \Phi^\varepsilon(s)) - \widetilde{F}_\varepsilon(s/\varepsilon, \widetilde{\Phi}^\varepsilon(s)) \right] ds \right\|_{B_m} \\
&\leq C(F, M) \int_0^t \|\Delta^\varepsilon(s)\|_{B_m} \, ds \\
&\quad + \left\| \int_0^t \mathrm{e}^{i(t-s)H_x} \left[F(s/\varepsilon, \Phi^\varepsilon(s)) - \widetilde{F}_\varepsilon(s/\varepsilon, \Phi^\varepsilon(s)) \right] ds \right\|_{B_m}.
\end{aligned} \tag{2.19}$$

We are thus led to estimating the second term on the right-hand-side of (2.19). To do so, we write, whenever $0 \leq t \leq T_0$,

$$\begin{aligned}
&\int_0^t \mathrm{e}^{i(t-s)H_x} \left[F(s/\varepsilon, \Phi^\varepsilon(s)) - \widetilde{F}_\varepsilon(s/\varepsilon, \Phi^\varepsilon(s)) \right] ds \\
&= \int_0^t \mathrm{e}^{i(t-s)H_x} F\left(\frac{s}{\varepsilon}, \Phi^\varepsilon(s)\right) ds - \int_0^1 \int_0^t \mathrm{e}^{i(t-s)H_x} F\left(\frac{s + \varepsilon \, T(\varepsilon) \, u}{\varepsilon}, \Phi^\varepsilon(s)\right) ds \, du \\
&= \int_0^t \mathrm{e}^{i(t-s)H_x} F\left(\frac{s}{\varepsilon}, \Phi^\varepsilon(s)\right) ds \\
&\qquad - \int_0^1 \int_0^t \mathrm{e}^{i(t-s)H_x} F\left(\frac{s + \varepsilon \, T(\varepsilon) \, u}{\varepsilon}, \Phi^\varepsilon(s + \varepsilon \, T(\varepsilon) u)\right) ds \, du + R_1^\varepsilon \\
&= \int_0^t \mathrm{e}^{i(t-s)H_x} F\left(\frac{s}{\varepsilon}, \Phi^\varepsilon(s)\right) ds - \int_0^1 \int_{\varepsilon \, T(\varepsilon) \, u}^{t + \varepsilon \, T(\varepsilon) \, u} \mathrm{e}^{i(t-s)H_x} F\left(\frac{s}{\varepsilon}, \Phi^\varepsilon(s)\right) ds \, du \\
&\qquad + R_1^\varepsilon + R_2^\varepsilon \\
&= \int_0^t \mathrm{e}^{i(t-s)H_x} F\left(\frac{s}{\varepsilon}, \Phi^\varepsilon(s)\right) ds - \int_0^1 \int_0^t \mathrm{e}^{i(t-s)H_x} F\left(\frac{s}{\varepsilon}, \Phi^\varepsilon(s)\right) ds \, du \\
&\qquad + R_1^\varepsilon + R_2^\varepsilon + R_3^\varepsilon.
\end{aligned}$$

Eventually, we have established

$$\int_0^t \mathrm{e}^{i(t-s)H_x} \left[F(s/\varepsilon, \Phi^\varepsilon(s)) - \widetilde{F}_\varepsilon(s/\varepsilon, \Phi^\varepsilon(s)) \right] ds = R_1^\varepsilon + R_2^\varepsilon + R_3^\varepsilon,$$

and we have postponed the task of estimating the remainders R_1^ε, R_2^ε, and R_3^ε, for the moment. The third remainder R_3^ε is easily estimated by

$$\begin{aligned}
\|R_3^\varepsilon\|_{B_m} &\leq \int_0^{\varepsilon \, T(\varepsilon)} \left\| F\left(\frac{s}{\varepsilon}, \Phi^\varepsilon(s)\right) \right\|_{B_m} ds + \int_t^{t + \varepsilon \, T(\varepsilon)} \left\| F\left(\frac{s}{\varepsilon}, \Phi^\varepsilon(s)\right) \right\|_{B_m} ds \\
&\leq C(F, M) \, \varepsilon \, T(\varepsilon).
\end{aligned}$$

Concerning R_1^ε, we write

$$\|R_1^\varepsilon\|_{B_m} \leq C(F,M)\ \varepsilon\, T(\varepsilon)\ \|\partial_t \Phi^\varepsilon(s)\|_{C^0([0,T_0+\varepsilon\, T(\varepsilon)];B_m)}.$$

Yet, the equation $i\partial_t \Phi^\varepsilon = H_x \Phi^\varepsilon + F(s/\varepsilon, \Phi^\varepsilon)$, together with the bounds at hand for Φ^ε clearly imply $\|\partial_t \Phi^\varepsilon(s)\|_{C^0([0,T_0+\varepsilon\, T(\varepsilon)];B_m)} \leq C$, for some $C > 0$ independent of ε. Eventually, we have established

$$\|R_1^\varepsilon\|_{B_m} \leq C\ \varepsilon\, T(\varepsilon),$$

for some $C > 0$ independent of ε. Concerning R_2^ε, we write in the similar spirit

$$\begin{aligned}\|R_2^\varepsilon\|_{B_m} &\leq (T + \varepsilon\, T(\varepsilon)) \left\| \left[\mathrm{e}^{i\varepsilon T(\varepsilon) u} - 1\right]\ F\left(\frac{s}{\varepsilon}, \Phi^\varepsilon(s)\right) \right\|_{C^0([0,1]\times[0,T_0+\varepsilon\, T(\varepsilon)];B_m)} \\ &\leq C(F,M)\, \varepsilon T(\varepsilon).\end{aligned}$$

We eventually deduce that for some $C > 0$ independent of ε.

$$\forall 0 \leq t \leq T_0, \quad \|\Phi^\varepsilon(t) - \widetilde{\Phi}^\varepsilon(t)\|_{B_m} \leq C\ \varepsilon\, T(\varepsilon), \tag{2.20}$$

<u>*Fourth step: conclusion*</u>
Gathering the above results, we recover

$$\forall 0 \leq t \leq T_0, \quad \|\Phi^\varepsilon(t) - \Phi(t)\|_{B_{m-2}} \leq C\ \left(\varepsilon\, T(\varepsilon) + \frac{\delta_M(\varepsilon)}{\varepsilon\, T(\varepsilon)}\right) \leq C\ \sqrt{\delta_M(\varepsilon)} \to 0,$$

provided we make the optimal choice

$$T(\varepsilon) = \sqrt{\delta_M(\varepsilon)}/\varepsilon. \tag{2.21}$$

□

3 Application: the cubic Schrödinger equation, with harmonic confinement

We apply our Main Theorem to the following simplest model of Bose condensation

$$i\partial_t \Psi^\varepsilon(t) = \left(-\Delta_x + x^2\right)\ \Psi^\varepsilon(t) + \frac{1}{\varepsilon}\ \left(-\Delta_z + z^2\right)\ \Psi^\varepsilon(t) + |\Psi^\varepsilon(t)|^2\ \Psi^\varepsilon(t). \tag{3.1}$$

In other words, we specify our discussion to the case

$$H_x = -\Delta_x + x^2, \quad H_z = -\Delta_z + z^2, \quad F(u) = +u.$$

We know from the Main Theorem that this model is asymptotically described by

$$\begin{aligned} i\partial_t \Phi(t) &= \left(-\Delta_x + x^2\right)\ \Phi(t) \\ &\quad + \lim_{T\to\infty} \frac{1}{T} \int_0^T \mathrm{e}^{+i\tau[-\Delta_z + z^2]}\ \left|\mathrm{e}^{-i\tau[-\Delta_z+z^2]}\ \Phi(t)\right|^2\ \mathrm{e}^{-i\tau[-\Delta_z+z^2]}\ \Phi(t)\, d\tau. \end{aligned} \tag{3.2}$$

Let us now give a more explicit form to (3.2). We know that the eigenelements of the harmonic oscillator $-\Delta_z + z^2$ are

$$E_p = (2p+1), \quad \text{and} \quad \chi_p(z) = H_p(z)\, \exp(-x^2/2),$$

where H_p is the p-th Hermite polynomial. Hence, introducing the quantities

$$\phi_p(t,x) = \langle \Phi(t,x,z)\,,\, \chi_p(z)\rangle\,, \qquad (p \in \mathbb{N}),$$

equation (3.2) readily becomes

$$\begin{aligned} i\partial_t \phi_p &= \left(-\Delta_x + x^2\right)\, \phi_p \\ &\quad + \lim_{T\to\infty} \frac{1}{T}\int_0^T \sum_{r,s,q\in\mathbb{N}} \phi_r(t)\,\phi_q(t)\,\overline{\phi_s(t)}\,\mathrm{e}^{-i\tau[E_q-E_s+E_r-E_p]}\, \langle \chi_q\,\chi_r\,,\,\chi_s\,\chi_p\rangle\, d\tau, \end{aligned}$$

Now, since the E_p's are integers, the $\displaystyle\lim_{T\to\infty}\frac{1}{T}\int_0^T \ldots$ simply becomes averaging over one period, namely $\displaystyle\frac{1}{2\pi}\int_0^{2\pi}\ldots$, and the latter integral transforms the sum $\displaystyle\sum_{q,r,s}\ldots$ into a sum over those integers such that $E_q + E_r = E_p + E_s$, or, in other words, $q + r = p + s$. We thus recover the averaged model

$$i\partial_t \phi_p(t,x) = \left(-\Delta_x + x^2\right)\, \phi_p + \sum_{r,s,q/q+r=p+s} A_{p,q,r,s}\; \phi_r\,\phi_q\,\overline{\phi_s} \tag{3.3}$$
$$\text{where} \quad A_{p,q,r,s} := \langle \chi_q\,\chi_r\,,\,\chi_s\,\chi_p\rangle\,.$$

This is an infinite system of cubic Schrödinger equations along the x plane. Note that we do not have any simple information about the behavior of the given coefficients $A_{p,q,r,s}$ entering the system, despite the fact that the eigenfunctions χ_p are explicitly known. This makes it definitely easier to deal directly with the equation on Φ (without projecting).

As a special case, equation (3.3) allows to recover the one mode situation treated in [BMSW]. Indeed, when the initial datum satisfies

$$\Phi(0,x,z) = \phi_0(0,x)\,\chi_0(z),$$

i.e. when $\Phi(0)$ lies entirely in the eigenspace associated with the lowest energy $E_0 = 1$, it is easily seen that the function

$$\Phi(t,x,z) = \phi_0(t,x)\,\chi_0(z)$$

solves the averaged system (3.2), provided $\phi_0(t,x)$ solves the one-mode problem

$$i\partial_t \phi_0(t,x) = \left(-\Delta_x + x^2\right)\, \phi_0 + A_{0,0,0,0}\, |\phi_0(t)|^2\, \phi_0(t). \tag{3.4}$$

One can even go a bit further, namely, when the initial datum is *any* one-mode function

$$\Phi(0,x,z) = \phi_p(0,x)\,\chi_p(z),$$

for some given index p, i.e. when $\Phi(0)$ lies entirely in the eigenspace associated with the energy E_p, it is easily seen that the function

$$\Phi(t,x,z) = \phi_p(t,x)\,\chi_p(z)$$

solves the averaged system (3.2), provided $\phi_p(t,x)$ solves the one-mode problem

$$i\partial_t \phi_p(t,x) = \left(-\Delta_x + x^2\right)\,\phi_p + A_{p,p,p,p}\,|\phi_p(t)|^2\,\phi_p(t). \tag{3.5}$$

Again, starting from the mode p, equation (3.3) can only feed the same mode p and no new mode is switched on. This observation extends the results of [BMSW] to *any* one-mode solution.

In the case where the initial datum contains at least *two* distinct modes, say p_0 and p_1, it is clear that equation (3.3) immediately allows to switch on the modes $2p_0 - p_1$, $2p_1 - p_0$, hence the modes $4p_0 - 3p_1$ and so on, so that eventually *an infinite number of modes* is switched on, and the need for a clean functional analytic framework to treat equation (3.3), namely the formulation (3.2), becomes transparent.

We wish to end this text with a last, bibliographical comment.

In [BaMSW], the above problem (3.3) has been *formally* derived. In that text, the authors study the existence and uniqueness for a simplified problem by proceeding to a *truncation of the modes.* Namely they considered the problem

$$i\partial_t \Phi_p = H_x\phi_p + \sum_{r,s,q/q+r=p+s \text{ and } p,q,r,s\leq L} A_{p,q,r,s}\phi_r\,\phi_q\,\overline{\phi_s}. \tag{3.6}$$

Needless to say, the truncated problem (3.6) is considerably simpler than (3.3), in that all the convergence issues of the series expansion are then removed. Unfortunately, the approach of [BaMSW] seemingly does not allow the construction of solutions to the whole limit problem when $L \to \infty$. It turns out that our approach actually allows to onstruct the solution of all *truncated* problems at once, and to show that it is indeed a good approximation of the untruncated one in B_m ($m > 3/2$), as $L \to \infty$.

References

[Ba] G. Bastard, **Wave mechanics applied to semiconductor heterostructures**, Les éditions de physique, 1996.

[BaMSW] W. Bao, P. Markowich, C. Schmeiser, R. Weishäupl, *On the Gross-Pitaevski equation with strongly anisotropic confinement: formal asymptotics and numerical experiments*, preprint (2004).

[BM] N. Ben Abdallah, F. Méhats, *Semiclassical analysis of the Schrödinger equation with a partially confining potential*, J. Math. Pures Appl. **84**, pp. 580-614 (2005).

[BMSW] N. Ben Abdallah, F. Méhats, C. Schmeiser, R. M. Weishäupl, *The nonlinear Schrödinger equation with strong anisotropic harmonic potential*, to appear in SIAM J. Math. Anal.

[BMP] N. Ben Abdallah, F. Méhats, O. Pinaud, *The adiabatic approximation of the Schrödinger Poisson system with a partial confinement*, SIAM J. Math. Anal. **36**, N.3, pp. 986-1013 (2005).

[BCD] B. Bidégaray, F. Castella, P. Degond, *From Bloch model to the rate equations*, Disc. Cont. Dyn. Sys., Vol. 11, N. 1, pp. 1-26 (2004).

[BCDG] B. Bidégaray-Fesquet, F. Castella, E. Dumas, M. Gisclon, *From Bloch model to the rate equations II: the case of almost degenerate energy levels,* Math. Mod. Meth. Appl. Sci., Vol. 14, pp. 1785-1817 (2004).

[BC] J.-M. Bony, J.-Y. Chemin, *Espaces fonctionnels associés au calcul de Weyl-Hörmander*, Bull. Soc. Math. France, Vol. 122, N. 1, pp. 77-118 (1994).

[CDG1] F. Castella, P. Degond, T. Goudon, *Diffusion dynamics of classical systems driven by an oscillatory force*, J. Stat. Phys., Vol. 124, N. 2-4, pp. 913-950 (2006).

[CDG2] F. Castella, P. Degond, T. Goudon, *Large time dynamics of a classical system subject to a fast varying force*, to appear in Comm. Math. Phys. (2007).

[FG] D. K. Ferry, S. M. Goodnick, **Transport in nanostructures**, Cambridge Univ. Press, 1997.

[Gr] E. Grenier, *Oscillatory perturbations of the Navier-Stokes equations*, J. Math. Pures Appl., Vol. 76, pp. 477-498 (1997).

[HJ] G. A. Hagedorn, A. Joye, *A time-dependent Born-Oppenheimer approximation with exponentially small error estimates*, Comm. Math. Phys. **223**, N. 3, pp. 583-626 (2001).

[He] B. Helffer, **Théorie spectrale pour des opérateurs globalement elliptiques**, Astérisque, Vol. 112, Soc. Math. de France (1984).

[HN] B. Helffer, F. Nier, **Hypoelliptic estimates and spectral theory for Fokker-Planck operators and Witten Lalacians**, Springer, to appear (2005).

[HR] B. Helffer, D. Robert, *Caclcul fonctionnel par la transformation de Mellin et opérateurs admissibles*, J. Funct. Anal., Vol. 53, N. 3, pp. 246-268 (1983).

[Ho] L. Hörmander, **The analysis of linear partial differential operators**, Springer (1985).

[K] T. Kato, **Perturbation Theory for Linear Operators**, Springer-Verlag, Berlin, Heidelberg, 1966.

[LZ] B.M. Levitan, V.V. Zhikov, **Almost periodic functions and differential equations**, Cambridge University Press (1982).

[LM] P. Lochak, C. Meunier, **Multiphase averaging for classical systems. With applications to adiabatic theorems**, Applied Mathematical Sciences, Vol. 72. Springer-Verlag. (1988).

[Ma] A. Martinez, **An introduction to Semicassical and Microlocal Analysis**, Universitext, Springer-Verlag, New York (2002).

[MS] G. Métivier, S. Schochet, *Averaging Theorems for conservative systems and the weakly compressible Euler equations*, J. Diff. Eq., Vol. 187, N. 1, pp. 106-183 (2003).

[Mi] K. A. Mitchell, *Geometric phase, curvature, and Extrapotentials in Constrained Quantum Systems*, preprint arXiv:quant-ph/0001059, 18 Jan 2000.

[P] O. Pinaud, *Adiabatic approximation of the Schrödinger-Poisson system with a partial confinement: the stationary case*, J. Math. Phys. **45**, N. 5, pp. 2029-2050 (2004).

[PB] E. Polizzi, N. Ben Abdallah, *Self-consistent three dimensional model for quantum ballistic transport in open systems*, Phys. Rev B. **66**, pp. 245301-245309 (2002).

[RS] M. Reed, B. Simon, **Methods of Modern Mathematical Physics**, Academic Press, New York, San Francisco and London, 1975.

[SV] J.A. Sanders, F. Verhulst, **Averaging methods in nonlinear dynamical systems**, Applied Mathematical Sciences, Vol. 59 (Springer-Verlag, 1985).

[Sc] S. Schochet, *Fast singular limits of hyperbolic PDEs*, J. Diff. Eq., Vol. 114, N. 2, pp. 476-512 (1994).

[ST] H. Spohn, S. Teufel, *Adiabatic decoupling and time-dependent Born-Oppenheimer theory*, Comm. Math. Phys. **224**, N. 1, pp. 113-132. (2001).

[T] S. Teufel, *Adiabatic Perturbation Theory in Quantum Dynamics*, Habilitationsschrift, Zentrum Mathematik Technische Universität München, 2002.

[W1] W.-M. Wang, *Pure point spectrum of the Floquet Hamiltonian for the quantum harmonic oscillator under time quasi-periodic perturbations*, Preprint.

[W2] W.-M. Wang, *Quasi-periodic solutions of nonlinearly perturbed quantum harmonic oscillator*, Preprint.