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Xavier Blanc

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Fast rotating Bose-Einstein condensates and Bargmann transform

X. Blanc
Université Pierre et Marie Curie (Paris 6),
UMR 7598, Laboratoire Jacques-Louis Lions,
Paris, F-75005.

Abstract

When a Bose-Einstein condensate (BEC) is rotated sufficiently fast, it nucleates vortices. The system is only stable if the rotational velocity Ω is lower than a critical value Ω_c. Experiments show that as Ω approaches Ω_c, the condensate nucleates more and more vortices, which become periodically arranged. We present here a mathematical study of this limit. Using Bargmann transform and an analogy with semi-classical analysis in second quantization, we prove that the system necessarily has an infinite number of vortices and provide an ansatz for the solution. This summarizes two joint works, with A. Aftalion (LJLL, Univ. Paris 6) and J. Dalibard (LKB, Ecole Normale Supérieure), on the one hand, and with A. Aftalion and F. Nier (IRMAR, Univ. Rennes I) on the other hand.

1 Introduction

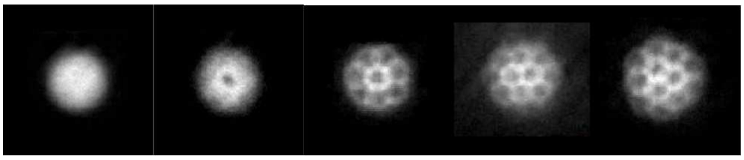
The existence of Bose-Einstein condensates (BEC for short) was first predicted in 1925 by Einstein on the ground of Bose's work on quantum gases. He proved the following fact: for an non-interacting quantum gas, if the temperature T is lower than some critical temperature T_c, then a macroscopic fraction of the gas collapses in the ground state. At that time, there was no experimental evidence of this fact.

It is only in 1995 that the first Bose-Einstein condensate is achieved [7] by the Jila group (University of Colorado). In the following years, several research teams in the world succeeded in making condensates. Therefore, theoretical as well as experimental study of these quantum macroscopic objects was renewed, as is testified by the great amount of literature on the subject (see for instance [3, 34, 35] and the references therein). In relation with this recent scientific interest, the Noble prize was awarded in 2001 to Cornell, Wiemann and Ketterle [17].

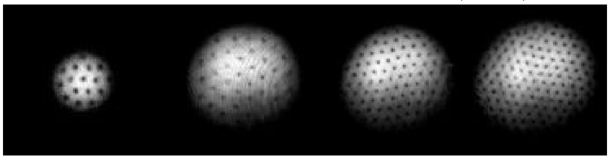
One of the special features of BEC is its superfluidity. This implies in particular the existence of quantized vortices. This is the case for instance when a BEC is rotated: should it be a classical fluid, its velocity field would be governed by solid body rotation, i.e the velocity field would be

$$v = \overrightarrow{\Omega} \times x, \tag{1.1}$$

where $\overrightarrow{\Omega}$ is the rotation vector. On the contrary, for a quantum fluid, the velocity field is the gradient of the phase of the wave function, and thus cannot match (1.1) unless it has some singularities around which we have a circulation of the phase. These singularities are the vortices. The wave function being in H^1, it must cancel at these points. This is why the wave function of a rotated BEC cancels



Dalibard et al., PRL **84**, 806 (2000)



Ketterle et al., PRL **87**, 210402 (2001)

Figure 1: Nucleation of vortices in a rotated condensate

at some points, which are the vortices. They have been observed experimentally [1, 11, 30, 32], as is shown in figure 1: if the rotation speed is sufficiently small, then no vortices are observed. When the rotation velocity grows, the number of vortices increases, and for very high velocities they seem to arrange themselves in a lattice [1, 16, 36].

The present work is concerned with the mathematical justification of this lattice of vortices. It has been extensively studied by physicists, starting with the seminal paper of Ho [21] and very recently of Fischer and Baym [18], Baym and Pethick [10], Cooper, Komineas and Read [14], Watanabe, Baym and Pethick [42], Sheehy and Radzihovsky [38].

The article is organized as follows: Section 2 describes the mathematical model associated with rotating Bose-Einstein condensates, and explains how it reduces to the minimization problem (2.22)-(2.23), which we recall here for the convenience of the reader:

$$\inf\left\{G^h(f),\, f \quad \text{is entire}, \quad f(z)e^{-\frac{|z|^2}{h}} \in L^2(\mathbb{C}), \quad \int_{\mathbb{C}} |f(z)|^2 e^{-\frac{|z|^2}{h}} = 1\right\},$$

where

$$G^h(f) = \int_{\mathbb{C}} |z|^2 |f(z)|^2 e^{-\frac{|z|^2}{h}} + \frac{g\Omega^2}{2} |f(z)|^4 e^{-\frac{2|z|^2}{h}},$$

and $\Omega = \sqrt{1-h^2}$ is the rotational velocity. The positive real number h is thus a small parameter as Ω reaches the (renormalized) critical value $\Omega_c = 1$ (recall that the system is stable only if $\Omega < \Omega_c$).

Then, we show in Section 3 that this minimization problem is related to the semi-classical analysis of a one dimensional harmonic oscillator in the second quantization framework. Section 4 is devoted to the mathematical results on this problem. The proofs are omitted, since they may be found in [6]. The last section accounts for some related open problem which seem of interest. Note that for the convenience of the reader, there is no need of reading Section 2 before going to the remaining parts of the paper, which are more concerned with mathematical results, but are self-contained.

2 Quantum N-body problem and Gross-Pitaevskii energy

2.1 Quantum N-body problem

A Bose-Einstein condensate is a quantum object with many particles. It is therefore described by a wave function

$$\Psi \in L^2_s(\mathbb{R}^{3N}), \tag{2.1}$$

where N is the number of particles, and the subscript s stands for the symmetry assumption: since we deal with bosons, $\Psi(x_1, \ldots, x_N)$ must be unchanged when two variables x_i and x_j are exchanged. Moreover, we assume that the condensate is rotated along the vector e_3, which is the third vector of the canonical basis of $\mathbb{R}^3$, with a rotational velocity $\Omega > 0$: the rotation vector is $\overrightarrow{\Omega} = \Omega e_3$.

On the space defined by (2.1), we therefore define the N-body hamiltonian:

$$H^N_\Omega = -\frac{1}{2}\sum_{j=1}^N \Delta_j + \frac{1}{2}\sum_{j=1}^N V(x_j) - i\sum_{j=1}^N \overrightarrow{\Omega}(x_j \times \nabla_j) + \sum_{1\leq j<k\leq N} v_N(x_j - x_k), \tag{2.2}$$

where V is the confining potential: in the present case, we will consider the harmonic potential $V(x) = |x|^2$, $x \in \mathbb{R}^3$, but other cases are possible [40]. Moreover, the function v_N is the interaction potential, which depends on N but is not explicitly given by physics. However, its "scattering length" a_N (see for instance [27]) is given, and it is sufficient for modelling the case we have in mind. We are interested in the ground state of the system, that is,

$$E^{Q,\Omega}_N = \inf\left\{\left\langle H^N_\Omega \Psi|\Psi\right\rangle, \quad \Psi \in H^1_s(\mathbb{R}^{3N}), \quad \sum_{i=1}^N |x_i|^2|\Psi|^2 \in L^1(\mathbb{R}^{3N}), \quad \int_{\mathbb{R}^{3N}} |\Psi|^2 = 1\right\}, \tag{2.3}$$

where H^1_s is the subspace of H^1 consisting of symmetric functions. This value, and the minimizing wave function Ψ, are rather difficult to compute. In addition, the number N is large (typically of the order of 10^7), hence it is a natural approximation to look for the limit of (2.3), together with the limit of its minimizer, as N goes to infinity.

2.2 The Gross-Pitaevskii energy

It is actually proved in [26] (see also [27, 28] for the simpler case $\Omega = 0$) that if v_N depends in a suitable way on N, the energy per particle $E^{Q,\Omega}_N/N$ converges as N tends to infinity to the Gross-Pitaevskii energy:

$$\lim_{N\to\infty} \frac{E^{Q,\Omega}_N}{N} = e^{GP},$$

where

$$e^{GP} = \inf\left\{E^{GP}(\phi), \quad \phi \in H^1(\mathbb{R}^3), \quad x\phi \in L^2(\mathbb{R}^3), \quad \int_{\mathbb{R}^3} |\phi|^2 = 1\right\}, \tag{2.4}$$

and the energy E^{GP} is given by

$$E^{GP}(\phi) = \int_{\mathbb{R}^3} \frac{1}{2}|\nabla\phi|^2 + \frac{1}{2}|x|^2|\phi|^2 - i\Omega\left[e_3(x\times\nabla)\phi\right]\overline{\phi} + \frac{1}{2}g|\phi|^4. \tag{2.5}$$

Here, the parameter g is related to the interaction potential v, and is defined by $g = 8\pi \lim_{N\to\infty}(a_N N)$. More precisely, it is assumed in [26] that

$$v(x) = \left(\frac{N}{8\pi g}\right)^2 v_1\left(\frac{Nx}{8\pi g}\right), \tag{2.6}$$

where v_1 is any fixed smooth potential decaying sufficiently fast at infinity, which has a scattering length equal to 1.

Moreover, it may be proved (here again, we refer to [26] for the details) that the minimizer Ψ_N of (2.3) is well approximated in the limit $N \longrightarrow \infty$ by the N-tensor product of the minimizer of (2.4):

$$\Psi_N(x_1, \dots x_N) \approx \prod_{j=1}^{N} \phi(x_j).$$

This result is made precise in [26] by the convergence of the density matrix of Ψ_N.

Note that the problem is now considerably simpler since the dimension fell from $3N$ to 3. However, the problem is now nonlinear, since the energy (2.5) is no more quadratic, due to the last term.

2.3 Reduction to a 2-dimensional model

Another simplification is the reduction from a 3D model to a 2D one. This is heuristically justified by the fact that we are interested in the rapid rotation regime, in which the centrifugal force grows stronger and stronger, and thus makes the condensate expand in the plane $\{x_3 = 0\}$ perpendicular to the rotation vector $\overrightarrow{\Omega}$. This is not proved rigourously for now, except in the special case of an anisotropic trap with a very large confinment in the x_3 direction with no rotation [37].

Such a two-dimensional model then reads:

$$e_{2D}^{GP} = \inf\left\{E_{2D}^{GP}(\phi), \quad \phi \in H^1(\mathbb{R}^2), \quad x\phi \in L^2(\mathbb{R}^2), \quad \int_{\mathbb{R}^2} |\phi|^2 = 1\right\}, \tag{2.7}$$

$$E_{2D}^{GP}(\phi) = \int_{\mathbb{R}^2} \frac{1}{2}|\nabla\phi|^2 + \frac{1}{2}|x|^2|\phi|^2 - i\Omega x(\nabla^\perp\phi)\overline{\phi} + \frac{1}{2}g|\phi|^4, \tag{2.8}$$

where for any $y = (y_1, y_2) \in \mathbb{R}^2$, $y^\perp$ denotes the vector $y^\perp = (y_2, -y_1)$.

2.4 Lowest Landau level approximation

The first terms of (2.8) may be seen as the beginning of a perfect square, so that E_{2D}^{GP} also reads

$$E_{2D}^{GP}(\phi) = \int_{\mathbb{R}^2} \frac{1}{2}|\nabla\phi - i\Omega x^\perp\phi|^2 + \frac{1}{2}(1-\Omega^2)|x|^2|\phi|^2 + \frac{1}{2}g|\phi|^4, \tag{2.9}$$

In order for the energy to be bounded below, we need to have $\Omega < 1$, which means that the trapping potential remains stronger than the rotating force. As $\Omega < 1$ approaches one, the extension of the condensate increases.

The first term in the energy (2.9) is identical to the energy of a particle placed in a uniform magnetic field 2Ω. It is also reminiscent of type II superconductors near the second critical field H_{c2}. The minimizers for

$$\int_{\mathbb{R}^2} \frac{1}{2}|\nabla\phi - i\Omega x^\perp \phi|^2 \text{ under } \int_{\mathbb{R}^2} |\phi|^2 = 1 \tag{2.10}$$

are well known [24, 29] through the study of the eigenvalues of the operator $-(\nabla - i\Omega x^\perp)^2$. The minimum is Ω and it is achieved in a space of infinite dimension called the lowest Landau level (LLL). This space is the closure for the L^2 norm of the space spanned by

$$\phi(x_1, x_2) = P(z)e^{-\Omega|z|^2/2} \text{ with } z = x_1 + ix_2 \tag{2.11}$$

where P varies in the space of polynomials. The other eigenvalues are $(2k+1)\Omega$, $k \in \mathbb{N}$.

We will see that as Ω approaches 1, the second and third term in the energy (2.9) produce a contribution of order $\sqrt{1-\Omega}$, which is much smaller than the gap between two eigenvalues of $-(\nabla - i\Omega x^\perp)^2$, namely 2Ω. Thus, it is natural, as a first step, to restrict to the minimizers of (2.10) and minimize the energy (2.9) in this reduced infinite dimensional space. Since we want to keep the same space as Ω varies, we will use the rescaled wave function

$$\psi(x) = \frac{1}{\sqrt{\Omega}}\phi\left(\frac{x}{\sqrt{\Omega}}\right), \tag{2.12}$$

which satisfies the condition $\int |\psi|^2 = 1$. Therefore, the energy (2.9) provides $E(\phi) = \tilde{E}(\psi)$ with

$$\tilde{E}(\psi) = \int_{\mathbb{R}^2} \frac{1}{2}|\nabla\psi - ix^\perp\psi|^2 + \frac{1-\Omega^2}{2\Omega}|x|^2|\psi|^2 + \frac{1}{2}g\Omega|\psi|^4, \tag{2.13}$$

and the condition (2.11) becomes

$$\psi(x) = P(z)e^{-|z|^2/2} \text{ with } P(z) = A\prod_{i=1}^{n}(z - z_i) \text{ and } z = x_1 + ix_2. \tag{2.14}$$

For such a ψ, the first term of the energy is equal to Ω. Hence, we find that the energy $\tilde{E}(\psi)$ is equal to

$$\tilde{E}(\psi) = \Omega + \int_{\mathbb{R}^2} \frac{1-\Omega^2}{2\Omega}|x|^2|\psi|^2 + \frac{g\Omega}{2}|\psi|^4 := E_{LLL}(\psi). \tag{2.15}$$

In [8], we have performed numerical computations, fixing an upper bound on the number of zeroes and using a conjugate gradient on the z_i to find a minimizer of the energy. This provides the pattern for vortices illustrated in Figure 2. On the left, we have plotted the z_i and on the right $|\psi|$ where ψ is related to the z_i through (2.14): in a central region, vortices are located on a regular triangular lattice, while the lattice is distorted towards the edges. The density plot of $|\psi|$ shows that the only visible vortices are the central ones in the regular lattice part, the outer ones being in a region of very low density.

In [4], we have constructed a wave function corresponding to the intuition given by the above remark on numerics. This wave function has zeroes on a lattice which is distorted towards the edges of the condensates. Computing the energy in the limit of an infinite number of zeroes (together with $\Omega \to 1$) gives an energy depending on the distortion. Minimizing it gives the expected distortion, and the upper bound for the energy (4.4). This method is more explicit, but provides no information on the minimizer, unlike for instance Theorem 4.3 below.

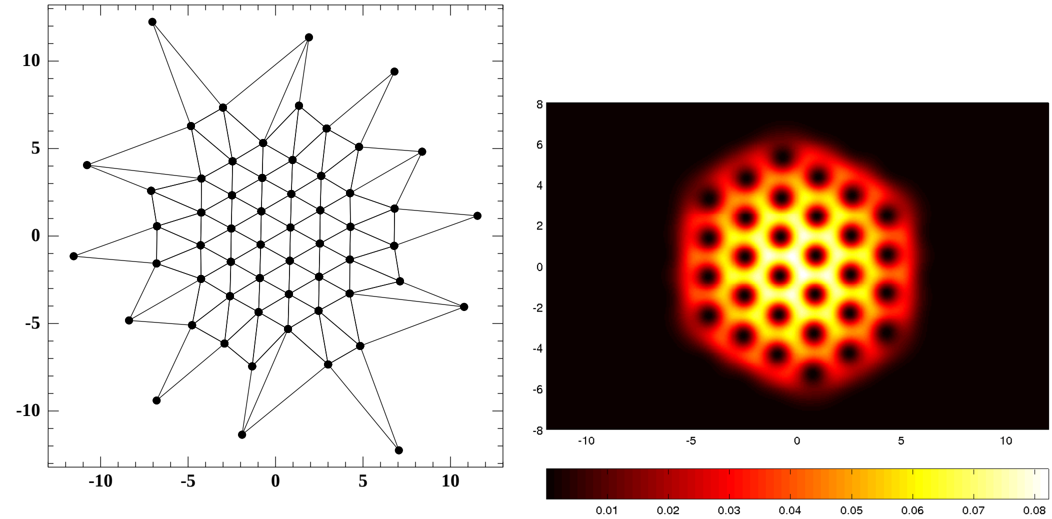


Figure 2: An example of (left): a configuration of z_i minimizing the energy for $\Omega = 0.999$, $g = 3$ and $n = 58$. (right): density plot of $|\psi|$

2.5 Bargmann space

The last step of the present discussion about modelling is to rescale once more the wave function: as pointed out in [4, 8], the wave function ψ has a support of the order of $(1-\Omega)^{-1/4}$, which goes to infinity as Ω tends to 1, therefore making ψ wide-spread in the limit. Hence, in order to keep a non-vanishing wave function, we need to rescale it, defining

$$u(x) = \frac{1}{(1-\Omega^2)^{\frac{1}{4}}}\psi\left(\frac{x}{(1-\Omega^2)^{\frac{1}{4}}}\right). \tag{2.16}$$

Then, setting

$$h = \sqrt{1-\Omega^2}, \tag{2.17}$$

the ansatz (2.14) is equivalent to

$$u(z) = f(z)e^{-\frac{|z|^2}{2h}}, \quad f \text{ is a polynomial.} \tag{2.18}$$

In addition, we find for (2.15) the following expression:

$$E_{LLL}(\psi) = \frac{h}{\Omega}\int_{\mathbb{C}}\left(|z|^2|u|^2 + \frac{g\Omega^2}{2}|u|^4\right)L(dz), \tag{2.19}$$

where $L(dz)$ denotes the Lebesgue measure $L(dz) = dx_1dx_2$ with $z = x_1 + ix_2$, and u is related to ψ by (2.16). Hence, introducing the Fock-Bargmann space

$$\mathcal{F}_h = \left\{f \in L^2(\mathbb{C}, e^{-\frac{|z|^2}{h}}L(dz)),\ \text{s.t } f \text{ entire}\right\} \tag{2.20}$$

$$\text{with} \qquad \|f\|^2_{\mathcal{F}_h} = \int_{\mathbb{C}}|f(z)|^2e^{-\frac{|z|^2}{h}}L(dz), \tag{2.21}$$

we can rephrase the problem of minimizing (2.15) over the set (2.14) as follows: find the minimizer(s) of

$$\inf \left\{ G^h(f),\ f \in \mathcal{F}_h,\ \|f\|_{\mathcal{F}_h} = 1 \right\}, \tag{2.22}$$

where

$$G^h(f) = \int_{\mathbb{C}} |z|^2 |f(z)|^2 e^{-\frac{|z|^2}{h}} + \frac{g\Omega_h^2}{2} |f(z)|^4 e^{-\frac{2|z|^2}{h}}, \tag{2.23}$$

and $\Omega_h = \sqrt{1-h^2}$.

3 Bargmann transform and semi-classical analysis

We present in this section some simple properties of the Fock-Bargmann space (2.20), and its link with semi-classical analysis. For this purpose, we first define the Bargmann transform [9]:

$$[B_h \varphi](z) = \frac{1}{(\pi h)^{3/4}} e^{\frac{z^2}{2h}} \int_{\mathbb{R}} e^{-\frac{(\sqrt{2}z-y)^2}{2h}} \varphi(y)\ dy\,,$$

with $z = \frac{x-i\xi}{\sqrt{2}} \in \mathbb{C}$ and $\varphi \in \mathcal{S}'(\mathbb{R})$. Other normalizations are possible:

- In [31], the standard semiclassical FBI transform is defined as:

  $$[T_h\varphi](x,\xi) = \frac{1}{2^{1/2}(\pi h)^{3/4}} e^{-\xi^2/2h} \int_{\mathbb{R}} e^{-\frac{(x-i\xi-y)^2}{2h}} \varphi(y)\ dy.$$

  Other normalizations or extensions can be found in [15, 39].

- In [19], the Bargmann transform is defined as

  $$[B\varphi](z) = 2^{1/4} \int_{\mathbb{R}} e^{2\pi z y - \pi y^2 - (\pi/2) z^2} \varphi(y)\ dy.$$

Elementary calculations lead to :

$$\begin{aligned} & [B_h\varphi](z) = 2^{1/2} e^{\frac{x^2+\xi^2}{4h}} e^{-i\frac{x\xi}{2h}} [T_h\varphi](z) = 2^{1/2} e^{|z|^2/2h} e^{-i\frac{x\xi}{2h}} [T_h\varphi](z) \\ \text{and} \quad & [B_h\varphi](z) = \frac{1}{(\pi h)^{1/4}} [B(2\pi h)^{1/4} \varphi((2\pi h)^{1/2}.)] \left(\frac{z}{(\pi h)^{1/2}} \right)\,. \end{aligned}$$

We mainly refer to the presentation of Martinez which already contains the small parameter $h > 0$, but the reader can make the relationship with other results by applying the previous change of variables. We simply list the classical properties of the Bargmann transform and the Fock-Bargmann space and refer to [9, 15, 19, 31, 39] for proofs.

3.1 Isometry property

For any $h > 0$, the transform T_h defines an isometry between $L^2(\mathbb{R}, dy)$ into $L^2(\mathbb{C}, dxd\xi)$ and onto the space $L^2(\mathbb{C}, dxd\xi) \cap e^{-\xi^2/2h}\mathcal{H}(\mathbb{C})$ where $\mathcal{H}(\mathbb{C})$ denotes the space of entire functions. Here the holomorphy of $B_h\varphi$ directly comes from its definition. Moreover, B_h defines a unitary transform from $L^2(\mathbb{R}, dy)$ onto $\mathcal{F}^h$ (note that our normalization gives $L(dz) = \frac{dxd\xi}{2}$).

Moreover, the product $B_h^* B_h$ is the identity on $L^2(\mathbb{R}, dy)$ while $B_h B_h^* = \Pi_h$ is the orthogonal projection from $L^2(\mathbb{C}, e^{-|z|^2/h} L(dz))$ onto $\mathcal{F}^h$. The adjoint of B_h is given by

$$[B_h^* f](y) = \frac{1}{(\pi h)^{3/4}} \int_{\mathbb{C}} e^{\frac{\overline{z'}^2}{2h}} e^{\frac{-(y-2^{1/2}\overline{z'})^2}{2h}} e^{-\frac{|z'|^2}{h}} f(z')\ L(dz')\ .$$

A simple gaussian integration w.r.t. $y \in \mathbb{R}$ yields

$$[\Pi_h f](z) = [B_h B_h^* f](z) = \frac{1}{\pi h} \int_{\mathbb{C}} e^{\frac{z\overline{z'}}{h}} e^{-\frac{|z'|^2}{h}} f(z')\ L(dz')$$

for all $f \in L^2(\mathbb{C}, e^{-|z|^2/h} L(dz))$.

3.2 Coherent states

The phase space $\mathbb{R}^2_{x,\xi}$ is endowed with the symplectic form

$$\sigma((x_1,\xi_1),(x_2,\xi_2)) = \xi_1 x_2 - x_1 \xi_2,$$

The associated unitary phase translations on $L^2(\mathbb{R}, dy)$ are given by

$$[\tau^h_{(x,\xi)} u](y) = e^{i\frac{\xi(2y-x)}{2h}} u(y-x), \quad \tau^h_{(x,\xi)} = e^{i\frac{\xi y - x(hD_y)}{h}}$$

and satisfy the Weyl relation

$$\tau^h_{X_1} \circ \tau^h_{X_2} = e^{i\frac{\sigma(X_1,X_2)}{h}} \tau^h_{X_1+X_2}, \quad X_k = (x_k, \xi_k)\ .$$

The coherent states are the normalized $L^2(\mathbb{R}, dy)$ functions given by:

$$\Phi^h_0(y) = \frac{1}{(\pi h)^{1/4}} e^{-\frac{y^2}{2h}}$$

$$\text{and} \qquad \Phi^h_{(x,\xi)}(y) = [\tau^h_{(x,\xi)} \Phi_0](y) = \frac{1}{(\pi h)^{1/4}} e^{i\frac{\xi(2y-x)}{2h}} e^{-\frac{(y-x)^2}{2h}}\ .$$

By recalling $z = \frac{x-i\xi}{\sqrt{2}}$ we get

$$[B_h\varphi](z) = \frac{1}{(\pi h)^{3/4}} e^{\frac{(x-i\xi)^2}{4h}} \int_{\mathbb{R}} e^{-\frac{(y-x+i\xi)^2}{2h}} \varphi(y)\ dy = \frac{1}{(\pi h)^{1/2}} e^{\frac{|z|^2}{2h}} \int_{\mathbb{R}} \overline{\Phi^h_{x,\xi}(y)} \varphi(y)\ dy,$$

hence

$$[B_h\varphi](z) = \frac{1}{(\pi h)^{1/2}} e^{\frac{|z|^2}{2h}} \langle \Phi^h_{(x,\xi)} \,|\, \varphi \rangle\ . \tag{3.1}$$

The identity $B_h^* B_h = \mathrm{Id}$ becomes the standard identity resolution on $L^2(\mathbb{R}, dy)$

$$\int_{\mathbb{R}^2} |\Phi^h_{x,\xi}\rangle\langle\Phi^h_{x,\xi}|\ \frac{dxd\xi}{(2\pi h)}\ .$$

>From the previous relation, we conjugate the action of $\tau^h_{(x_0,\xi_0)}$ via B_h:

$$\begin{aligned}
[B_h\tau^h_{(x_0,\xi_0)}\varphi](z) &= \frac{e^{\frac{|z|^2}{2h}}}{(\pi h)^{1/2}}\langle\Phi^h_{(x,\xi)}\,|\,\tau^h_{(x_0,\xi_0)}\varphi\rangle \\
&= \frac{e^{\frac{|z|^2}{2h}}}{(\pi h)^{1/2}}\langle\tau^h_{-(x_0,\xi_0)}\tau^h_{(x,\xi)}\Phi^h_0\,|\,\varphi\rangle \\
&= \frac{e^{\frac{|z|^2}{2h}}}{(\pi h)^{1/2}}e^{-i\frac{\sigma((-x_0,-\xi_0),(x,\xi))}{2h}}\langle\Phi^h_{(x-x_0,\xi-\xi_0)}\,|\,\varphi\rangle \\
&= e^{\frac{|z|^2}{2h}}e^{\frac{2i(\xi_0 x-x_0\xi)}{4h}}e^{-\frac{|z-z_0|^2}{2h}}[B_h\varphi](z-z_0) = e^{\frac{\overline{z_0}(2z-z_0)}{2h}}[B_h\varphi](z-z_0)\ ,
\end{aligned}$$

with $z=\frac{x-i\xi}{\sqrt{2}}$ and $z_0=\frac{x_0-i\xi_0}{\sqrt{2}}$. With our normalization, the Bargmann transform of the function Φ^h_0 is the constant function $(\pi h)^{-1/2}$ and we get more generally

$$[B_h\Phi^h_{(x_0,\xi_0)}](z) = (\pi h)^{-1/2}e^{\frac{\overline{z_0}(2z-z_0)}{2h}}\ .$$

Hence the relation

$$h\partial_z[B_h\Phi^h_{(x_0,\xi_0)}] = \overline{z_0}[B_h\Phi^h_{(x_0,\xi_0)}]\ ,$$

holds for all $z_0=\frac{x_0-i\xi_0}{\sqrt{2}}\in\mathbb{C}$.

3.3 Harmonic oscillator

The harmonic oscillator (or number operator in the Fock representation) is the self adjoint operator on $L^2(\mathbb{R},dy)$ given by:

$$\begin{aligned}
\tilde{N}_h &= \frac{1}{2}(-h^2\partial_y^2+y^2-h) = a_h^*a_h \\
D(\tilde{N}_h) &= \left\{u\in L^2(\mathbb{R},dy),\ y^\alpha D_y^\beta u\in L^2(\mathbb{R},dy),\ \alpha+\beta\le 2\right\},
\end{aligned}$$

where the annihilation and creation operators, $a_h=\frac{1}{\sqrt{2}}(h\partial_y+y)$ and $a_h^*=\frac{1}{\sqrt{2}}(-h\partial_y+y)$, satisfy the CCR $[a_h,a_h^*]=h$. The normalized Hermite functions are then given by

$$H_0^h(y)=\frac{1}{(\pi h)^{1/4}}e^{-\frac{y^2}{h}} \qquad H_n^h=\frac{1}{h^{n/2}\sqrt{n!}}(a_h^*)^nH_0 \quad \text{for } n\in\mathbb{N}\,, \tag{3.2}$$

and form an orthonormal basis of eigenfunctions with

$$\tilde{N}_hH_n^h = nhH_n^h\,.$$

An integration by parts shows

$$[B_hh\partial_y\varphi](z) = [B_hy\varphi](z)-\sqrt{2}z[B_h\varphi](z)$$

which yields

$$zB_h = B_h\circ(\frac{-h\partial_y+y}{\sqrt{2}}) = B_h\circ a_h^*\ . \tag{3.3}$$

We then differentiate $B_h\varphi$ with respect to z, and we obtain

$$h\partial_z[B_h\varphi](z) = -z[B_h\varphi](z) + \sqrt{2}[B_h y\varphi](z)$$

which leads to

$$(h\partial_z)\circ B_h = B_h \circ (\frac{h\partial_y + y}{\sqrt{2}}) = B_h \circ a_h \ . \tag{3.4}$$

>From this we recover

$$a_h\Phi^h_{(x_0,\xi_0)} = \overline{z_0}\Phi^h_{(x_0,\xi_0)} \qquad \text{with } z_0 = \frac{x_0 - i\xi_0}{\sqrt{2}} \ , \tag{3.5}$$

while (3.3) and (3.2) imply

$$\begin{aligned} B_h[H_n^h] &= \frac{1}{(\pi h)^{1/2} h^{n/2}\sqrt{n!}} z^n \ , \\ \tilde{N}_h &= B_h^*[z(h\partial_z)]B_h \ . \end{aligned}$$

Thus, we set

$$N_h = B_h \tilde{N}_h B_h^* = z(h\partial_z)\,.$$

Using (3.4), (3.5) and (3.1), we see that any element $f = B_h\varphi$ of $\mathcal{F}^h$ satisfies

$$h\partial_z f = h\partial_z(\Pi_h f) = \Pi_h(\overline{z}f) \ .$$

We also note

$$z(h\partial_z)f = h\partial_z(zf) - hf = h\partial_z\Pi_h(zf) - hf = \Pi_h(|z|^2 - h)\Pi_h f.$$

Since $B_h = \Pi_h B_h$, this provides another useful writing of the operator $\tilde{N}_h$:

$$\tilde{N}_h = B_h^*[|z|^2 - h]B_h \ , \quad N_h = \Pi_h(|z|^2 - h)\Pi_h \ .$$

3.4 h-Pseudo-differential operators

We simply recall the link with the Anti-Wick quantization[1]. The Anti-Wick quantization of a symbol $b(x,\xi)$ can be defined as

$$b^{A-Wick}(y, hD_y) = \int_{\mathbb{R}^2} b(x,\xi)|\Phi^h_{(x,\xi)}\rangle\langle\Phi^h_{(x,\xi)}| \ \frac{dxd\xi}{2\pi h} \ .$$

It is a positive quantization in the sense that

$$(b \geq 0) \Rightarrow (b^{A-Wick}(y, hD_y) \geq 0)$$

and this implies

$$\left\| b^{A-Wick}(y, hD_y) \right\| \leq \|b\|_{L^\infty} \ .$$

Another simple consequence of its definition

$$\left\| b^{A-Wick}(y, hD_y) \right\|_{\mathcal{L}^1} \leq \frac{1}{2\pi h} \|b\|_{L^1}$$

[1]The "Anti-Wick" name corresponds to the fact that the quantized symbol $|z|^2 = z\overline{z} = \overline{z}z$ equals $a_h a_h^*$.

where $\mathcal{L}^1(L^2(\mathbb{R}, dy))$ denotes the space of trace-class operators.
The Anti-Wick quantization is close to the Weyl quantization due to the relation

$$b^{A-Wick}(y, hD_y) = (\frac{e^{-|z|^2/h}}{\pi h} * b)^W(y, hD_y)$$

For symbols in $S(1, dx^2 + d\xi^2)$ this leads to

$$\left\| b^{A-Wick}(y, hD_y) - b^W(y, hD_y) \right\| = O(h)$$

which allows to write in this class of symbols

$$\begin{aligned} b_1^{A-Wick}(y, hD_y) \circ b_2^{A-Wick}(y, hD_y) &= (b_1 b_2)(y, hD_y) + O_{\mathcal{L}(L^2)}(h) \\ \frac{i}{h}[b_1^{A-Wick}(y, hD_y), b_2(y, hD_y)] &= \{b_1, b_2\}^{A-Wick}(y, hD_y) + O_{\mathcal{L}(L^2)}(h) \ . \end{aligned}$$

Such results can be extended to some Hörmander classes (see [31, Chap XVIII] or [22]) or even to symbols with low regularity (see [25]). We note also the estimates

$$\left\| b^{A-Wick}(y, hD_y) \right\| \leq \|b(y, hD_y)\|_{\mathcal{L}^2} \leq (2\pi h)^{-1/2} \|b\|_{L^2}$$

deduced from the relation with the Weyl quantization.
Finally we translate the action of $b^{A-Wick}(y, hD_y)$ on the Fock space $\mathcal{F}^h$. From the relationship between the Bargmann transform and the coherent states, we get the relations

$$\begin{aligned} & b^{A-Wick}(y, hD_y) = B_h^* \circ b(x, \xi) \circ B_h \\ \text{and} \quad & B_h b^{A-Wick}(y, hD_y) B_h^* = \Pi_h \circ b(x, \xi) \circ \Pi_h \end{aligned}$$

where $b(x, \xi)$ simply denotes the multiplication by the function $b(x, \xi)$ in $L^2(\mathbb{C}, e^{-|z|^2/h} \ L(dz))$. Hence $b^{A-Wick}(y, hD_y)$ acts on $\mathcal{F}^h$ as a Toeplitz operator.

It is also possible to introduce an analytic pseudodifferential calculus. For this aspect, we refer the reader to [15, 31, 39].

4 Mathematical results

This section presents the results detailed in [6]. We give them here without proofs, refering the interested reader to [6].

4.1 Existence of a minimizer

First of all, let us define the spaces

$$\mathcal{F}_h^s = \left\{ f \text{ entire, s.t. } \int_{\mathbb{C}} \langle z \rangle^{2s} |f(z)|^2 e^{-\frac{|z|^2}{h}} L(dz) < \infty \right\}, \tag{4.1}$$

where $\langle z \rangle = \sqrt{1 + |z|^2}$. The fact that $\mathcal{F}_h^1$ is compactly imbedded into $\mathcal{F}_h$ allows to prove the following:

Theorem 4.1 *For fixed $h > 0$, the minimization problem* (2.22) *admits a solution in $\mathcal{F}_h^1$. Any minimizer is a solution of the Euler-Lagrange equation*

$$\Pi_h \left[\left(|z|^2 + g\Omega_h^2 e^{-\frac{|z|^2}{h}} |f|^2 - \lambda \right) f \right] = 0 \tag{4.2}$$

where $\lambda \in \mathbb{R}$ is the Lagrange multiplier and is bounded independently of h. The Euler-Lagrange equation can also be written as

$$zh\partial_z f + \frac{g\Omega_h^2}{2} \bar{f}(h\partial_z)[f^2(2^{-1}.)] - (\lambda - h)f = 0, \text{ in } \mathcal{F}_h^{-1} \tag{4.3}$$

the operator $\bar{f}(h\partial_z)$ being defined as the limit $\lim_{K\to\infty} \sum_{k=0}^{K} \overline{a_k}(h\partial_z)^k$ if $f(z) = \sum_{k=0}^{\infty} a_k z^k$.

The compactness property allows to pass to the limit in the constraint for any minimizing sequence (passing to the limit in the energy is not a problem since it is convex). Moreover, in order to write down the Euler-Lagrange equation (4.2), one needs to check that the energy is sufficiently regular. The first term is a continuous bilinear form in $\mathcal{F}_h^1$, which causes no problem. However, proving that the second term is differentiable is not that obvious. It is done in [6] using a hypercontractivity property of the semi-group associated to the operator $N_h = zh\partial_z$. The proof of this hypercontractivity property may be found in [12, 20]

Concerning the energy, we have the following result:

Theorem 4.2 *The minimum*

$$e_{LLL}^h = \inf \left\{ G^h(f), \quad f \in \mathcal{F}_h, \quad \|f\|_{\mathcal{F}_h} = 1 \right\}$$

satisfies

$$\frac{2\Omega_h}{3}\sqrt{\frac{2g}{\pi}} < e_{LLL}^h \leq \frac{2\Omega_h}{3}\sqrt{\frac{2gb}{\pi}} + o_G(h^0) \tag{4.4}$$

where the parameter b describes the contribution of the vortex lattice and is related to a minimization problem in Theorem 4.4 below. Moreover, the Lagrange multiplier λ satisfies the uniform estimates $e_{LLL}^h \leq \lambda \leq 2e_{LLL}^h$.

In order to prove the lower bound in (4.4), one minimizes the energy (2.23) over $L^2\left(\mathbb{C}, e^{-\frac{|z|^2}{2h}} L(dz)\right)$, which contains $\mathcal{F}_h$. Then it is easily seen that the minimizer is the inverted parabola

$$|u_{min}|^2(z) = \frac{2}{\pi R_h^2}\left(1 - \frac{|z|^2}{R_h^2}\right)\mathbf{1}_{\{|z|\leq R_h\}}, \quad R_h = \sqrt{\lambda} = \left(\frac{2g\Omega_h^2}{\pi}\right)^{1/4}. \tag{4.5}$$

The first point is that this explicit minimizer gives the expected lower bound, and the second point is that u_{min} satisfying (4.5) cannot be in $\mathcal{F}_h$ since it has compact support. This explains why the left-hand side inequality is strict in (4.4).

Concerning the upper bound in (4.4), it is related to choosing a good test function f. We will see below that it is related to Jacobi's Θ function.

Using the Euler-Lagrange equation, we are able to prove the following result:

Theorem 4.3 *Let f be a minimizer of (2.22). If h is sufficiently small, then f has an infinite number of zeroes.*

Actually, the term "sufficiently small" is made precise in [6]. It is explicit in the sense that if one has an estimate on λ (which is numerically tractable through an estimate on e^h_{LLL}), then the condition amounts to $h < h_0$, where h_0 depends explicitly on g and λ.

Theorem 4.3 implies in particular that the minimizer cannot be a polynomial. However, we have also proved in [6] that, for h fixed, the minimizer of G^h over the subspace of polynomials of degree lower than K converges, as K goes to infinity, to a minimizer of problem (2.22).

4.2 Theta function

We consider in this section the question of choosing a good test function f in order to find the upper bound announced in (4.4). As indicated by the numerical simulations (and by the experiments), it is likely that a good approximation of the minimizer is given by a vortex configuration consisting of a lattice. We therefore introduce the following notation for a lattice:

$$\mathfrak{L} = \frac{1}{\nu}\left(\mathbb{Z} \oplus \tau\mathbb{Z}\right), \quad \nu \in \mathbb{R}_+^*, \quad \tau = \tau_R + i\tau_I, \quad \tau_I > 0, \quad |\tau| \geq 1, \quad -\frac{1}{2} \leq \tau_R < \frac{1}{2}. \tag{4.6}$$

This provides a description for all lattices with smallest period $1/\nu$.

As is stated in Theorem 4.4 below, if we want the function f to be holomorphic, have simple zeroes and vanish on each site of a given lattice $\mathfrak{L}$, it is necessarily related to Jacobi's theta function, which is defined as follows (see for instance [13]):

$$\Theta(v,\tau) = \frac{1}{i}\sum_{n=-\infty}^{+\infty} (-1)^n e^{i\pi\tau(n+1/2)^2} e^{(2n+1)\pi i v}, \quad v \in \mathbb{C}. \tag{4.7}$$

Such an ansatz was introduced by Abrikosov (see for example [2, 41]) in the context of superconductors modelling. Before stating the corresponding theorem, we recall the definition of the average of a periodic function (here, Q is any cell of the lattice):

$$⨍ |u|^n = \frac{\int_Q |u|^n(z)\ L(dz)}{\int_Q L(dz)} = \lim_{R\to\infty} \frac{\int_{|z|\leq R} |u|^n\ L(dz)}{\int_{|z|\leq R} L(dz)}.$$

Theorem 4.4 *Let $\mathfrak{L}$ be a lattice given by (4.6). If the function f is entire, if its zeroes are exactly the points of $\mathfrak{L}$ and are simple, and if $\left|e^{-\frac{|z|^2}{h}} f(z)\right|$ is $\mathfrak{L}$-periodic, then the lattice parameter ν and the function f satisfy*

$$\nu = \sqrt{\frac{\tau_I}{\pi h}} \quad \textit{and} \quad f(z) = c f_\tau(z), \quad c \in \mathbb{C}^*$$

$$\textit{with} \qquad f_\tau(z) = e^{\frac{z^2}{2h}}\Theta\left(\frac{\sqrt{\tau_I}}{\sqrt{\pi h}}z, \tau\right). \tag{4.8}$$

The function $f_\tau(z)$ solves the equation

$$\Pi_h\left(e^{-\frac{|z|^2}{h}}|f_\tau|^2 f_\tau\right) = \lambda_\tau f_\tau, \quad \textit{in } \mathcal{F}_h^s,\ s < -1, \tag{4.9}$$

$$\textit{with} \qquad \lambda_\tau = \frac{⨍ |u_\tau|^4}{\left(⨍ |u_\tau|^2\right)} = \frac{1}{\sqrt{2\tau_I}}\sum_{k\in\mathbb{Z}}\sum_{\ell\in\mathbb{Z}} e^{-\frac{\pi}{\tau_I}|k\tau-\ell|^2} \tag{4.10}$$

where $u_\tau(z) = e^{-\frac{|z|^2}{2h}} f_\tau(z)$. *Moreover, for the quantity* $\gamma(\tau)$ *defined by*

$$\gamma(\tau) = \frac{\fint |u_\tau|^4}{\left(\fint |u_\tau|^2\right)^2}, \tag{4.11}$$

we have

$$\gamma(\tau) = \sum_{k\in\mathbb{Z}} \sum_{\ell\in\mathbb{Z}} e^{-\frac{\pi}{\tau_I}|k\tau-\ell|^2}. \tag{4.12}$$

Note that equation (4.9) may be seen as equation (4.2) without the confining term $\Pi_h|z|^2\Pi_h f = zh\partial_z f + hf$.

In [33], the following result is proved:

Theorem 4.5 (Optimal lattice, [33]) *The complex number* $\tau = j = e^{\frac{2i\pi}{3}}$, *corresponding to the hexagonal lattice, is the unique minimizer of* $\gamma(\tau)$ *in the fundamental domain*

$$\left\{\tau = \tau_R + i\tau_I \in \mathbb{C},\ \tau_I > 0,\ |\tau| \geq 1,\ -\frac{1}{2} \leq \tau_R < \frac{1}{2}\right\}$$

and $b = \gamma(j) \sim 1.15959$.

The problem of minimizing the quantity $\gamma(\tau)$ with respect to τ was already addressed in [23]. One can find there some numerical evidence of the fact that $\tau = j$ is the minimizer. The authors of [33] *prove* that $\tau = j$, i.e the hexagonal lattice, is indeed the unique minimizer.

4.3 Limit as $h \longrightarrow 0$

Despite the preceding results, the ansatz (4.8) is not sufficient to obtain the upper bound in (4.4): we need to take into account the fact that the solution should in some sense "look like" the inverted parabola (4.5). Actually, we expect that when h is sufficiently small, any minimizer of (2.22) is close in some sense to $f_\tau(z)\alpha(z)$, where f_τ is the function described in Theorem 4.4 which varies on a characteristic size $\sqrt{h}$, and α is a slow varying profile which optimizes the energy. We are not able to prove such a result but the converse: $f_\tau(z)\alpha(z)$ can be approximated, as h tends to 0, by the element $\Pi_h(\alpha f_\tau)$ of $\mathcal{F}_h$, which is almost a solution of (4.2):

Theorem 4.6 *Let* $\tau \in \mathbb{C}\setminus\mathbb{R}$, *let* $\alpha \in C^{0,\frac{1}{2}}(\mathbb{C};\mathbb{C})$ *be such that* $\mathrm{supp}(\alpha) \subset K$ *for some compact set* K *and* $\int |\alpha|^2 = 1$. *For* f_τ *defined by (4.8), we set*

$$g^h_{\alpha,\tau} = \|\Pi_h(\alpha f_\tau)\|^{-1}_{\mathcal{F}_h} \Pi_h(\alpha f_\tau), \quad \text{and} \quad v^h_{\alpha,\tau}(z) = g^h_{\alpha,\tau}(z) e^{-\frac{|z|^2}{2h}}. \tag{4.13}$$

Then we have

$$G^h\left(g^h_{\alpha,\tau}\right) = \int_{\mathbb{C}} \left(|z|^2|\alpha(z)|^2 + \frac{Na\gamma(\tau)}{2}|\alpha(z)|^4\right) L(dz) + O(h^{1/4}) \tag{4.14}$$

where $\gamma(\tau)$ *is given by (4.11) and* $O(h^{1/4})$ *depends only on* $\|\alpha\|_{C^{0,1/2}}$, τ, λ *and* K. *Moreover, for any* $\lambda \in \mathbb{C}$,

$$\Pi_h\left(\left(|z|^2 - \lambda + Na\Omega_h^2 \left|v^h_{\alpha,\tau}\right|^2\right) g^h_{\alpha,\tau}\right) = \Pi_h\left(\left(|z|^2 - \lambda + Na\gamma(\tau)|\alpha|^2\right) g^h_{\alpha,\tau}\right) + O_{\mathcal{F}_h}(h^{1/4}), \tag{4.15}$$

where $O_{\mathcal{F}_h}(h^{1/4})$ *depends only on* $\|\alpha\|_{C^{0,1/2}}$, τ, λ *and* K.

In order to approximate a minimizer of (2.22), we need to pick the optimal function α. Minimizing the right-hand side of (4.14) with respect to τ and α under the constraint $\int |\alpha|^2 = 1$ yields

$$\tau = j \text{ and } |\alpha(z)|^2 = \frac{1}{Na\gamma(\tau)} \left(\sqrt{\frac{2Na\gamma(\tau)}{\pi}} - |z|^2 \right)_+ , \tag{4.16}$$

where the first equality is a consequence of Theorem 4.5. This provides in particular a test function for the upper bound of the energy, and makes precise the remainder estimate in the upper bound of (4.4), which is an improvement of the results of [4].

With this choice of α and τ, and if in addition λ in (4.15) is such that $\lambda = \sqrt{2Na\gamma(\tau)/\pi}$, (4.15) implies that

$$\Pi_h \left(\left(|z|^2 - \lambda + Na\Omega_h^2 \left| v_{\alpha,\tau}^h \right|^2 \right) g_{\alpha,\tau}^h \right) = O\left(h^{1/4}\right) \quad \text{in} \quad \mathcal{F}_h. \tag{4.17}$$

In other words, $g_{\alpha,\tau}^h$ is a solution of (4.2) up to an error term of order $h^{1/4}$. Furthermore, it is proved in [6] that, as h tends to 0, $g_{\alpha,\tau}^h$ is very close to $f_\tau(z)\alpha(z)$. This implies that the zeroes of $g_{\alpha,\tau}^h$ are located on an almost regular triangular lattice in the support of α. We do not have much information though, on the zeroes located outside the support of α, the "invisible vortices".

5 Perspectives

We have presented here a brief account of the work [6]. Note that further results may be found there. Among other things, we provide an error analysis of the numerical results displayed in Figure 2. We also derive a dynamical version of the above static picture. However, some open questions remain.

First, we have an upper bound in (4.4) which does not mach the lower bound. Should it be the case, we would probably be able to prove a convergence result on the minimizer. The commonly accepted conjecture is that the lower bound should contain the parameter b and thus be equal to the upper bound we already prove. Showing this more or less amounts to prove that a minimizer exhibits a lattice of zeroes. This intuition is corroborated by numerical simulations, but proving it seems far from being easy.

Another problem concerns modelling rather than the semi-classical analysis we have done here: we have approximated the original problem (2.7) by (up to rescaling) the LLL problem (2.22), but we have no rigourous justification of this fact. In particular, it would be of interest to prove that any solution of (2.7) is close to a minimizer of (2.22) after proper rescaling.

One last problem is to derive more precise estimates on the solutions of (2.22) as $h \to 0$. All we know about it for now is that it is bounded in $\mathcal{F}_h^1$ and that the corresponding $u(z) = f(z)e^{-\frac{|z|^2}{2h}}$ is bounded in $L^4(\mathbb{C})$. For instance it is a natural thing to look for a bound in $L^\infty(\mathbb{C})$, but we are not able to prove it for now.

Finally, let us mention the main drawback of the present work: we are only able to prove that the ansatz (4.13) with α and τ given by (4.16) is almost a solution of the equation in the sense of (4.17). A natural question is, can we prove that it is close (in some sense) to a solution, or even close to a local minimizer?

References

[1] J.R. Abo-Shaeer, C. Raman, J.M. Vogels, W. Ketterle, *Observation of Vortex Lattices in Bose-Einstein Condensates,* Science **292**, pp 476–479, 2001.

[2] A.A. Abrikosov, *Magnetic properties of group II Superconductors,* J. Exptl. Theoret. Phys. (USSR) **32**(5), pp 1147–1182, 1957.

[3] A. Aftalion, *Vortices in Bose Einstein condensates,* Progress in nonlinear differential equations, Vol 67, Birkhauser, 2006.

[4] A. Aftalion, X. Blanc *Vortex lattices in rotating Bose Einstein condensates,* to appear in SIAM J. Math. Anal.

[5] A. Aftalion, X. Blanc, F. Nier, *Vortex distribution in the lowest Landau level,* Phys. Rev. A **73**, p 011601(R), 2006.

[6] A. Aftalion, X. Blanc, F. Nier, *Lowest Landau level and Bargmann transform in Bose-Einstein condensates*, in preparation.

[7] A. Anderson, J. R. Ensher, M. R. Matthews, C. E. Wiemann, E. A. Cornell, *Observation of Bose-Einstein condensation in a dilute atomic vapor*, Science **269**, p 198, 1995.

[8] A. Aftalion, X. Blanc and J. Dalibard, Vortex patterns in fast rotating Bose Einstein condensates, Phys. Rev. A **71**, p 023611, 2005.

[9] V. Bargmann, *On a Hilbert space of analytic functions and an associated integral transform.* Comm. Pure Appl. Math. **14**, pp 187–214, 1961.

[10] G. Baym, C.J. Pethick, *Vortex core structure and global properties of rapidly rotating Bose-Einstein condensates,* Phys. Rev. A **69**, 2004.

[11] V. Bretin, S. Stock, Y. Seurin, J. Dalibard, *Fast Rotation of a Bose-Einstein Condensate,* Phys. Rev. Lett. **92**, 050403, 2004.

[12] E. Carlen, *Some Integral Identities and Inequalities for Entire Functions and Their Application to the Coherent State Transform,* J. Funct. Analysis **97**, 1991.

[13] K. Chandrasekharan *Elliptic Functions* Grundlehren der mathematischen Wissenschaften 281, Springer, 1985.

[14] N. R. Cooper, S. Komineas, N. Read, *Vortex lattices in the lowest Landau level for confined Bose-Einstein condensates,* Phys. Rev. A **70**, p 033604, 2004.

[15] J.M. Delort *FBI transformation. Second microlocalization and semilinear caustics.* Lect. Notes in Math. 1522, Springer-Verlag, 1992.

[16] P. Engels et al, *Observation of Long-lived Vortex Aggregates in Rapidly Rotating Bose-Einstein Condensates,* Phys. Rev. Lett. **90**, p 170405, 2003.

[17] E. A. Cornell, W. Ketterle, C. E. Wiemann, *Les Prix Nobel. The Nobel Prizes 2001,* Editor Tore Frängsmyr, [Nobel Foundation], Stockholm, 2002.

[18] U.R. Fischer, G. Baym, *Vortex states of rapidly rotating dilute Bose-Einstein condensates,* Phys. Rev. Lett. **90**, p 140402, 2003.

[19] G.B Folland *Harmonic Analysis in Phase Space,* Princeton University Press, 1989.

[20] L. Gross *Hypercontractivity on complex manifolds,* Acta Math. **182**, 1999.

[21] T. L. Ho, *Bose-Einstein Condensates with Large Number of Vortices,* Phys. Rev. Lett. **87**, p 060403, 2001.

[22] L. Hörmander. *Lectures on Nonlinear Hyperbolic Differential Equations.* Mathématiques et Applications 26, Springer, 1997.

[23] W.H. Kleiner, L.M. Roth, S.H Autler, *Bulk Solutions of Ginzburg-Landau Equations for type II Superconductors: Upper Critical Fiel Region,* Phys. Rev. **133**(5A), pp 1226–1227, 1964.

[24] L. D. Landau,E. M. Lifschitz, *Quantum Mechanics* (Oxford: Pergamon), 1965.

[25] N. Lerner *The Wick calculus of pseudo-differential operators and energy estimates,* New trends in microlocal analysis (Tokyo 1995), Springer, Tokyo, 1997.

[26] E. H. Lieb, R. Seiringer, *Derivation of the Gross-Pitaevskii equation for rotating Bose gases,* Arxiv:math-ph/0504042, 2005.

[27] E. H. Lieb, R. Seiringer, J. Yngvason, *Bosons in a trap: A rigorous derivation of the Gross-Pitaevskii energy functional,* Phys. Rev. A **61**, p 0436021, 2000.

[28] E. H. Lieb, R. Seiringer, J. Yngvason, *A rigourous derivation of the Gross-Pitaevskii energy functional for a two-dimensional Bose gas,* Comm. Math. Phys. **224**, pp 17–31, 2001.

[29] K. Lu, X. B. Pan, *Gauge invariant eigenvalue problems in R^2 and in R^2_+*. Trans. Amer. Math. Soc. **352**, no. 3, pp 1247–1276, 2000.

[30] K. Madison K, Chevy F, Bretin V, Dalibard J, *Vortex Formation in a Stirred Bose-Einstein Condensate,* Phys. Rev. Lett. **84**, p 806, 2000

[31] A. Martinez *An Introduction to Semiclassical Analysis and Microlocal Analysis,* Universitext, Springer-Verlag, 2002.

[32] M.R. Matthews et al., *Vortices in a Bose-Einstein Condensate,* Phys. Rev. Lett. **83**, p 2498, 1999.

[33] S. Nonnenmacher, A. Voros, *Chaotic Eigenfunctions in Phase Space*, J. Stat. Phys. **92**, pp 431–518, 1998.

[34] C. Pethick, H. Smith, *Bose-Einstein condensation in dilute gases*, Cambridge University Press, 2001.

[35] L. Pitaevskii, S. Stringari, *Bose-Einstein condensation*, Oxford University Press, 2003.

[36] V. Schweikhard, I. Coddington, P. Engels, V. P. Mogendorff, E. A. Cornell, *Rapidly Rotating Bose-Einstein Condensates in and near the Lowest Landau Level,* Phys. Rev. Lett. **92**, p 040404 2004.

[37] K. Schnee, J. Yngvason, *Bosons in Disc-Shaped Traps: From 3D to 2D*, Erwin Schrödinger Institute of Vienna, preprint n° 1716, 2005.

[38] D. E. Sheehy, L. Radzihovsky, *Vortices in Spatially Inhomogeneous Superfluids,* Phys. Rev. A **70**, p 063620 2004.

[39] J. Sjöstrand *Singularités analytiques microlocales.* Soc. Math. France, Astérisque **95**, pp 1–166, 1982.

[40] S. Stock, V. Bretin, F. Chevy, J. Dalibard, *Shape oscillation of a rotating Bose-Einstein condensate,* Europhys. Lett. **65**, p 594, 2004.

[41] O. Törnkvist, *Ginzburg-Landau theory of the electroweak phase transition and analytical results,* arXiv:hep-ph/9204235, 1992.

[42] G. Watanabe, G. Baym and C. J. Pethick, *Landau levels and the Thomas-Fermi structure of rapidly rotating Bose-Einstein condensates,* Phys. Rev. Lett. **93**, p 190401 2004.