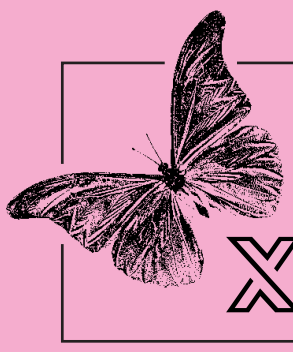

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# Semiclassical Limit of the cubic nonlinear Schrödinger Equation concerning a superfluid passing an obstacle

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**Abstract**

In this paper, we study the semiclassical limit of the cubic nonlinear Schrödinger equation with the Neumann boundary condition in an exterior domain. We prove that before the formation of singularities in the limit system, the quantum density and the quantum momentum converge to the unique solution of the compressible Euler equation with the slip boundary condition as the scaling parameter approaches 0.

Mathematics Subject Classification (1995): 35L65, 81

Keywords: Semiclassical limit, Schrödinger equation, compressible Euler equation.

## 1 Introduction

Here we consider the local in time semi-classical limit of the cubic Schrödinger equation in the exterior of a two dimensional domain in $\mathbb{R}^2$. More precisely, let $\Omega$ be an exterior domain in $\mathbb{R}^2$ such that $\partial\Omega$ is a bounded, smooth curve, and let $\nu(x)$ be the unit outward normal vector to $\partial\Omega$ at $x \in \partial\Omega$. We study the following equations when the

parameter $\epsilon$ goes to zero:

$$\begin{cases} i\,\epsilon\,\partial_t\,\psi^\epsilon = -\dfrac{\epsilon^2}{2}\,\Delta\psi^\epsilon + \left(|\psi^\epsilon|^2 - 1\right)\psi^\epsilon, & \text{in } \Omega\times\mathbb{R}_+ \\ \psi^\epsilon(t=0,x) = \sqrt{\rho_0^\epsilon(x)}\,\exp\left(\dfrac{i}{\epsilon}S_0^\epsilon(x)\right), & \\ \dfrac{\partial\psi^\epsilon}{\partial\nu}\,\Big|_{\partial\Omega} = 0, \quad \psi^\epsilon(t,x)\to\exp\left(\dfrac{i}{\epsilon}\,S^\infty(x)\right) & \text{as } |x|\to\infty, \end{cases} \tag{1.1}$$

where $\epsilon$ is a small positive parameter, $S^\infty(x) = u^\infty\cdot x$, and $u^\infty$ is a constant two-vector.

The motivation to study the problem (1.1) comes from many intresting issues concerning a superfluid passing an obstacle, see for example [FPR] and [JP]. The nonlinear Schrödinger equation (0.1), which is also called the Gross-Pitaevskii equation, has been proposed and studied as the fundamental equation for understanding superfluids, see Ginzburg-Pitaevskii [GP], Landau-Lifschitz [LL], Gross [G] and many others. It has also been used to model phenomena in the Bose-Einstein condensates. The model mathematical problem for a superfluid passing an obstacle is as follows:

$$\begin{aligned} &-i\,\psi_t = \Delta\psi + \psi(1-|\psi|^2) \quad \text{in } \mathbb{R}^2\setminus\mathbb{B}_R \\ &\text{with } \frac{\partial\psi}{\partial\nu}\Big|_{\partial\mathbb{B}_R} = 0 \quad \text{and} \quad \psi(x,0)\approx e^{i\,v_0\cdot x} \end{aligned} \tag{1.2}$$

at $|x| = +\infty$. Here $\mathbb{B}_R$ denotes the obstacle. In (1.2), one has normalized the equation in such a way the Planck constant becomes 1. Thus the size $R$ is often much larger than the unity. The well-known Madelung transform (see [M]) is to introduce two real variables $\rho\ge 0$ and $\phi$ such that $u = \sqrt{\rho}\,e^{i\,\phi}$. Then under a suitable condition one can show that (1.2) is equivalent to the fluid-type equations.

$$\begin{cases} \dfrac{\partial\rho}{\partial t} + \text{div }(\rho u) = 0 \\ \dfrac{\partial}{\partial t}\,(\rho u) + \text{div}\,(\rho\,(u\otimes u)) + \nabla\left(\dfrac{\rho^2}{2}\right) = \rho\nabla\left(\dfrac{\Delta\sqrt{\rho}}{\sqrt{\rho}}\right). \end{cases} \tag{1.3}$$

Here $u = \nabla\phi$. We note also that the phase dynammics according to

$$\frac{\partial\phi}{\partial t} = \frac{\Delta\sqrt{\rho}}{\sqrt{\rho}} - |\nabla\phi|^2 + (1-\rho). \tag{1.4}$$

The term on the right-hand side of the second equation in (1.3) is called the quantum pressure. It can be formally argued that this quantum pressure term can be neglected in a limiting process when the obstacle size $\mathbb{B}_R$ (or $R$) is much larger compared with the microscopic scale of the Gross-Pitaevskii equation (which is normalized to be 1), and when one is interested in only "long-wave" approximations (see [FPR]). Indeed, set $R = \frac{1}{\epsilon}$, and consider $\psi_\epsilon(x,t) = \sqrt{\rho^\epsilon(x,t)}\,e^{\frac{i}{\epsilon}\,S^\epsilon(x,t)}$ with $\nabla\,S^\epsilon(x,\bullet)\simeq u^\infty$ at $|x| = \infty$,

then after a proper scaling of spatial and time-variables, one reduces to study (1.1) and its associated fluid type equation:

$$\begin{cases} \partial_t \rho^\epsilon + \operatorname{div}(\rho^\epsilon u^\epsilon) = 0 \\ \partial_t(\rho^\epsilon\, u^\epsilon) + \operatorname{div}(\rho^\epsilon\, u^\epsilon \otimes u^\epsilon) + \dfrac{1}{2}\,\nabla(\rho^\epsilon)^2 = \dfrac{\epsilon^2}{2}\,\rho^\epsilon \nabla\left(\dfrac{\Delta\sqrt{\rho^\epsilon}}{\sqrt{\rho^\epsilon}}\right) \\ \rho^\epsilon(t=0,x) = \rho_0^\epsilon(x), \quad u^\epsilon(t=0,x) = \nabla S_0^\epsilon(x) \end{cases} \tag{1.5}$$

where $u^\epsilon = \nabla S^\epsilon$. The domain $\Omega$ is now given by $\mathbb{R}^2 \setminus B_1$, and the boundary conditions can be written in the following equivalent form:

$$\begin{cases} \epsilon \dfrac{\partial\sqrt{\rho^\epsilon}}{\partial\nu}\Big|_{\partial\Omega} = 0, \quad u^\epsilon \cdot \nu\Big|_{\partial\Omega} = 0, \qquad \text{and} \quad \rho^\epsilon(t,x) \to 1, \\ u(t,x) \to u^\infty \qquad \text{as} \quad |x| \to \infty. \end{cases} \tag{1.6}$$

Thus, the formal WKB-limit as $\epsilon \to 0$ of (1.5)–(1.6) is given by the following compressible Euler equation:

$$\begin{cases} \partial_t\, \rho + \operatorname{div}(\rho u) = 0, \\ \partial_t(\rho u) + \operatorname{div}(\rho u \otimes u) + \dfrac{1}{2}\,\nabla\rho^2 = 0 \\ u(t=0,x) = u_0(x), \quad \rho(t=0,x) = \rho_0(x) \end{cases} \tag{1.7}$$

with the slip boundary condition.

$$u \cdot \nu\Big|_{\partial\Omega} = 0, \quad \text{and} \quad \rho(t,x) \to 1, \quad u(t,x) \to u^\infty \quad \text{as} \quad |x| \to \infty. \tag{1.8}$$

Of course, it is necessary to assume that $(\rho_0^\epsilon(x), \nabla S_0^\epsilon(x))$ converges to $(\rho_0(x), u_0(x))$ in some appropriate sense. It should be noted that the first boundary condition in (1.6), that is, $\epsilon\frac{\partial}{\partial\nu}\left(\sqrt{\rho\,\epsilon}\right)\big|_{\partial\Omega} = 0$, disappears in the limiting process $\epsilon \to 0^+$. Otherwise it would lead to an additional boundary condition for the limit system (1.7) which would be undesirable.

Before presenting the main result of this paper, let us recall some known results on the semiclassical limit of nonlinear Schrödinger equation.

Firstly when $\Omega = \mathbb{R}^d$ and if there is no super fluid at the infinity, the nonlinear term $(|\psi^\epsilon|^2 - 1)\psi^\epsilon$ in (1.1) is often replaced by $g(|\psi^\epsilon|^2)\psi^\epsilon$ with $g'(\cdot) > 0$. If, in addition, the phase function $S_0^\epsilon$ is independent of $\epsilon$, and the amplitude is given by the expansion: $\sum_{j=0}^N a_j(x)\epsilon^j + \epsilon^N r_N(x,\epsilon)$ with $\lim_{\epsilon\to 0} \|r_N(\cdot,\epsilon)\|_{H^s} = 0$ for $s$ large enough, Grenier ([Grenier98]) obtained a similar expansion for the solution of (1.1) in a small time. His main idea is that: instead of looking, as usual, for solutions $\psi^\epsilon$ of the form:

$$\psi^\epsilon(t,x) = a^\epsilon(t,x) e^{i\frac{S(t,x)}{\epsilon}} \tag{1.9}$$

with $S(t,x)$ independent of $\epsilon$, $a^\epsilon(t,x)$ a real valued function, he looks for solutions $\psi^\epsilon$ of the form:

$$\psi^\epsilon(t,x) = a^\epsilon(t,x)\, e^{i\frac{S^\epsilon(t,x)}{\epsilon}} = \left(a_1^\epsilon(t,x) + i a_2^\epsilon(t,x)\right) e^{i\frac{S^\epsilon(t,x)}{\epsilon}} \tag{1.10}$$

with $a_1^\epsilon, a_2^\epsilon, S^\epsilon$ being real valued functions. By plugging (1.10) into (1.1), separating the real and imaginary part, one can get the governing equations for $a_1^\epsilon, a_2^\epsilon$ and $\nabla S^\epsilon$. Then the standard energy estimate for symmetric hyperbolic system can be used to solve the resulting problem. But unfortunately, this method cannot be applied here. The main difficulty lies in the Neumann boundary condition in (1.1). In fact, if we assume the solution has the form (1.10), then the boundary condition $\frac{\partial \psi^\epsilon}{\partial \nu} \mid_{\partial\Omega} = 0$ can be rewritten in the following equivalent form:

$$\left(\epsilon \frac{\partial a_1^\epsilon}{\partial \nu} - a_2^\epsilon \frac{\partial S^\epsilon}{\partial \nu}\right)\Big|_{\partial\Omega} = 0, \quad \left(\epsilon \frac{\partial a_2^\epsilon}{\partial \nu} + a_1^\epsilon \frac{\partial S^\epsilon}{\partial \nu}\right)\Big|_{\partial\Omega} = 0.$$

With these nonlinear boundary conditions, all known existing methods for the energy estimates do not seem to work.

On the other hand, when one considers the semiclassical limit of Schrödinger-Poisson equation:

$$\begin{cases} i\,\epsilon\,\partial_t\,\psi^\epsilon = -\dfrac{\epsilon^2}{2}\,\Delta\psi^\epsilon + V^\epsilon\psi^\epsilon, & \text{in} \qquad \mathbb{R}^d \times \mathbb{R}_+ \\ -\Delta V^\epsilon = |\psi^\epsilon|^2 - b(x), \\ \psi^\epsilon(t=0,x) = \sqrt{\rho_0^\epsilon(x)}\,\exp\left(\dfrac{i}{\epsilon} S_0^\epsilon(x)\right). \end{cases} \tag{1.11}$$

Here $V^\epsilon$ can not be written as $g(|\psi^\epsilon|^2)$ with $g'(\cdot) > 0$, the method in [Grenier98] can not be applied to study (1.11) as $\epsilon$ approaches 0. Motivated by a work of Brenier [Brenier2000] in the studying of the convergence of the scaled Vlasov-Poisson system to the incompressible Euler system, the second author [ZP1] (more general nonlinearity in [ZP2]) uses Wigner measure and modifies the modulated energy estimate to prove the convergence of the quantum density and quantum momentum to the solution of compressible Euler equations before the formation of singularities in the limit system as $\epsilon$ approaches 0.

Indeed, in 1932, E. Wigner [Wigner] introduced the following transform in quantum mechanics:

$$f^\epsilon(t,x,\xi) = \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} e^{-i\xi y}\psi^\epsilon(t, x+\frac{\epsilon y}{2})\overline{\psi^\epsilon(t,x-\frac{\epsilon y}{2})}\,dy, \tag{1.12}$$

then a direct calculation [LP] by applying (1.12) to (1.11) shows that $f^\epsilon$ satisfies the so-called Wigner equation:

$$\partial_t f^\epsilon + \xi\cdot\nabla_x f^\epsilon + \theta^\epsilon[V^\epsilon]f^\epsilon = 0, \qquad x,\xi\in\mathbb{R}^d, t>0, \tag{1.13}$$

where $\theta^\epsilon[V^\epsilon]f^\epsilon$ is a pseudo-differential operator defined by

$$\theta^\epsilon[V^\epsilon]f^\epsilon = \frac{i}{(2\pi)^d}\int\!\!\int_{\mathbb{R}^{2d}_{\eta,y}} \frac{V^\epsilon(x+\frac{\epsilon y}{2}) - V^\epsilon(x-\frac{\epsilon y}{2})}{\epsilon} f^\epsilon(t,x,\eta)e^{-i(\xi-\eta)y}\,d\eta\,dy.$$

Formally passing $\epsilon \to 0$ in (1.13), one recovers the classical Vlasov-Poisson equation. However, the rigorous justification is much more difficult. The main difficulty lies in the fact that: the authors in [Gerard] and [LP] prove that the limit to the Wigner transform is only a positive Radon measure. Actually, this measure is called semi-classical measure by the author in [Gerard] and is called Wigner measure by authors in [LP]. So far, the only known general global limit from (1.13) to Vlasov-Poisson equation is in 1-D [ZZM], where the authors crucially used the properties of BV functions in two space dimension.

In particular, the main ingredient in [ZP1] (and slightly different one in [ZP2]) is to study the evolution of the following functional:

$$H^\epsilon(t) = \int_{\mathbb{R}^d} h^\epsilon\,(t,x)\,dx = \int_{\mathbb{R}^d} \frac{1}{2}\left(\int_{\mathbb{R}^d} |\xi - u(t,x)|^2\, f^\epsilon(t,x,\xi)\,d\xi + |\nabla\triangle^{-1}(\rho^\epsilon - \rho)|^2\right) dx, \tag{1.14}$$

where $(\rho, u)$ is the unique local smooth solution to the limit system. Here, since we work on the exterior domain $\Omega$, we do not know how to globally define Wigner transform of $\psi^\epsilon$ on $\Omega$. (As pointed by P. Gerard and L. Miller, one might localize the wave function $\psi^\epsilon$ inside $\Omega$, and define a localized version of Wigner transform. However, this modified Wigner transform will not be useful in the following calculations, one may check the proof of (2.9) in [LZ] for more details.) Hence (1.14) cannot be directly applied. Fortunately, we observe that by (3.24) in [ZP1] one has

$$\int_{\mathbb{R}^d} |\xi - u(t,x)|^2\, f^\epsilon(t,x,\xi)\,d\xi = |(\epsilon\nabla_x - iu)\,\psi^\epsilon|^2.$$

In other words what really was used in [ZP1] (or [ZP2]) is in fact $\int_{\mathbb{R}^d} h^\epsilon\,(t,x)\,dx$ with

$$h^\epsilon(t) =: \frac{1}{2}\left(|(\epsilon\nabla_x - iu)\,\psi^\epsilon|^2 + \left|\nabla\Delta^{-1}(\rho^\epsilon - \rho)\right|^2\right).$$

Note by the special nonlinearity in (1.1), we need to replace the second term above by $|\rho^\epsilon - \rho|^2$. Therefore, we shall consider the following functional:

$$H^\epsilon(t) =: \frac{1}{2}\int_\Omega |(\epsilon\nabla_x - i\,u)\,\psi^\epsilon|^2\,dx + \frac{1}{2}\int_\Omega |\rho^\epsilon - \rho|^2\,dx. \tag{1.15}$$

It can be viewed as a defect measure in studying weakly convergent sequences of solutions. We shall prove that $H^\epsilon(t)$ satisfies a Gronwall-type growth estimate. Thus, if $H^\epsilon(0) \to 0$ as $\epsilon \to 0^+$, then $H^\epsilon(t) \to 0$ for $t$ in an interval of considerations.

This argument can actually be used also to simplify part of proofs in [ZP1] and [ZP2]. It also avoids the use of a much more sophisticated analytic tool–Wigner measures.

It should be mentioned that a similar idea was also used in a recent work [MP1] to study the quasi-neutral limit of the scaled Schrödinger-Poisson equation to the incompressible Euler equation in a periodic domain.

In this text, we need the following assumptions :

(A1) $\left(\sqrt{\rho_0^\epsilon(x)}\ \exp\left(\frac{i}{\epsilon}S_0^\epsilon(x)\right) - \exp(\frac{iu^\infty\cdot x}{\epsilon}) \in H^3(\Omega)\right)$, and $\nabla\sqrt{\rho_0^\epsilon(x)}$, $\exp\left(\frac{i}{\epsilon}\ S_0^\epsilon\right) - \exp\left(\frac{iu^\infty\cdot x}{\epsilon}\right)$, $\sqrt{\rho_0^\epsilon(x)}\,\nabla S_0^\epsilon(x)$ are uniformly bounded in $L^2(\Omega)$;

(A2) both $\rho_0^\epsilon(x) - \rho_0(x)$ and $\sqrt{\rho_0^\epsilon(x)}\,(\nabla S_0^\epsilon(x) - u_0(x))$ converge to 0 in $L^2(\Omega)$.

To guarantee the local existence of smooth solution to the (1.7) and (1.8), we need the following compatibility conditions for the initial data:

(A3) let $\frac{1}{2} \le \rho_0(x)$, $(\rho_0(x) - 1, u_0(x) - u^\infty) \in H^3(\Omega)$, then $\nu\cdot\partial_t^k u(0)\big|_{\partial\Omega} = 0$, $0 \le k \le 2$, with $\partial_t^k u(0)$ the $k^{th}$ time derivative at $t = 0$ of any solution of (1.7) and (1.8). These derivatives can be calculated from the second equation of (1.7) to yield a condition in terms of $\rho_0$ and $u_0$.

Here is our main Theorem from [LZ]:

**Theorem 1.1.** *Let the initial datum $(\rho_0^\epsilon(x), S^\epsilon(x)), (\rho_0(x), u_0(x))$ satisfy (A1–A3). $\psi^\epsilon(t,\ x), (\rho(t,x), u(t,x))$ be the solutions to* (1.1) *and* (1.7)–(1.8) *respectively. Then there exists a positive constant $T^*$ such that for all $T < T^*$, $(\rho(t,x) - 1, u(t,x) - u^\infty) \in \bigcap_{j=0}^2 C^j([0,T], H^{3-j}(\Omega))$, furthermore,*

$$|\psi^\epsilon(t,x)|^2 - \rho(t,x) \to 0 \ \textit{in}\ L^\infty\left([0,T], L^2(\Omega)\right), \tag{1.16}$$

$$\epsilon Im\left(\overline{\psi^\epsilon}(t,x)\nabla\psi^\epsilon(t,x)\right) \to (\rho u)(t,x) \ \textit{in}\ L^\infty\left([0,T], L^1_{loc}(\Omega)\right), \tag{1.17}$$

*as $\epsilon \to 0$.*

**Remark 1.1.**

1) Comparing the above Theorem with the results in [ZP1] and [ZP2], we improved the convergence in (1.17). In [ZP1] and [ZP2], one only proved: for any fixed $t < T^*$, there holds

$$\epsilon\mathrm{Im}(\overline{\psi^\epsilon(t,\cdot)}\nabla\psi^\epsilon(t,\cdot)) \rightharpoonup (\rho u)(t,\cdot) \text{ in the sense of measure.}$$

2) By modifying the proof a little bit, we can show Theorem 1.1 for a more general nonlinearity, $f(|\psi^\epsilon|^2)\psi^\epsilon$ with $f'(\cdot) > 0$, and in exterior domain of general space dimension, provided that (1.1) still has a global unique smooth solution with this nonlinear term. Actually under the condition that $f'(\cdot) > 0$, then we can prove the limit system (1.7)-(1.8) but with $\frac{1}{2}\rho^2$ there replaced by $F(\rho) = \int_0^\rho f(\tau)\,d\tau$ still has local well-posedness. Then instead of study the time derivative of the functional $H^\epsilon(t)$ defined in (1.1), we consider the time derivative of

$$\overline{H}^\epsilon(t) =: \frac{1}{2}\int_\Omega |(\epsilon\nabla_x - i\,u)\,\psi^\epsilon|^2\,dx + \int_\Omega (F(\rho^\epsilon) - F(\rho) - f(\rho)(\rho^\epsilon - \rho))\,dx.$$

For a clear presentation, we are not going to pursue that here.

Finally, we would like to point out that for the classical fluids, it is well-known (see [DD]) there is a critical speed $v_0$ of the fluids at infinity such that whenever $|u^\infty| < v_0$, there is a steady state solution of (1.7). More precisely, there is a smooth solution of

$$\operatorname{div}(\rho \nabla \phi) = 0 \ \text{ in } \Omega, \quad \nabla \phi_{(\infty)} = u^\infty, \tag{1.18}$$

with $\rho = 1 - |\nabla \phi|^2 > 0$ in $\Omega$ (see also (1.4)). Solutions of (1.18) have maximum of $|\nabla \phi|$ achieved somewhere on $\partial\Omega$.

On the other hand, when $|u^\infty| > v_0$, then there is no smooth solution to (1.18). The flow (1.7) with such initial data would develop shock in a later time.

One often refers to the former case as subsonic and the latter case as supersonic. One consequence, of our convergence theorem (1.1) for the semiclassical limit, is that in this limiting process the same picture remains valid in the subsonic case. Since a superfluid is by definition frictionless, there cannot be shock waves developed for (1.1). (In particular the flow (1.1) is time reversible.) What would be the substitution for "shock" has been addressed in [FPR] and [JP]. However, a precisely mathematical proof has not been found particularly for the transonic case, that is when $|u^\infty| \simeq v_0$.

# 2 Outline of the proof of Theorem 1.1

**step 1.** The local well-posedness of (1.7-1.8).

In this section, we will prove the local existence of smooth solution to the exterior problem of the limit system (1.7-1.8). Actually we will study the problem with more general pressure term than that in (1.7):

$$\begin{cases} \partial_t \rho + \operatorname{div}(u\rho) = 0, \qquad x \in \Omega, \quad t \geq 0, \\ \partial_t u + u \cdot \nabla u + \nabla P(\rho) = 0, \\ (\rho(t=0,x), u(t=0,x)) = (\rho_0(x), u_0(x)), \end{cases} \tag{2.1}$$

with the boundary conditions:

$$u \cdot \nu \,|_{\partial\Omega} = 0, \quad \rho(t,x) \to \rho^\infty, u \to u^\infty, \text{ as } |x| \to \infty. \tag{2.2}$$

To guarantee the strict hyperbolicity of (2.1), we need the assumption that

$$P'(\cdot) > 0. \tag{2.3}$$

When $\rho^\infty = 0$, $u^\infty = 0$ and $\Omega$ is a bounded domain, this problem has been studied by Beirao in [Bei81] and [Bei92]. And the local existence of smooth solutions to the full ideal gas dynamics equations in a bounded domain has been studied by Schochet in [Sch86]. In this section, we are going to modify the arguments in [Bei81], [Bei92] and [Sch86] to yield the local well-posedness of (2.1–2.2).

For convenience, let us denote $\bigcap_{j=0}^{k} C^j([0,T], H^{k-j}(\Omega))$ by $X_{k,T}$, with the norm $|||w|||_{k,T} = \sup_{0\le t\le T} |||w(t)|||_k$ and $|||w(t)|||_k = \sum_{j=0}^{k} \|\partial_t^j w(t,\cdot)\|_{H^{k-j}(\Omega)}$. As a convention in this section, $C(\cdot,\cdot,\cdots)$ will be constants, which are nondecreasing functions of their variables and they may change from line to line.

Then the following Theorem is the main result of this step:

**Theorem 2.1.** *Let $(\rho_0(x)-\rho^\infty, u_0(x)-\bar u(x)) \in H^3(\Omega)$, and satisfy the compatibility condition (A3) in the introduction, where $\bar u(x) \in C^\infty(\Omega)$, with*

$$\bar u(x) = \begin{cases} 0, & \text{if } x \in \{x : |x| \le R\}, \\ u^\infty, & \text{if } x \in \{x : |x| \ge 2R\}, \end{cases}$$

*for a sufficiently large $R$ so that $\Omega \subset \{x : |x| \le R\}$. Then there exists a positive constant $T^*$, such that (2.1–2.2) has a unique local smooth solution $(\rho, u)$ with $(\rho(t,x)-\rho^\infty, u(t,x)-\bar u(x)) \in X_{3,T}$, for any $T < T^*$.*

**Remark 2.1.** 1) It should be noted here that with smoother initial data and along with compatibility conditions, we can get a more smooth solution. And the proof of the Theorem is not only for space dimension 2, but it works for general space dimension greater than 1.

2) Generally smooth solution to (2.1) will blow up in finite time even in the whole space case, see [Si85] for example.

The main idea of the proof is to modify the classical arguments in [Bei81] and [Bei92] to our case here, and we omit the details.

**step 2.** The global existence of solution to (1.1).

For simplicity, let us set $\epsilon = 1$ in (1.1). More precisely, let $\Omega$ be an exterior domain of $\mathbb{R}^2$, with $\partial\Omega$ bounded and smooth.Suppose $u^\infty = (u_1^\infty, u_2^\infty)$ is a constant two vector, we consider the global existence of smooth solutions to the following initial boundary value problem:

$$\begin{cases} i\,\partial_t\,\psi = -\dfrac{1}{2}\,\Delta\psi + \big(|\psi|^2 - 1\big)\,\psi, \qquad x \in \Omega, \quad t \ge 0, \\ \psi(t=0,x) = \psi_0(x), \quad \psi_0(x) \to e^{iu^\infty\cdot x} \quad \text{as} \quad |x| \to \infty, \\ \dfrac{\partial\psi}{\partial\nu}\,|_{\partial\Omega} = 0. \end{cases} \tag{2.4}$$

Comparing with the problems in [BG] and [TS], one of the main difficulties here is that: since $\psi_0(x) \to e^{iu^\infty\cdot x}$ as $|x| \to \infty$, $\psi_0(\cdot)$ and $\nabla\psi_0(\cdot) \notin L^2(\Omega)$. We will actually prove the existence of more regular solutions than those obtained in [BG] and [TS]. The main result can be stated as the following:

**Theorem 2.2.** *Let $s \geq 2$ be a positive integer, $\psi_0(x) - e^{iu^\infty \cdot x} \in H^s(\Omega)$. Then (2.4) has a unique global smooth solution $\psi(t,x)$ such that $\partial_t^j \partial_x^\alpha \left(\psi(t,x) - e^{i(u^\infty \cdot x - \frac{|u^\infty|^2}{2}t)}\right) \in L^\infty\left([0,T],\, H^{s-2j-|\alpha|}(\Omega)\right)$ for all $T < \infty$ and $1 \leq 2j + |\alpha| \leq s$.*

**step 3.** Modified Madelung's fluid dynamic equation and *a priori* estimate.

In this step, we will employ and improve some arguments in [ZP1] and [ZP2] to prove Theorem 1.1. If the initial data of (1.1) satisfies (A1) in the introduction, by Theorem 2.2 in the Appendix, we know that (1.1) has a unique global smooth solution $\psi^\epsilon(t,x)$ such that $\partial_t^j \partial_x^\alpha (\psi^\epsilon(t,x) - A^\epsilon(t,x)) \in L^\infty([0,T], H^{s-2j-|\alpha|}(\Omega))$ for all $T < \infty, 1 \leq 2j + |\alpha| \leq 3$, where $A^\epsilon(t,x) = \chi(x)\, e^{\frac{i}{\epsilon}(u^\infty \cdot x - \frac{|u^\infty|^2}{2}t)}$, and $\chi(x) \in C^\infty(\mathbb{R}^2)$ with $\chi(x) \begin{cases} 0, & \text{for } |x| \leq R \\ 1, & \text{for } |x| \geq 2R, \end{cases}$ and $R$ is big enough such that $\Omega^c \subset B_R(0)$.

Before we proceed further, let us first modify the Madelung's fluid dynamic equation to the following form, see [LX].

**Lemma 2.1.** *Let $\rho^\epsilon(t,x) =: |\psi^\epsilon(t,x)|^2$, $J_j^\epsilon(t,x) =: \epsilon Im(\overline{\psi^\epsilon}\, \partial_j\, \psi^\epsilon)$. Then there holds*

*1)*

$$\partial_t\, \rho^\epsilon + div\, J^\epsilon = 0, \tag{2.5}$$

$$\partial_t\, J_j^\epsilon + \frac{\epsilon^2}{4} \sum_{k=1}^{2} \partial_k \left(4\Re(\partial_j\, \psi^\epsilon\, \partial_k\, \overline{\psi^\epsilon}) - \partial_j\, \partial_k |\psi^\epsilon|^2\right) + \frac{1}{2}\, \partial_j (\rho^\epsilon)^2 = 0. \tag{2.6}$$

*2) Let $R$ be large enough such that $|x + u^\infty T^*| \leq R$ for all $x \in \partial\Omega$, then for $0 \leq t \leq T^*$, there holds*

$$\int_\Omega \left(\epsilon^2(1-\chi)|\nabla\psi^\epsilon|^2 + \chi|\epsilon\nabla\psi^\epsilon - iu^\infty\, \psi^\epsilon|^2\right) dx + \int_\Omega \left(\rho^\epsilon(t,x) - 1\right)^2\, dx \leq Ce^{Ct}, \tag{2.7}$$

*where $C$ is a constant depending only on $\chi(x)$ and various constants in the assumptions (A1)–(A3) in the introduction.*

**Remark 2.2.** Notice by the argument at the beginning of this section, we know that $\psi^\epsilon(t,x) - A^\epsilon(t,x) \to 0$ as $|x| \to \infty$. However, we do not know how to obtain the uniform estimate for $\int_\Omega |\epsilon\nabla(\psi^\epsilon - A^\epsilon)|^2\, dx + \int_\Omega (\rho^\epsilon - 1)^2\, dx$, as we did in step 2 for $\epsilon$ fixed case.

**step 4.** The time derivative of $H^\epsilon(t)$.

Let $(\rho_0(x), u_0(x))$ satisfies (A3) in the introduction, then by Theorem 2.1, (1.9–1.10) has a unique local smooth solution $(\rho(t,x), u(t,x))$ with $(\rho(t,x)-1,\, u(t,x)-u^\infty) \in X_{3,T}$,

for any $T < T^*$. As it was pointed out in the introduction, we shall study the evolution of the following functional:

$$H^\epsilon(t) =: \frac{1}{2}\int_\Omega |(\epsilon\nabla_x - iu)\psi^\epsilon|^2\,dx + \frac{1}{2}\int_\Omega |\rho^\epsilon - \rho|^2\,dx, \tag{2.8}$$

for $0 < t < T^*$.

The key ingredient in the proof of Theorem 1.1 will then be the following lemma:

**Lemma 2.2.** *Let $H^\epsilon(t)$ be defined as (2.8). Then we have*

$$\begin{aligned}\frac{d}{dt}\,H^\epsilon(t) = &-\sum_{j,k=1}^{2}\int_\Omega \partial_j\,u_k \Re\left(\left(\epsilon\partial_{x_j} - iu_j\right)\psi^\epsilon\overline{\left(\epsilon\partial_{x_k} - iu_k\right)\psi^\epsilon}\right)dx\\ &-\frac{1}{2}\int_\Omega div\,u\,(\rho^\epsilon-\rho)^2\,dx + \frac{\epsilon^2}{4}\int_\Omega \nabla\rho^\epsilon(\nabla\,div\,u)\,dx.\end{aligned} \tag{2.9}$$

We should point out that all the integration by parts used in the proof of this lemma is consistent with the Neumann boundary condition for (1.1), and boundary conditions (1.8) for the limit system.

**step 5.** The complete proof of Theorem 1.1.

*Proof of Theorem 1.1.* First of all we have, by (2.7), that

$$\begin{aligned}&\epsilon^2\left|\int_\Omega \nabla\rho^\epsilon\cdot(\nabla\mathrm{div}\,u)\,dx\right|\\ &= \epsilon\left|\int_\Omega\left(\psi^\epsilon\,\overline{(\epsilon\nabla\psi^\epsilon - iu^\infty\psi^\epsilon)} + (\epsilon\nabla\psi^\epsilon - iu^\infty\psi^\epsilon)\,\overline{\psi^\epsilon}\right)\nabla\mathrm{div}\,u\,dx\right|\\ &\le 2\epsilon\left(\int_\Omega|\epsilon\nabla\psi^\epsilon - iu^\infty\psi^\epsilon|^2\,dx\right)^{\frac12}\left(\int_\Omega\left(|\psi^\epsilon|^2-1\right)|\nabla\mathrm{div}\,u|^2\,dx + \int_\Omega|\nabla\mathrm{div}\,u|^2\,dx\right)^{\frac12}\\ &\le C\epsilon\left(\left(\int_\Omega\left(|\psi^\epsilon|^2-1\right)^2\,dx\right)^{\frac14}\left(\int_\Omega|\nabla\mathrm{div}u|^4\,dx\right)^{\frac14} + \|\mathrm{div}\,u\|_{H^1}\right)\\ &\le C\epsilon\|\nabla u(t,\cdot)\|_{H^2(\Omega)}.\end{aligned} \tag{2.10}$$

Next, from (2.9), (2.10) and the Gronwall inequality, we obtain

$$H^\epsilon(t) \le C(T)\,e^{\int_0^t\|\nabla u(s,\cdot)\|_{L^\infty}\,ds}\left(H^\epsilon(0)+\epsilon\right), \qquad 0 < t \le T < T^*. \tag{2.11}$$

Finally by assumption (A2) in the introduction, we conclude

$$\begin{aligned}H^\epsilon(0) &= \frac12\int_\Omega\left|\left(\epsilon\nabla_x - iu_0\right)\psi_0^\epsilon\right|^2\,dx + \frac12\int_\Omega\left(\rho_0^\epsilon-\rho_0\right)^2\,dx\\ &\le \frac12\int_\Omega\rho_0^\epsilon\,|u_0-\nabla S^\epsilon|^2\,dx + \epsilon\int_\Omega\left|\nabla\sqrt{\rho_0^\epsilon}\right|^2\,dx\\ &\quad+\frac12\int_\Omega\left(\rho_0^\epsilon-\rho\right)^2\,dx = o(1), \qquad \text{as } \epsilon\to 0.\end{aligned} \tag{2.12}$$

Therefore, one has

$$\lim_{\epsilon\to 0} H^\epsilon(t) = 0, \quad 0 \le t \le T < T^*. \tag{2.13}$$

In particular, (2.13) implies that

$$\rho^\epsilon(t,x) - 1 \to \rho(t,x) - 1 \text{ in } L^\infty\left([0,T], L^2(\Omega)\right) \text{ as } \epsilon \to 0, \tag{2.14}$$

and that

$$\begin{aligned} J^\epsilon(t,x) - (\rho u)(t,x) &= \epsilon \mathrm{Im}\left(\overline{\psi^\epsilon}\,\nabla\psi^\epsilon\right)(t,x) - (\rho u)(t,x) \\ &= \epsilon \mathrm{Im}\left(\overline{\psi^\epsilon}(\nabla - iu)\psi^\epsilon\right)(t,x) + \epsilon \mathrm{Im}\left(\left(|\psi^\epsilon|^2 - \rho\right)u\right)(t,x) \\ &\longrightarrow 0, \text{ in } L^\infty\left([0,T], L^1_{\mathrm{loc}}(\Omega)\right) \text{ as } \epsilon \to 0. \end{aligned} \tag{2.15}$$

This completes the proof of the Theorem. □


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# References


[Bei81] H. Beirao Da Veiga, On the barotropic motion of compressible perfect fluids, *Ann. Sc. Norm. Sup. Pisa*, **8** (1981), 417–351.

[Bei92] H. Beirao Da Veiga, Data dependence in the mathematical theory of compressible inviscid fluids, *Arch. Rational Mech. Anal.*, **119** (1992), 109–127.

[Brenier2000] Y. Brenier, Convergence of the Vlasov-Poisson system to the incompressible Euler equations, *Comm. Partial Diff. Equations*, **25** (2000), 737–754.

[BG] H. Brezis and T. Gallouet, Nonlinear Schrödinger evolution equations, *Nonlinear Analysis, TMA*, **4** (1980), 677–681.

[DD] G. C. Dong, Nonlinear second order parabolic equations. Translated from the Chinese by Kai Seng Chou [Kaising Tso]. Translations of Mathematical Monographs, 95. American Mathematical Society, Providence, RI, 1991. viii+251 pp .

[FPR] T. Frisch, Y. Pomeau and S. Rica, Transition to dissipation in a model of superflow, *Phys. Rev. Lett.* **69**, No. 11, (1992), 1644–1647.

[Gerard] P. Gérard, Mesures semi-classiques et ondes de Bloch, *Séminaire sur les équations aux Dérivées Partielles*, 1990–1991, Exp. No. XVI, 19 École Polytech., Palaiseau, 1991.

[G] E. P. Gross, *J. Math. Phys.* **4** (1963), 195–

[GP] V. L. Ginzburg and L. P. Pitaevskii, On the theory of superfluidity, *Sov. Phys. JETP* **7** (1958), 585.

[Grenier98] E. Grenier, Semiclassical limit of the nonlinear Schrödinger equation in small time, *Proc. Amer. Math. Soc.*, **126** (1998), 523–530.

[JP] C. Josserand and Y. Pomeau, Nonlinear aspects of the theory of Bose-Einstein condensates, *Nonlinearity* **14** (2000), R25–R62.

[LL] L. D. Landau and E. M. Lifschits, *Fluid Mechanics, Course of Theoretical Physics* **6** London-New York, Pregamon Press (1989).

[LX] F. H. Lin and J. Xin, On the Incompressible Fluid Limit and the Vortex Motion Law of the Nonlinear Schrödinger Equation, *Comm. Math. Phys.* **200** (1999), 249–274.

[LZ] F. H. Lin and Ping Zhang, Semiclassical limit of the Gross-Pitaevskii equation in an exterior domain, *Arch. Rational Mech. Anal.*, (to appear).

[LP] P. L. Lions and T. Paul, Sur les mesures de Wigner, *Rev. Mat. Iberoamericana*, **9** (1993), 553–618.

[M] E. Madelung, Quanten theorie in Hydrodynamic Form, *Z. Physik* **40** (1927), 322.

[MP1] M. Puel, Convergence of the Schrödinger-Poisson system to the incompressible Euler equations, *Comm. Partial Diff. Equations*, **27** (2002), 2311–2331.

[Sch86] S. Schochet, The compressible Euler equations in a bounded domain: existence of solutions and the incompressible limit, Comm. Math. Phys., **104** (1986), 49–75.

[Si85] T. C. Sideris, Formation of singularities in three-dimensional compressible fluids, Comm. Math. Phys., **101** (1985), 475–485.

[TS] M. Tsutsumi, On smooth solutions to the initial-boundary value problem for the nonlinear Schrödinger equation in two space diemnsion, *Nonlinear Analysis, TMA*, **13** (1989), 1051–1056.

[Wigner] E. Wigner, On the quantum correction for the thermodynamic equivalium, *Phys. Rev.*, **40** (1932), 742-759.

[ZZM] Ping Zhang, Yuxi Zheng, and N. J. Mauser, The limit from the Schr?dinger-Poisson to the Vlasov-Poisson equations with general data in one dimension, *Comm. Pure Appl. Math.,* **55** (2002), 582–632.

[ZP1] Ping Zhang, Wigner measure and the semiclassical limit of Schrödinger-Poisson Equations, *SIAM. J. Math. Anal.*, **34** (2002), 700–718.

[ZP2] Ping Zhang, Semiclassical limit of nonlinear Schrödinger equation (II), *J. Partial Diff. Eqs.*, **15** (2002), 83–96.