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Renormalization of exponential sums and matrix cocycles

RENORMALIZATION OF EXPONENTIAL SUMS AND MATRIX COCYCLES

ALEXANDER FEDOTOV AND FRÉDÉRIC KLOPP

ABSTRACT. In this paper, we present a new point of view on the renormalization of some exponential sums stemming from number theory. We generalize this renormalization procedure to study some matrix cocycles arising in spectral problems of quantum mechanics.

RÉSUMÉ. Dans cet article, nous présentons un nouveau point de vue sur la renormalisation de certaines sommes exponentielles issues de la théorie des nombres. Nous généralisons cette procédure pour étudier certains cocycles matriciels liés à des problèmes spectraux de la mécanique quantique.

0. INTRODUCTION

The work we discuss here was motivated by a talk by Michel Mendès-France given at the University Paris 12 in January, 2005. In his talk, M. Mendès-France discussed geometrical aspects of some exponential sums coming up in number theory (see e.g. [7, 13] and references therein).

0.1. **Renormalization of exponential sums.**

On the complex plane, one plots the sequence of points representing the successive values of the sums

$$(0.1) \qquad S(N, a, k) = \sum_{0 \leq n \leq N-1} e\left(-\frac{an^k}{k}\right), \quad N = 1, 2, 3 \ldots, \quad e(z) = e^{2\pi i z},$$

where $0 < a < 1$ and $k > 1$ are fixed parameters. The points $S_1(a)$, $S_2(a)$, ... , being successively connected by segments of strait lines, form beautiful curves having a self similar structure, see Fig. 1. The explanation of some features of this structure given during Mendès-France's talk was based on the Poisson summation formula,

$$(0.2) \qquad \sum_n g(n) = \sum_m \hat{g}(m), \quad g(m) = \int_{-\infty}^{\infty} e(-mx)g(x)\, dx,$$

which, being applied formally, leads the "relations"

$$(0.3) \qquad S(N, a, 2) \sim \frac{e(-1/8)}{\sqrt{a}} S(N_1, a_1, 2), \quad N_1 = aN, \quad a_1 = -\frac{1}{a};$$

$$(0.4) \qquad S(N, a, 3) \sim \frac{e(-1/8)}{(4a)^{1/4}} S(N_1, a_1, 3/2), \quad N_1 = a(N-1)^2, \quad a_1 = -\frac{1}{\sqrt{a}}$$

and so on.

For a given N, if $a < 1$ (and this can always be arrange for by the 1-periodicity of $z \mapsto e(z)$), the quadratic sum in the left hand side of (0.3) is expressed in terms of a quadratic sum with

(a) The graph of the exponential sum for $k = 2$ and $a = \pi^3$ (500 points)

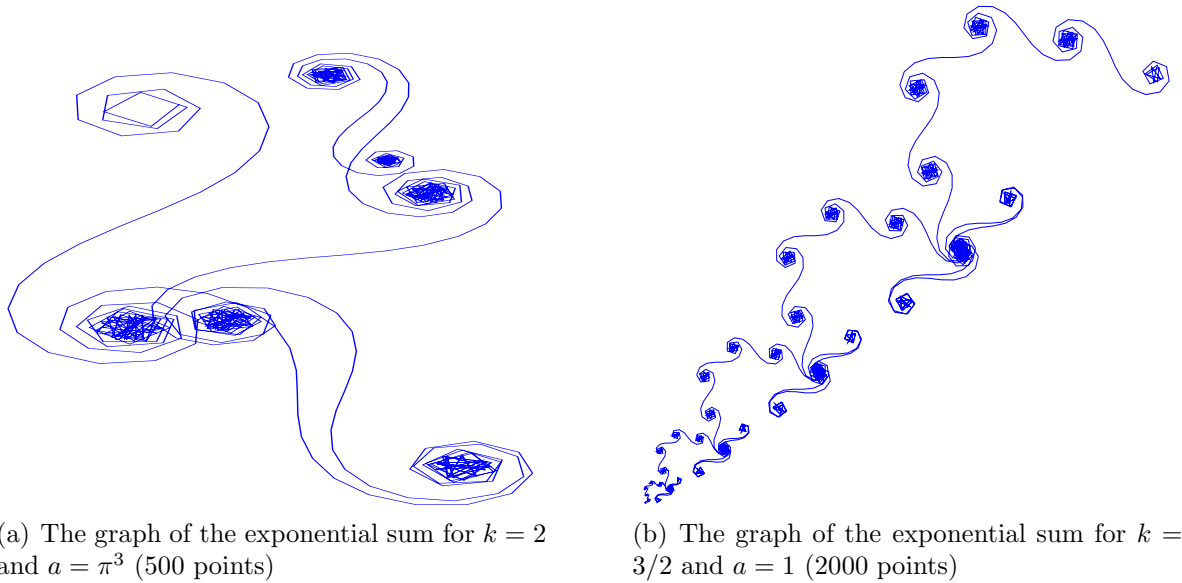(b) The graph of the exponential sum for $k = 3/2$ and $a = 1$ (2000 points)

Figure 1: The graphs of exponential sums

a smaller number of terms in the right hand side. In (0.4), if $a \ll 1/N$, one sees the same kind of phenomenon.

The described renormalization corresponds to "erasing of small geometric details" of the complex curves representing the exponential sums and to a "rough description" of the long segments of these curves in terms of shorter segments of curves of similar origin. Note that, if a is small, the complex curves contain typical "smooth" curlicues (see [1]) as in Fig. 1(b) and 2.

There are three natural questions related to the formulae (0.3)- (0.4). First, is there a simple analytic description of the correction terms in these formulae? Second, are these correction terms really error terms with respect the exponential sums, i.e. are they smaller than the renormalized sum? And, last, what is the behavior of these terms for small values of a?

Though the observations we present in the sequel successfully work in the general case, in this note, we concentrate on quadratic exponential sums i.e. we assume that, in (0.1), the constant k is always equal to 2. In this case, one can write a simple exact renormalization formula; in the general case, one gets an asymptotic formula for $N \to \infty$. It appears that the correction terms are given by a new special function.

Figure 2: For $k = 2$ and $a = \pi/1000$

To characterize this function, we note that the exponential sums appearing in (0.3) can be described in terms to the solutions of the following difference equations

$$(0.5) \qquad s(z) - s(z - 1) = e\left(\frac{z^2}{2a}\right) \quad \text{and} \quad s(z + a) - s(z) = e\left(-\frac{z^2}{2a}\right), \quad z \in \mathbb{C}.$$

The special function mentioned above satisfies the first of these equations; multiplied by a simple exponential factor, it becomes a solution of the second one, see Lemma 1.1. Moreover, as $a \to 0$, the leading term of the asymptotics of this function contains the Fresnel integral
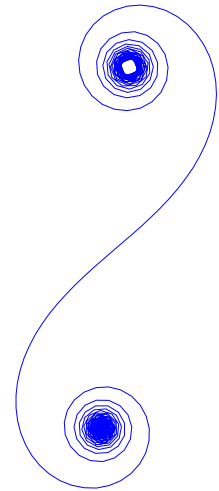
responsible for the curlicue structures, see Fig. 4.

The renormalization of exponential sums has been intensively studied; some of the first papers were [12, 15, 16, 14]. The articles [6] and [8] are recent papers on the same topic; for more references and historical remarks, we refer to them. In [12], the authors have estimated the correction terms; and, in [6] and [8], to study the fine properties of the correction terms, the Poisson formula was used and a fine geometric analysis of the curves related to the exponential sums was carried out. In the present note, use of the difference equations (0.5) will immediately lead to an exact renormalization formula. To our knowledge, the existence of such formulas was unknown.

In section 1.3, we write down the asymptotics of the special function for $a \to 0$; this immediately gives a simple analytic description of the curlicues.

0.2. **Renormalization of matrix cocycles.** The second motivation for our work was the paper [3]. There, the author has studied the difference equation

$$(0.6) \qquad \psi(k+1) = M_{3/2}(k)\psi(k), \quad M_{3/2}(k) = I + k^{-1/2}L_{3/2}(k), \quad k = 1, 2, 3 \ldots$$

where $\psi : \mathbb{N} \to SL(2, \mathbb{C})$ is an unknown matrix valued function, and $L_{3/2} : \mathbb{N} \to SL(2, \mathbb{C})$ is a matrix valued function of the form

$$(0.7) \qquad L_{3/2}(k) = \begin{pmatrix} \alpha_1 & \beta_1 \, e(2a_1 k^{3/2}/3) \\ \bar{\beta}_1 \, e(-2a_1 k^{3/2}/3) & \bar{\alpha}_1 \end{pmatrix},$$

and α and β are two complex constants. Such equations appear naturally in spectral problems of quantum mechanics as, for example, in the case of the Kronig-Penney electron in a constant electric field studied in [3].

In [3], it appeared that, in a certain sense, this difference equation is equivalent to the analogous difference equation with the matrices

$$(0.8) \qquad M_3(k) = I + k^{-1/2}L_3(k), \quad L_3(k) = \begin{pmatrix} \tilde{\alpha} & \tilde{\beta} \, e(-ak^3/3) \\ \bar{\beta} \, e(ak^3/3) & \bar{\alpha} \end{pmatrix}$$

instead of $M_{3/2}$ and $L_{3/2}$. The constants a_1 and a are related by the same relation as in (0.4). This result was obtained by an analog of the classical real WKB method (see [9]); the matrix M_3 is the transition matrix relating basis solutions having simple asymptotic behavior on intervals separated by a turning point.

The behavior of solutions of (0.7) and (0.8) are determined respectively by the behavior of the matrix products

$$M_{3/2}(N_1) \cdots M_{3/2}(2)M_{3/2}(1) \quad \text{and} \quad M_3(N) \cdots M_3(2)M_3(1).$$

If we admit that "products are similar to sums", then, we have to agree that these two matrix products are similar to the exponential sums in (0.4): both are defined by the exponentials $e(ak^3/3)$ and $e(2a_1 k^{3/2}/3)$ and, as already noted, the law of the transformation $a \mapsto a_1$ is the same.

The above observations lead us to the idea that the "quadratic exponential matrix products" defined in terms of matrices of the form

$$(0.9) \qquad M_2(k) = \begin{pmatrix} \alpha & \beta \, e(-ak^2/2) \\ \bar{\beta} \, e(ak^2/2) & \bar{\alpha} \end{pmatrix}$$

can be exactly renormalized just as the quadratic exponential sums can be; and, that, in result of this renormalization, one again obtains quadratic exponential matrix products (with new constant parameters). If this is the case, then, one can expect that such matrix products can be as effectively analyzed as the quadratic exponential sums. The main result presented in the present note is the exact renormalization formula for such quadratic matrix cocycles. It

is described in section 2. This result finds interesting applications in the spectral theory of ergodic operators [10].

Finally, we note that to renormalize the matrix product, we generalize the ideas of the monodromization method, a renormalization approach developed recently to study almost periodic equations, see, for example, [4] and [11]. The exact renormalization formula for the quadratic exponential sums can also be obtained using the monodromization idea; it is a "one dimensional" version of the exact renormalization formula for the matrix products.

1. Renormalizing quadratic exponential sums

1.1. The special function $\mathcal{F}$.
Consider the function $\mathcal{F} : \mathbb{C} \to \mathbb{C}$ defined by the formula:

$$(1.1) \qquad \mathcal{F}(\xi, a) = \int_{\gamma(\xi)} \frac{e\left(\frac{p^2}{2a}\right) dp}{e(p - \xi) - 1},$$

where the contour $\gamma(\xi)$ is going up from infinity along $l(\xi)$, the strait line $\xi + e^{i\pi/4}\mathbb{R}$, coming infinitesimally close to the point ξ, then, going around this point in the anti-clockwise direction along an infinitesimally small semi-circle, and, then, going up to infinity again along $l(\xi)$ (see Fig. 3).

The function $\mathcal{F}$ is the special function mentioned in the introduction. One proves:

Figure 3: $l(\xi)$ and $\gamma(\xi)$

Lemma 1.1. *For each $a > 0$, $\mathcal{F}$ is an entire function of ξ, and, for all $\xi \in \mathbb{C}$, one has*

$$(1.2) \qquad \mathcal{F}(\xi, a) - \mathcal{F}(\xi - 1, a) = e\left(\frac{\xi^2}{2a}\right);$$

$$(1.3) \qquad \mathcal{G}(\xi + a, a) - \mathcal{G}(\xi, a) = e\left(-\frac{\xi^2}{2a}\right),$$

$$(1.4) \qquad \mathcal{G}(\xi, a) = c(a)\, e\left(-\frac{\xi^2}{2a}\right) \mathcal{F}(\xi, a), \quad c(a) = e(-1/8)\, a^{-1/2}.$$

Proof. The first relation (1.2) follows from the residue theorem. The second relation (1.3) becomes obvious after the change of variable $z = p - \xi$ in the integral defining $\mathcal{F}$. ☐

1.2. The exact renormalization formula.
We now state the exact renormalization formula for the quadratic exponential sum $S(N, a, 2)$. One has

Theorem 1.1. *Let $N \in \mathbb{Z}$ and $a \in \mathbb{R}$ be two positive numbers. Let*

$$(1.5) \qquad \xi = \{aN\}, \quad N_1 = [aN], \quad a_1 = -\frac{1}{a}.$$

Then,

$$(1.6) \qquad S(N, a, 2) = c(a) \left[S(N_1, a_1, 2) + e\left(-\frac{aN^2}{2}\right) \mathcal{F}(\xi, a) - \mathcal{F}(0, a) \right].$$

Remark. *Note that $S(N, a_1, 2)$ is 2-periodic in a_1. Therefore, in (1.6), one can replace a_1 by the number defined by*
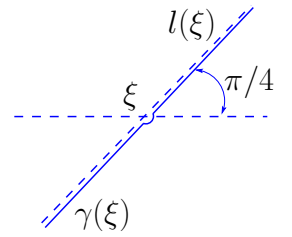
$$a_1 = -\frac{1}{a}\,(\,mod\,2\,), \quad -1 < a_1 \leq 1.$$

Proof. By (1.3), we get $\mathcal{G}(Na, a) = S(N, a, 2) + \mathcal{G}(0, a) = S(N, a, 2) + c(a)\mathcal{F}(0, a)$. And, by (1.2),

$$\mathcal{F}(Na, a) = \sum_{k=0}^{N_1-1} e\left(\frac{(aN-k)^2}{2a}\right) + \mathcal{F}(\xi, a) = e\left(\frac{aN^2}{2}\right) S(N_1, a_1, 2) + \mathcal{F}(\xi, a).$$

Now, (1.4) implies (1.6). □

1.3. The asymptotics of $\mathcal{F}$ and the structures of the curlicues.
To understand the curlicue structure seen in Fig. 1 and 2, we first derive the asymptotics of $\mathcal{F}$ for a small. The curlicues are then discussed.

1.3.1. *The asymptotics of $\mathcal{F}$.* First, discuss the asymptotics of $\mathcal{F}$ for $a \to 0$. One has

Proposition 1.1. *Let $-1/2 \le \xi \le 1/2$ and $0 < a \le 1$. Then, $\mathcal{F}$ admits the representation:*

$$(1.7) \qquad \mathcal{F}(\xi, a) = e(1/8)\, f(a^{-1/2}\xi) + O(a^{1/2}), \quad f(t) = e(t^2/2) \int_{-\infty}^{t} e(-\tau^2/2)\, d\tau,$$

where $O(a^{1/2})$ is bounded by $C\, a^{1/2}$, where C is an absolute constant.

Note that, when $0 \le \xi < 1$ as in Theorem 1.1, we can bring ourselves back to the case of Proposition 1.1 using (1.2).

Proof. We represent $\mathcal{F}$ in the form:

$$(1.8) \qquad \mathcal{F}(\xi, a) = \frac{1}{2\pi i} \int_{\gamma(\xi)} \frac{e\left(\frac{p^2}{2a}\right) dp}{p - \xi} + \int_{\gamma(\xi)} g(p - \xi)\, e\left(\frac{p^2}{2a}\right) dp,$$

where

$$g(p - \xi) = \frac{1}{e(p-\xi)-1} - \frac{1}{2\pi i(p-\xi)}.$$

As $-1/2 \le \xi \le 1/2$, the integration contour in the second integral can be deformed into the curve $\gamma(0)$ without intersecting any pole of the integrand. Then, the distance between the integration contour and these poles becomes bounded from below by $1/2^{3/2}$, and one easily gets $|g(p-\xi)| \le Const$ uniformly in $-1/2 \le \xi \le 1/2$ and in $p \in \gamma(0)$. This immediately implies that the second term in (1.8) is bounded by $Const\, a^{1/2}$. Finally, it is easily seen that first term satisfies the equation $I'(\xi) = -e(1/8)a^{-1/2} + \xi a^{-1} I(\xi)$, and that it tends to 0 when $\xi \to -\infty$ along $\mathbb{R}$. This implies that this term is equal to $e(1/8)\, f(a^{-1/2}\xi)$ and completes the proof of Proposition 1.1. □

Remark 1.1. Applying the saddle point method to the second term in (1.8), one can get the asymptotic expansion for $\mathcal{F}$ up to any given order of $a^{1/2}$.

1.3.2. *The curlicues.* Now, we can turn to the discussion of the curlicue structures observed in Fig. 1 and 2. We obtain their asymptotic description for $a \to 0$. We describe the curlicues up to terms of order $O(1)$. Remark 1.1 made above allows to control all these terms.

We shall write $A \sim B$ if $|A - B|$ is bounded by an absolute constant.

Let N_0 and N_1 be two positive integers such that

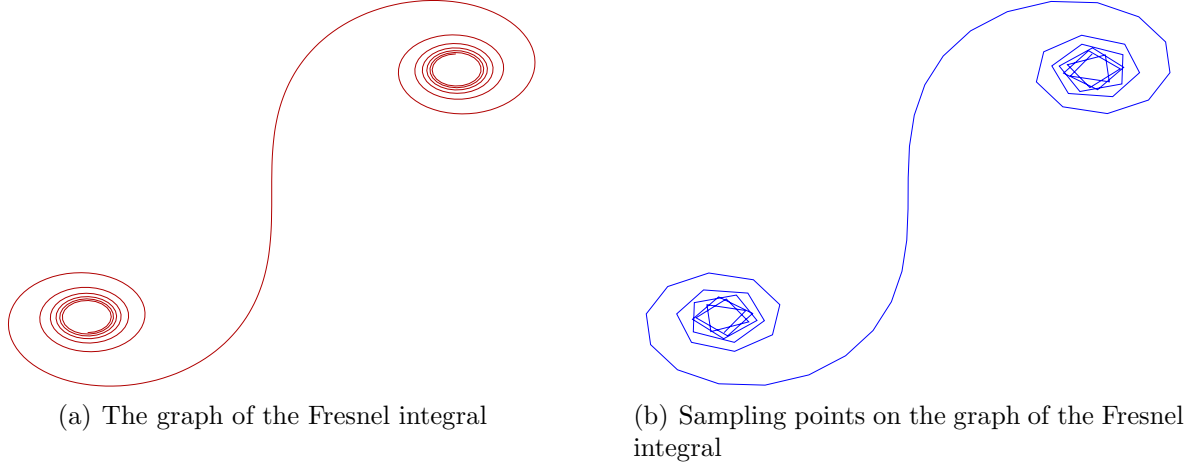$$N_1 = [aN_0], \qquad a(N_0 - 1) < N_1.$$

We check

(a) The graph of the Fresnel integral

(b) Sampling points on the graph of the Fresnel integral

Figure 4: The graph of the Fresnel integral

Lemma 1.2. *Let N_0 and N_1 be fixed as above. If*

$$-1/2 \leq aN - N_1 \leq 1/2,$$

then, for $a \to 0$, one has

$$(1.9) \qquad S(N, a, 2) \sim Const + a^{-1/2} e\left(\frac{N_1^2}{2a}\right) \int_{-\infty}^{(aN-N_1)/a^{1/2}} e(-\tau^2/2) \, d\tau$$

where Const denotes an expression independent of N.

Proof. We prove (1.9) only for $aN - N_1 \geq 0$. For the negative values of $aN - N_1$, the arguments are similar.

The number $[aN]$ stays equal to N_1 for all $N \geq N_0$ as long as $\xi = \{aN\} < 1$. For these values of N, the quadratic exponential sum in the right hand side of (1.6) stays constant, and the variations of $S(N, a, 2)$ are described by the expression $E = c(a)e\left(-\frac{aN^2}{2}\right) \mathcal{F}(\xi, a)$ where $\xi = aN - N_1$. As

$$e\left(-\frac{aN^2}{2}\right) e\left(\frac{\xi^2}{2a}\right) = e\left(-\frac{(aN)^2}{2a} + \frac{(aN - N_1)^2}{2a}\right) = e\left(-NN_1 + \frac{N_1^2}{2a}\right) = e\left(\frac{N_1^2}{2a}\right),$$

formula (1.7) implies that

$$E \sim a^{-1/2} e\left(\frac{N_1^2}{2a}\right) \int_{-\infty}^{\xi/a^{1/2}} e(-\tau^2/2) \, d\tau, \quad \xi = aN - N_1.$$

This and (1.6) imply (1.9). $\qquad\qquad \square$

Lemma 1.2 shows that the curlicue structures of the graphs of the quadratic exponential sums on the complex plane are described by the Fresnel integral $F(t) = \int_{-\infty}^{t} e(-\tau^2/2) \, d\tau$. To our knowledge, this was not known (see e.g. [1, 6, 8]). In Fig. 4(a), we show the graph of the Fresnel integral, and in Fig. 2, we show a typical segment of a quadratic exponential sum. One can also compare Fig. 4(b) and Fig. 1(a): in the first picture, we sample some points on the graph of the Fresnel integral and interpolate linearly between consecutive points ; the second figure is the graph of the sum $N \mapsto S(N, \pi^3, 2)$.

2. Renormalization of the matrix cocycles

2.1. The matrix cocycle.
Consider the cylinder $\mathbb{R} \times \mathbb{T}$, where $\mathbb{T}$ is the one dimensional torus of length 1. Let M_2 be the matrix defined in (0.9). We interpret the matrix product

$$(2.1) \qquad\qquad M_2(N-1) \cdots M_2(2)\, M_2(1)\, M_2(0)$$

as the matrix cocycle defined by the matrix valued function $\mathcal{M} : \mathbb{R} \times \mathbb{T} \to SL(2, \mathbb{C})$,

$$(2.2) \qquad\qquad \mathcal{M}(x) = \begin{pmatrix} \alpha & \beta\, e(-x_2) \\ \bar{\beta} e(+x_2) & \bar{\alpha} \end{pmatrix}, \quad x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{R} \times \mathbb{T}.$$

and $T(a)$, the skew shift on the cylinder i.e. the automorphism of the cylinder defined by

$$T(a)x = Jx + ae_1, \quad J = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \quad e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

It is intimately related to the quadratic exponential sums, see [5]. We can write the matrix product (2.1) in the form

$$(2.3) \qquad\qquad \mathcal{M}(T^{N-1}(a)\, x) \cdots \mathcal{M}(T(a)\, x)\, \mathcal{M}(x)$$

with $x_1 = a/2$, and $x_2 = 0$, i.e. consider it as a matrix cocycle defined on $\mathbb{R} \times \mathbb{T}$ by the pair $(\mathcal{M}, T)$. We denote this cocycle by $(\mathcal{M}, T)$.

2.2. The monodromy matrix.
The analysis of the cocycle $(\mathcal{M}, T)$ is equivalent to the analysis of the equation

$$(2.4) \qquad\qquad \Psi(T(a)x) = \mathcal{M}(x)\Psi(x), \quad x \in \mathbb{R} \times \mathbb{T}.$$

To analyze this equation we generalize the ideas of the monodromization method, see, for example, [4] and [11]. The idea is that, on the cylinder, the map $T(a)$ and the translation by e_1 commute; so, the space of solutions of (2.4) is invariant with respect to the translation $f(\cdot) \to f(\cdot + e_1)$. Hence, $\Psi(\cdot)$ and $\Psi(\cdot + e_1)$ both satisfy this equation. One can easily see that the space of solutions of (2.4) is a two dimensional module over the ring of functions invariant with respect to the transformation $f(\cdot) \to f(T(a)\cdot)$. Assume that Ψ is a fundamental solution to (2.4), i.e., that $\det \Psi(x) = 1$ for all x. Then, this implies that, $\forall x \in \mathbb{R} \times \mathbb{T}$,

$$(2.5) \qquad\qquad \Psi(x + e_1) = \Psi(x)\tilde{\mathcal{M}}^t(x),$$

where ^{t} denotes the transposition, and $\tilde{\mathcal{M}} : \mathbb{R} \times \mathbb{T} \to SL(2, \mathbb{C})$ is a matrix valued function satisfying the relations

$$(2.6) \qquad\qquad \tilde{\mathcal{M}}(T(a)\, x) = \tilde{\mathcal{M}}(x), \quad \det \tilde{\mathcal{M}}(x) = 1, \quad x \in \mathbb{R} \times \mathbb{T}.$$

Define s_1, a shift on the cylinder, by $s_1(x) = x + e_1$. There is a simple relation between the matrix cocycles $(\mathcal{M}, T)$ and $(\tilde{\mathcal{M}}, s_1)$. To describe it, we pick $x = (x_1, x_2) \in \mathbb{R} \times \mathbb{T}$ such that $0 \le x_1, x_2 < 1$. We fix N, a positive integer, and we represent $T^N(a)x$ in the form

$$T^N(a)x = me_1 + ne_2 + \xi,$$

where n, m are (non-negative) integers, $\xi \in \mathbb{T} \times \mathbb{R}$, and $0 \le \xi_1, \xi_2 < 1$. It is easy to see that

$$m = [Na + x_1].$$

The definition of the monodromy matrix (2.5) immediately implies

Lemma 2.1. *Let Ψ be a fundamental solution to equation (2.4), and let $\tilde{\mathcal{M}}$ be the corresponding monodromy matrix. Then,*

$$(2.7) \quad \mathcal{M}(T^{N-1}(a)x)\cdots\mathcal{M}(T(a)x)\mathcal{M}(x)$$

$$= \Psi(\xi)\left[\,\tilde{\mathcal{M}}(\xi + (m-1)e_1)\cdots\tilde{\mathcal{M}}(\xi + e_1)\tilde{\mathcal{M}}(\xi)\,\right]^t\Psi(x)^{-1}.$$

This formula will play a crucial role for the renormalization of the matrix cocycles.

2.3. **Self-similarity of the quadratic exponential matrix cocycle.** To formulate our results, we introduce more notations. Consider the matrix defined by (2.2). As this matrix is unimodular, one has $|\alpha|^2 = 1 + |\beta|^2$. Without loss of generality, we can and do assume that $\beta \geq 0$ as the matrix with parameters α and $|\beta|$ is similar to the matrix with parameters α and β. Then, we get

$$\beta = (|a|^2 - 1)^{1/2}.$$

With this in mind, we denote our matrix by $\mathcal{M}(x, a)$.
Our main technical result is

Theorem 2.1. *Let $0 < a < 1$. There exists Ψ, a fundamental (unimodular matrix) solution to (2.4) with $\mathcal{M} = \mathcal{M}(x, a)$, such that the corresponding monodromy matrix is*

$$\tilde{\mathcal{M}} = \begin{pmatrix} Au & Bv \\ \bar{B}v^{-1} & \bar{A}u^{-1} \end{pmatrix}, \qquad u = e\left(-\frac{x_1}{a}\right), \quad v = e\left(x_2 - \frac{x_1^2}{2a} - \frac{x_1}{a} + \frac{x_1}{2}\right),$$

where

$$A = -e\left(-1/(2a)\right)\,(\bar{\alpha})^{1/a}, \quad B = \left(|\alpha|^{2/a} - 1\right)^{1/2}.$$

The solution $x \mapsto \Psi(x)$ is an entire function of x.

Note that A and B are independent of x. The proof of Theorem 2.1 is not elementary. In the present note, we only note that the construction of Ψ is closely related to the theory of the minimal entire solutions of difference equations on the complex plane (as developed in [2]).

By means of Theorem 2.1 and relation (2.7), we get the main result of this note:

Theorem 2.2. *Let $0 < a < 1$. Define x, m and ξ as for Lemma 2.1. Then,*

$$\mathcal{M}\left(T^{N-1}(a)\,x, \alpha\right)\cdots\mathcal{M}\left(T(a)\,x, \alpha\right)\mathcal{M}\left(x, \alpha\right)$$

$$= \Psi(\xi)\,D\left(\xi_1\right)^{-1}$$

(2.8)

$$\cdot\left[\,\mathcal{M}\left(T^{m-1}(a_1)\,y, \alpha_1\right)\cdots\mathcal{M}\left(T(a_1)\,y, \alpha_1\right)\mathcal{M}(y, \alpha_1)\,\right]^t$$

$$\cdot D\left(\xi_1 + m\right)\Psi(x)^{-1},$$

where $D(t) = \text{diag}\left(e(-t^2/(2a)), e(t^2/(2a))\right)$ and

$$a_1 = -1/a\ (mod\,2), \quad -1 < a_1 \leq 1,$$

$$m = [Na + x], \quad \alpha_1 = -(\bar{\alpha})^{1/a},$$

$$y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}, \quad y_1 = -(\xi_1/a + 1/2)\,(mod\,1), \quad y_2 = -(\xi_1^2/(2a) + \xi_1/2 + \xi_2)\,(mod\,1)$$

Relation (2.8) relates the matrix cocycles $(\mathcal{M}(\cdot, \alpha), T(a))$ and $(\mathcal{M}(\cdot, \alpha_1), T(a_1))$. It is a "two-dimensional" analog to (1.6), the exact renormalization formula relating the exponential sums. Of course, to make (2.8) useful for applications, one has to obtain an effective description of the solution Ψ. And, this is possible! Let us outline the idea. To study the input matrix

product for large N, one makes many consecutive renormalizations; if a is irrational, at each step, one obtains a new constant a. One can carry out the renormalizations so that each of these constants satisfies $0 < a < 1$. But, then, as the absolute value of the input constant α is greater than 1, the absolute values of the new constants α will grow. Using standard results from the metric theory of numbers, one can see that, for almost all values of the input a, the sequence of the new α tends to infinity. In result, roughly, one can replace the solution Ψ by their asymptotics for $\alpha \to \infty$. We use this idea in [10].

Finally, we note that the exact renormalization formula for the quadratic exponential sums can also be obtained using the monodromization idea. Then, the exponential $e(-a_1 z^2/2)$ arises from the "one dimensional" analog of the monodromy matrix, and the function $\mathcal{F}$ controlling the correction terms in (0.3) plays the role of the solution Ψ.

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