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Asymptotic stability of solitary waves for nonlinear Schrödinger equations

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Introduction

The goal of the present work is to extend to the multidimensional case the results of [9].

Consider the nonlinear Schrödinger equation

$$i\psi_t = -\triangle\psi + F(|\psi|^2)\psi, \quad (t,x) \in \mathbb{R}\times\mathbb{R}^d,\ d \geq 3. \tag{1}$$

For suitable F it possesses important solutions of special form - solitary waves (or, shortly, solitons):

$$e^{i\Phi}\varphi(x-b(t),E),$$

$$\Phi = \omega t + \gamma + \frac{1}{2}x\cdot v,\ b(t) = vt + c,\ E = \omega + \frac{|v|^2}{4} > 0,$$

where ω, $\gamma \in \mathbb{R}$, v, $c \in \mathbb{R}^d$ are constants and φ is a ground state that is a smooth positive spherically symmetric, exponentially decreasing solution of the equation

$$-\triangle\varphi + E\varphi + F(\varphi^2)\varphi = 0. \tag{2}$$

We consider the Cauchy problem for equation (1) with initial data close to a sum

$$\sum_{j=1}^{N} e^{i\beta_{0j}+i\frac{x\cdot v_{0j}}{2}}\varphi(x-b_{0j},E_{0j}),$$

assuming that the initial solitons $e^{i\beta_{0j}+i\frac{x\cdot v_{0j}}{2}}\varphi(x-b_{0j},E_{0j})$, $j=1,\ldots,N$ are well separated either in the original space or in Fourier space: for $j\neq k$, either $|v^0_{jk}|$ or $\min\limits_{t\geq 0}|b^0_{jk}(t)|$ is sufficiently large, where $v^0_{jk} = v_{0j}-v_{0k}$, $b^0_{jk}(t) = b_{0j}-b_{0k}+v^0_{jk}t$. In the second case we shall assume that the "collision time" $t^0_{jk} = -\frac{b^0_{jk}(0)\cdot v^0_{jk}}{|v^0_{jk}|^2}$ is "bounded" from above, see subsection 1.4, (1.5) for the exact formulation. We show that under some suitable assumptions on the spectral structure of the one soliton linearizations, the large time asymptotics of the solution is given by a sum of solitons with slightly modified parameters plus a small dispersive term. In [9] this was proved in the case $d=1$, $N=2$. The main new ingredient in the analysis is a combination of the estimates for the linear one soliton evolution obtained by Cuccagna in [3] and the ideas of Hagedorn [6].

1 Background and statement of the results

1.1 Assumptions on F

Consider the nonlinear Schrödinger equation (1). We assume the following.

Hypothesis H1. *F is a smooth function, $F(0) = 0$, F satisfies the estimates*

$$F(\xi) \geq -C\xi^q, \quad |F^{(\alpha)}(\xi)| \leq C\xi^{p-\alpha}, \ \ \alpha = 0, 1, 2,$$

where $C > 0$, $\xi \geq 1$, $q < \frac{2}{d}$, $p < \frac{2}{d-2}$.

Set $g(\xi) = E\xi + F(\xi^2)\xi$.

Hypothesis H2.

(i) *There exists $\xi_0 > 0$ such that $g(\xi) > 0$ for $\xi < \xi_0$, $g(\xi) < 0$ for $\xi > \xi_0$ and $g'(\xi_0) < 0$.*

(ii) *There exists $\xi_1 > 0$ such that $\int_0^{\xi_1} dsg(s) = 0$.*

Further assumptions are given in terms of the function

$$I(\xi, \lambda) = -\lambda\xi g'(\xi) + (\lambda + 2)g(\xi).$$

We consider ξ_0 of (H2) and assume:

Hypothesis H3. *For any $\xi > \xi_0$ there exists a $\lambda(\xi) > 0$, continuously depending on ξ, such that $I(t, \lambda) \geq 0$ for $0 < t < \xi$ and $I(t, \lambda) \leq 0$ for $t > \xi$.*

We suppose hypotheses (H2,3) to be true for E in some open interval $\mathcal{A} \subset \mathbb{R}_+$.

Under these assumptions equation (2), for $E \in \mathcal{A}$, has a unique positive spherically symmetric smooth exponentially decreasing solution $\varphi(x, E)$, see [1, 7]. More precisely, as $|x| \to \infty$

$$\varphi(x, E) \sim Ce^{-\sqrt{E}|x|}|x|^{-\frac{(d-1)}{2}}.$$

This asymptotic estimate can be differentiated any number of times with respect to x and E.

The functions $w(x, \sigma) = \exp(i\beta + iv \cdot x/2)\varphi(x - b, E)$, $\sigma = (\beta, E, b, v) \in \mathbb{R}^{2d+2}$, will be called soliton states. $w(x, \sigma(t))$ is a solitary wave solution iff $\sigma(t)$ satisfies the system:

$$\beta' = E - \frac{|v|^2}{4}, \quad E' = 0, \quad b' = v, \quad v' = 0. \tag{1.1}$$

1.2 One soliton linearization

Consider the linearization of equation (1) on a soliton $w(x, \sigma(t))$:

$$\psi \sim w + \chi,$$

$$i\chi_t = (-\triangle + F(|w|^2))\chi + F'(|w|^2)(|w|^2\chi + w^2\bar{\chi}).$$

Introducing the function $\vec{f}$:

$$\vec{f} = \begin{pmatrix} f \\ \bar{f} \end{pmatrix}, \quad \chi(x,t) = \exp(i\Phi)f(y,t),$$

$$\Phi = \beta(t) + \frac{v \cdot x}{2}, \quad y = x - b(t),$$

one gets

$$i\vec{f}_t = L(E)\vec{f}, \quad L(E) = L_0(E) + V(E), \;\; L_0(E) = (-\triangle + E)\sigma_3,$$

$$V(E) = V_1(E)\sigma_3 + iV_2(E)\sigma_2, \quad V_1 = F(\varphi^2) + F'(\varphi^2)\varphi^2, \;\; V_2(E) = F'(\varphi^2)\varphi^2.$$

Here σ_2, σ_3 are the standard Pauli matrices

$$\sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

We consider L as an operator in $L_2(\mathbb{R}^d \to \mathbb{C}^2)$ defined on the domain where L_0 is self adjoint. L satisfies the relations

$$\sigma_3 L \sigma_3 = L^*, \quad \sigma_1 L \sigma_1 = -L,$$

where $\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. The continuous spectrum of $L(E)$ fills up two semi-axes $(-\infty, E]$ and $[E, \infty)$. In addition $L(E)$ may have finite and finite dimensional point spectrum on the real and imaginary axis.

Zero is always a point of the discrete spectrum. One can indicate $d+1$ eigenfunctions

$$\vec{\xi}_0 = \varphi \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \quad \vec{\xi}_j = \varphi_{y_j} \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \;\; j = 1, \dots d,$$

and $d+1$ generalized eigenfunctions

$$\vec{\xi}_{d+1} = -\varphi_E \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad \vec{\xi}_{d+1+j} = -\frac{1}{2} y_j \varphi \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \;\; j = 1, \dots d,$$

$$L\vec{\xi}_j = 0, \quad L\vec{\xi}_{d+1+j} = \vec{\xi}_j, \;\; j = 0, \dots, d.$$

Let M be the generalized null space of the operator L. Under assumptions (H1,2,3), the vectors $\vec{\xi}_j$, $j = 0, \dots, 2d+1$, span the subspace M iff

$$\frac{d}{dE}\|\varphi(E)\|_2^2 \neq 0,$$

see [13, 7, 3].

We shall assume that

Hypothesis H4. *The set $\mathcal{A}_0$ of $E \in \mathcal{A}$ such that*

(i) *zero is the only eigenvalue of the operator $L(E)$, and the dimension of the corresponding generalized null space is equal to $2d+2$;*

(ii) *$\pm E$ is not a resonance for $L(E)$;*
is nonempty.

Obviously, the set $\mathcal{A}_0$ is open.

Remark. $\pm E$ is said to be a resonance of $L(E)$ if there is a solution ψ of the equation $(L(E) \mp E)\psi = 0$ such that $< x >^{-s} \psi \in L_2$ for any $s > 1/2$ but not for $s = 0$. It is well known that $\pm E$ can never be a resonance if $d \geq 5$.

Consider the evolution operator e^{-itL}. One has the following proposition.

Proposition 1.1 *For $E \in \mathcal{A}_0$ and any $x_0, x_1 \in \mathbb{R}^d$,*

$$\| \langle x - x_0 \rangle^{-\nu_0} e^{-iL(E)t} \hat{P}(E) f \|_2 \leq C \langle t \rangle^{-d/2} \| \langle x - x_1 \rangle^{\nu_0} f \|_2, \quad \nu_0 > \frac{d}{2},$$

where $\hat{P}(E)$ is the spectral projection onto the subspace of the continuous spectrum of $L(E)$:

$$\operatorname{Ker} \hat{P} = M, \quad \operatorname{Ran} \hat{P} = (\sigma_3 M)^{\perp}.$$

The constant C here is uniform with respect to $x_0, x_1 \in \mathbb{R}^d$ and E in compact subsets of $\mathcal{A}_0$.

This proposition is an immediate consequence of the L_p- L_q estimates of $e^{-iLt}\hat{P}$ proved by Cuccagna [3], see also [10].

1.3 The nonlinear equation

We formulate here the necessary facts about the Cauchy problem for equation (1) with initial data in $H^1(\mathbb{R}^d)$.

Proposition 1.2 *Suppose that F satisfies* (H1). *Then the Cauchy problem for equation* (1) *with initial data $\psi(x,0) = \psi_0(x)$, $\psi_0 \in H^1(\mathbb{R}^d)$ has a unique solution ψ in the space $C(\mathbb{R} \to H^1)$, and ψ satisfies the conservation laws*

$$\int dx|\psi|^2 = const, \quad H(\psi) \equiv \int dx[|\nabla\psi|^2 + U(|\psi|^2)] = const,$$

where $U(\xi) = \int_0^{\xi} ds F(s)$. Furthermore, for all $t \in \mathbb{R}$

$$\|\psi(t)\|_{H^1} \leq c(\|\psi_0\|_{H^1})\|\psi_0\|_{H^1},$$

where $c : \mathbb{R}_+ \to \mathbb{R}_+$ is a smooth function.

The assertion stated here can be found in [4, 5], for example.

1.4 Description of the problem

Consider the Cauchy problem for equation (1) with initial data

$$\psi|_{t=0} = \psi_0 \in H^1 \cap L_1, \quad \psi_0 = \sum_{j=1}^{N} w(\cdot, \sigma_{0j}) + \chi_0, \tag{1.2}$$

$$\sigma_{0j} = (\beta_{0j}, E_{0j}, b_{0j}, v_{0j}), \quad \min_{j\neq k} |v_{jk}^0| \geq v_0 > 0. \tag{1.3}$$

Here $v_{jk}^0 = v_{0j} - v_{0k}$. Set $b_{jk}^0 = b_{0j} - b_{0k}$, $j \neq k$. Write b_{jk}^0 as the sum

$$b_{jk}^0 = r_{jk}^0 - t_{jk}^0 v_{jk}^0, \quad r_{jk}^0 \cdot v_{jk}^0 = 0,\ t_{jk}^0 = -\frac{b_{jk}^0 \cdot v_{jk}^0}{|v_{jk}^0|^2}. \tag{1.4}$$

For $j \neq k$ we define the effective small parameter ϵ_{jk}:

$$\epsilon_{jk} = \begin{cases} (\min\limits_{t\geq 0} |b_{jk}^0(t)| + |v_{jk}^0|)^{-1}, & \text{if } t_{jk}^0 \leq \kappa < r_{jk}^0 >, \\ |v_{jk}^0|^{-1} & \text{otherwise,} \end{cases} \tag{1.5}$$

where $b_{jk}^0(t) = b_{jk}^0 + t v_{jk}^0$, κ is a fixed positive constant.

Assume that

(T1) $\epsilon \equiv \max\limits_{j\neq k} \epsilon_{jk}$ *is sufficiently small*[1];

(T2) $E_{0j} \in \mathcal{A}_0$, $j = 1, \ldots, N$.

Our goal is to describe the asymptotic behavior of the solution ψ as $t \to +\infty$, provided χ_0 is sufficiently small in the following sense:

(T3) *for some* m', $\frac{1}{m} + \frac{1}{m'} = 1$, $m \geq 2p + 2$, $\frac{4}{d} + 2 < m < \frac{4}{d-2} + 2$ *if* $d \geq 4$, $4 \leq m < \frac{4}{d-2} + 2$ *if* $d = 3$, *the norm*

$$\mathcal{N} = \|\chi_0\|_1 + \|\hat{\chi}_0\|_{m'}$$

is sufficiently small.

Here $\hat{\chi}_0$ stands for the Fourier transform of χ_0.

Our main result is given by the following theorem.

Theorem 1.1 *For* $t \geq 0$ *the solution* ψ *of* (1), (1.2) *admits the representation*

$$\psi(t) = \sum_{j=1}^{N} w(\cdot, \sigma_j(t)) + \chi(t), \quad \sigma_j(t) = (\beta_j(t), E_j(t), b_j(t), v_j(t)),$$

where $|E_j(t) - E_{0j}|$, $|v_j(t) - v_{0j}|$, $j = 1, \ldots, N$, $\|\chi(t)\|_{L_2 \cap L_m}$ *are small uniformly w.r.t.* $t \geq 0$, *and as* $t \to +\infty$,

$$\|\chi(t)\|_m = O(t^{-d(\frac{1}{2} - \frac{1}{m})}).$$

Moreover, there exist vectors $\sigma_{+j} = (\beta_{+j}, E_{+j}, b_{+j}, v_{+j})$, *such that as* $t \to +\infty$,

$$|\sigma_j(t) - \sigma_{+j}(t)| = O(t^{-\delta}),$$

for some $\delta > 0$. *Here* $\sigma_{+j}(t)$ *is the trajectory of* (1.1) *with the initial data* $\sigma_{+j}(0) = \sigma_{+j}$.

[1] "Sufficiently small (large)" assumes constants that depend only on v_0, κ and $E_{0j}, j = 1, \ldots, N$.

2 Proof of theorem 1.1

In this section we outline the proof of theorem 1.1. The details can be found in [10]. Up to some technical modifications the main line of the proof repeats that of [9].

2.1 Splitting of the motions

Following [9] we decompose the solution ψ as follows.

$$\psi(x,t) = \sum_{j=1}^{N} w(x,\sigma_j(t)) + \chi(x,t). \tag{2.1}$$

Here $\sigma_j(t) = (\beta_j(t), E_j(t), b_j(t), v_j(t))$ is an arbitrary trajectory in the set of admissible values of parameters, it is not a solution of (1.1) in general.

We fix the decomposition (2.1) by imposing the orthogonality conditions

$$\left\langle \vec{f}_j(t), \sigma_3 \vec{\xi}_k(E_j(t)) \right\rangle = 0, \quad j = 1,\dots,N, \;\; k = 0,\dots,2d+1. \tag{2.2}$$

Here

$$\vec{f}_j = \begin{pmatrix} f_j \\ \bar{f}_j \end{pmatrix}, \quad \chi(x,t) = \exp(i\Phi_j) f_j(y_j,t),$$

$$\Phi_j = \beta_j(t) + v_j \cdot x/2, \quad y_j = x - b_j(t),$$

$< \cdot, \cdot >$ is the inner product in $L_2(\mathbb{R}^d \to \mathbb{C}^2)$.

Geometrically these conditions mean that for each t the vector $\vec{f}_j(t)$ belongs to the subspace of the continuous spectrum of the operator $L(E_j(t))$.

For ψ of the form (1.2) with $\min\limits_{\substack{j,k \\ j\neq k}}(|v_{jk}^0| + |b_{jk}^0|)$ sufficiently large, and with χ_0 sufficiently small in some L_p norm, the solvability of (2.2) is guaranteed by the non-degeneration of the corresponding Jacobi matrix. So, one can assume that initial decomposition (1.2) obeys (2.2). To prove the existence of a decomposition (2.1), (2.2) for $t > 0$, one can invoke a standard continuity type argument, see [10] for the details.

Rewriting (2.1) as an equation for χ one gets

$$i\vec{\chi}_t = H(\vec{\sigma}(t))\vec{\chi} + N, \tag{2.3}$$

Here

$$\vec{\chi} = \begin{pmatrix} \chi \\ \bar{\chi} \end{pmatrix}, \quad \vec{\sigma} = (\sigma_1,\dots,\sigma_N) \in \mathbb{R}^{(2d+2)N},$$

$$H(\vec{\sigma}) = -\triangle\sigma_3 + \sum_{j=1}^{N} \mathcal{V}(w_j),$$

$$\mathcal{V}(w) = (F(|w|^2) + F'(|w|^2)|w|^2)\sigma_3 + F'(|w|^2)\begin{pmatrix} 0 & w^2 \\ -\bar{w}^2 & 0 \end{pmatrix}, \quad w_j = w(x,\sigma_j).$$

The nonlinearity N is given by the following expression

$$N = N_0 + \sum_{j=1}^{N} e^{i\sigma_3 \Phi_j} l(\sigma_j) \vec{\xi}_0(y_j, E_j),$$

$$N_0 = F(|\psi_s + \chi|^2) \begin{pmatrix} \psi_s + \chi \\ -\bar{\psi}_s - \bar{\chi} \end{pmatrix} -$$

$$\sum_{j=1}^{N} \left(F(|w_j|^2) \begin{pmatrix} w_j \\ -\bar{w}_j \end{pmatrix} + \mathcal{V}(w_j)\vec{\chi} \right), \quad \psi_s = \sum_{j=1}^{N} w_j,$$

$$l(\sigma_j) = \gamma_j' + \frac{1}{2} v_j' \cdot y_j + i c_j' \cdot \nabla \sigma_3 - i E_j' \partial_E \sigma_3,$$

where γ_j, c_j are defined as follows.

$$\beta_j(t) = \int_0^t ds (E_j(s) - \frac{|v_j(s)|^2}{4} - \frac{v_j'(s) \cdot b_j(s)}{2}) + \gamma_j(t),$$

$$b_j(t) = \int_0^t ds v_j(s) + c_j(t).$$

In terms of parameters (γ, E, c, v) (1.1) takes the form

$$\gamma' = 0, \ E' = 0, \ c' = 0, \ v' = 0.$$

Substituting the expression for χ_t from (2.3) into the derivative of the orthogonality conditions, one gets for $j = 1, \ldots, N$

$$\begin{aligned}
e(E_j)E_j' &= -i \left\langle N_j, \sigma_3 e^{i\Phi_j} \vec{\xi}_0(\cdot - b_j, E_j) \right\rangle + \left\langle \vec{f}_j, l(\sigma_j) \vec{\xi}_0(E_j) \right\rangle, \\
n(E_j)v_j' &= \left(\left\langle N_j, \sigma_3 e^{i\Phi_j} \vec{\xi}_k(\cdot - b_j, E_j) \right\rangle + \left\langle \vec{f}_j, l(\sigma_j) \vec{\xi}_k(E_j) \right\rangle \right)_{k=1,\ldots,d}, \\
e(E_j)\gamma_j' &= \left\langle N_j, \sigma_3 e^{i\Phi_j} \vec{\xi}_{d+1}(\cdot - b_j, E_j) \right\rangle + \left\langle \vec{f}_j, l(\sigma_j) \vec{\xi}_{d+1}(E_j) \right\rangle, \qquad (2.4) \\
n(E_j)c_j' &= i \left(\left\langle N_j, \sigma_3 e^{i\Phi_j} \vec{\xi}_{d+1+k}(\cdot - b_j, E_j) \right\rangle + \left\langle \vec{f}_j, l(\sigma_j) \vec{\xi}_{d+1+k}(E_j) \right\rangle \right)_{k=1,\ldots,d}.
\end{aligned}$$

Here

$$N_j = N_0 + \sum_{k, k \neq j} \mathcal{V}(w_k)\vec{\chi} + \sum_{k, k \neq j} e^{i\sigma_3 \Phi_k} l(\sigma_k) \vec{\xi}_0(y_k, E_k), \quad j = 1, \ldots, N,$$

$$e = \frac{d}{dE} \|\varphi\|_2^2, \ n = \frac{1}{2} \|\varphi\|_2^2.$$

The right hand side of (2.4) also contain the derivative $\vec{\sigma}'$, which enters linearly in $l(\sigma_k)$. In principle, system (2.4) can be solved with respect to derivative and together with equation (2.3) constitutes a complete system for $\vec{\sigma}$ and χ:

$$i\vec{\chi}_t = H(\vec{\sigma}(t))\vec{\chi} + N(\vec{\sigma}, \vec{\chi}), \qquad (2.5)$$

$$\vec{\sigma}' = G(\vec{\sigma}, \vec{\chi}), \quad \chi|_{t=0} = \chi_0, \quad \sigma_j(0) = \sigma_{0j}. \qquad (2.6)$$

2.2 Integral representations for χ

In this subsection we follow closely the constructions of Hagedorn [6] (developed in order to prove the asymptotic completeness for the charge transfer model). We start by rewriting (2.5) as an integral equation

$$\vec{\chi}(t) = \mathcal{U}_0(t,0)\chi_0 - i\int_0^t \mathcal{U}_0(t,s)\left[\sum_{j=1}^{N}\mathcal{V}_j(s)\vec{\chi}(s) + N\right]ds, \tag{2.7}$$

Here $\mathcal{U}_0(t,\tau) = e^{i(t-\tau)\triangle\sigma_3}$, $\mathcal{V}_j = \mathcal{V}(w_j)$.

Next we introduce the one soliton adiabatic propagators $\mathcal{U}_j^A(t,\tau)$:

$$i\mathcal{U}_{j\,t}^A(t,\tau) = L_j(t)\mathcal{U}_j^A(t,\tau), \quad \mathcal{U}_j^A(t,\tau)|_{t=\tau} = I,$$

$$L_j(t) = -\triangle\sigma_3 + \tilde{\mathcal{V}}_j(t) + R_j(t), \quad R_j(t) = iT_{0j}(t)[P_j'(t), P_j(t)]T_{0j}^*(t),$$

$$\tilde{\mathcal{V}}_j(t) = T_{0j}(t)T_j(t)V(E_{0j})T_j^*(t)T_{0j}^*(t), \quad P_j(t) = T_j(t)\hat{P}(E_{0j})T_j^*(t).$$

Here

$$T_{0j}(t) = B_{\beta_{0j}(t),b_{0j}(t),v_{0j}}, \quad T_j(t) = B_{\theta_j(t),a_j(t),0},$$

$$\theta_j = \int_0^t ds\left(E_j(s) - E_{0j} + \frac{|v_j(s) - v_{0j}|^2}{4}\right), \quad a_j = \int_0^t ds(v_j(s) - v_{0j}),$$

$$(B_{\beta,b,v}f)(x) = e^{i\beta\sigma_3 + i\frac{v\cdot x}{2}\sigma_3}f(x-b),$$

$\sigma_{0j}(t) = (\beta_{0j}(t), E_{0j}, b_{0j}(t), v_{0j})$ being the solution of (1.1) with initial data $\sigma_{0j}(0) = \sigma_{0j}$. Obviously,

$$P_j^A(t)\mathcal{U}_j^A(t,\tau) = \mathcal{U}_j^A(t,\tau)P_j^A(\tau),$$

where

$$P_j^A(t) = T_{0j}(t)P_j(t)T_{0j}^*(t).$$

Write the solution χ as the sum:

$$\vec{\chi}(t) = \vec{h}_j(t) + \vec{k}_j(t), \quad \vec{h}_j(t) = P_j^A(t)\vec{\chi}(t).$$

Using the adiabatic evolution $\mathcal{U}_j^A(t,\tau)$ one can write the following representation for $h_j(t)$

$$\vec{h}_j(t) = \mathcal{U}_j^A(t,0)P_j^A(0)\vec{\chi}_0 - i\int_0^t \mathcal{U}_j^A(t,s)P_j^A(s)[\sum_{m,\,m\neq j}\mathcal{V}_m(s)\vec{\chi}(s) + D_j(s)]ds, \tag{2.8}$$

Here

$$D_j = N + (\mathcal{V}_j - \tilde{\mathcal{V}}_j)\vec{\chi} - R_j\vec{\chi}. \tag{2.9}$$

Combining (2.7), (2.8) one gets finally

$$\vec{\chi} = \text{(I)} + \text{(II)} + \text{(III)} + \text{(IV)}, \tag{2.10}$$

where

$$(\mathrm{I}) = \mathcal{U}_0(t,0)\vec{\chi}_0 - i\sum_j \int_0^t ds\mathcal{U}_0(t,s)\tilde{\mathcal{V}}_j(s)\mathcal{U}_j^A(s,0)P_j^A(0)\vec{\chi}_0,$$

$$(\mathrm{II}) = -\sum_{\substack{j,m\\ j\neq m}} \int_0^t dsK_j(t,s)\mathcal{V}_m(s)\vec{\chi}(s),$$

$$(\mathrm{III}) = -i\int_0^t ds\mathcal{U}_0(t,s)D,$$

$$(\mathrm{IV}) = -\sum_j \int_0^t dsK_j(t,s)D_j(s).$$

Here

$$D = N + \sum_j \left(\tilde{\mathcal{V}}_j\vec{k}_j + (\mathcal{V}_j - \tilde{\mathcal{V}}_j)\vec{\chi}\right), \tag{2.11}$$

$$K_j(t,s) = \int_s^t d\rho\mathcal{U}_0(t,\rho)\tilde{\mathcal{V}}_j(\rho)\mathcal{U}_j^A(\rho,s)P_j^A(s).$$

The relations (2.4), (2.7), (2.10) make up the final form of the equation which is used to prove theorem 1.1.

2.3 Estimates of solitons parameters

Following [2, 9] we consider (2.4), (2.7), (2.10) on some finite interval $[0,t_1]$ and then study the limit $t_1 \to +\infty$. On the interval $[0,t_1]$ we introduce a natural system of norms for the components of the solution ψ:

$$M_0(t) = \sum_{j=1}^N |\gamma_j(t) - \beta_{0j}| + |E_j(t) - E_{j0}| + |c_j(t) - b_{0j}| + |v_j(t) - v_{0j}|,$$

$$M_1(t) = \sum_{j=1}^N \| < y_j >^{-\nu} \chi(t)\|_2, \quad M_2(t) = \|\chi(t)\|_{2p+2}, \quad \nu > \frac{d+2}{2},$$

without loss of generality one can assume that $m = 2p+2$.

These norms generate the system of majorants

$$\mathbb{M}_0(t) = \sup_{0\le\tau\le t} M_0(\tau),\ \mathbb{M}_l(t) = \sup_{0\le\tau\le t} M_l(\tau)\rho^{-\mu_l}(\tau),\ l=1,2, \quad \hat{\mathbb{M}}_k = \mathbb{M}_k(t_1).$$

Here $1 < \mu_1 < \frac{3}{2}$ if $d=3$ and $1 < \mu_1 = \frac{dp}{2}$ if $d \ge 4$, $\mu_2 = d(\frac{1}{2} - \frac{1}{2p+2})$,

$$\rho(t) = < t >^{-1} + \sum_{\substack{j,k\\ j\neq k}} < t - t_{jk} >^{-1},$$

t_{jk} being "the collision times" that are defined as follows. We set $t_{jk}=0$ if $t^0_{jk}\leq 0$. For (j,k) such that $t^0_{jk}>0$, we define t_{jk} by the relation,

$$\int_0^{t_{jk}} ds \frac{\tilde{v}_{jk}(s)\cdot v^0_{jk}}{|v^0_{jk}|^2}=t^0_{jk},$$

where

$$\tilde{v}_{jk}(t)=\begin{cases} v_{jk}(t), & \text{if } t\leq t_1, \\ v_{jk}(t_1), & \text{if } t>t_1, \end{cases} \qquad v_{jk}(t)=v_j(t)-v_k(t).$$

Let us mention that

(i) t_{jk} are well defined provided $|v_{jk}(t)-v^0_{jk}|<v_0$, $0\leq t\leq t_1$;

(ii) the collision times t_{jk} belonging to the interval $[0,t_1]$ "do not depend on t_1".

It follows directly from the definition of M_0 that

$$|\theta'_j(t)|,\ |a'_j(t)|\leq M_0(t)+M_0^2(t),\ \ |b_j(t)-\tilde{b}_j(t)|\leq M_0(t), \tag{2.12}$$

$$|\Phi_j(x,t)-\tilde{\Phi}_j(x,t)|\leq M_0(t)<x-b_j(t)>+\mathbb{M}_0(t)\int_0^t ds|c'_j(s)|, \tag{2.13}$$

where

$$\tilde{b}_j(t)=b_{0j}(t)+a_j(t), \quad \tilde{\Phi}_j(x,t)=\beta_{0j}(t)+\theta_j(t)+v_{0j}\cdot x/2.$$

It is also easy to check that $\tilde{b}_{jk}=\tilde{b}_j-\tilde{b}_k$ admits the estimates

$$|\tilde{b}_{jk}(t)|\geq c|v^0_{jk}||t-t_{jk}|, \tag{2.14}$$

$$|\tilde{b}_{jk}(t)|\geq c(\min_{s\geq 0}|b^0_{jk}(s)|+|v^0_{jk}||t-t_{jk}|)-c, \quad t^0_{jk}\leq\kappa<r^0_{jk}> \tag{2.15}$$

provided $M_0(t)\leq c$ for $0\leq t\leq t_1$. Here and below c is employed for positive constants that depend only on v_0,κ and eventually on $E_j, j=1,\ldots,N$, in that case they can be chosen uniformly with respect to E_j in some finite vicinity of E_{0j}.

As an immediate consequence of (2.4), one gets

$$|\lambda_j(t)|\leq W(\mathbb{M})[\sum_{\substack{i,l\\ i\neq k}} e^{-c|b_{ik}(t)|}+\left(\mathbb{M}_1^2(t)+\mathbb{M}_2^2(t)\right)\rho^{2\mu_1}(t)]. \tag{2.16}$$

We use $W(\mathbb{M})$ as a general notation for functions of $\mathbb{M}_0$, $\mathbb{M}_1$, $\mathbb{M}_2$, which are bounded in some finite vicinity of the point $\mathbb{M}_l=0$, $l=0,1,2$, and may acquire $+\infty$ out some larger vicinity. They depend only on v_0, κ_0, E_{j0}, $j=1,\ldots,N$ and can be chosen to be spherically symmetric and monotone. In all the formulas where W appear it would not be hard to replace them by some explicit expressions but such expressions are useless for our aims.

Combining (2.13), (2.16) one gets

$$|\Phi_j(x,t)-\tilde{\Phi}_j(x,t)|\leq W(\mathbb{M})\mathbb{M}_0(t)<x-b_j(t)>. \tag{2.17}$$

Integrating (2.16) and taking into account (2.14), (2.15) we obtain

$$\mathbb{M}_0 \le W(\hat{\mathbb{M}})[\epsilon + \mathbb{M}_1^2 + \mathbb{M}_2^2]. \tag{2.18}$$

Consider the vectors $\vec{k}_j(t) = (I - P_j^A(t))\vec{\chi}(t)$,

$$\vec{k}_j(x,t) = \sum_{l=0}^{2d+1} k_{jl}(t) e^{i\tilde{\Phi}_j \sigma_3} \vec{\xi}_l(x - \tilde{b}_j(t), E_{0j}).$$

The orthogonality conditions (2.2) together with (2.12), (2.17) lead immediately to the estimate:

$$|k_{jl}(t)| \le W(\mathbb{M})\mathbb{M}_0(t)\|e^{-c|x-b_j(t)|}\chi(t)\|_2 \le W(\mathbb{M})\mathbb{M}_0(t)\mathbb{M}_1(t)\rho^{\mu_1}(t). \tag{2.19}$$

2.4 Linear estimates

To study the behavior of solutions of integral equation (2.10) we need some estimates of the evolution operators $\mathcal{U}_m^A(t,\tau)P_m^A(\tau)$. The necessary estimates are collected in this subsection, the complete proofs can be found in [10].

Lemma 2.1 *For any $x_0, x_1 \in \mathbb{R}^d$, $0 \le \tau \le t \le t_1$,*

$$\| \langle x - x_0\rangle^{-\nu_0} \mathcal{U}_j^A(t,\tau)P_j^A(\tau) f\|_2 \le W(\hat{\mathbb{M}}) \langle t-\tau\rangle^{-d/2} \| \langle x - x_1\rangle^{\nu_0} f\|_2. \tag{2.20}$$

The function W here is independent of x_0, x_1 and t_1.

This result is a simple consequence of proposition 1.1.

Remark. Due to the representation

$$\mathcal{U}_j^A(t,\tau)P_j^A(\tau)f = P_j^A(t)\mathcal{U}_0(t,\tau)f$$
$$-i\int_\tau^t ds\,\mathcal{U}_j^A(t,s)P_j^A(s)(\tilde{\mathcal{V}}_j(s) + R_j(s))\mathcal{U}_0(s,\tau)f,$$

and the estimate

$$|(R_j(t)f)(x)| \le W(\mathbb{M})e^{-c|x-b_j(t)|}\mathbb{M}_0(t)\|e^{-c|x-b_j(t)|}f\|_2, \tag{2.21}$$

(2.20) leads immediately to the inequality

$$\| \langle x - x_0\rangle^{-\nu_0} \mathcal{U}_j^A(t,\tau)P_j^A(\tau) f\|_2 \le W(\hat{\mathbb{M}}) \frac{(\|f\|_{p_1'} + \|f\|_{p_2'})}{|t-\tau|^{d(\frac{1}{2}-\frac{1}{p_1})}\langle t-\tau\rangle^{d(\frac{1}{p_1}-\frac{1}{p_2})}}, \tag{2.22}$$

where $2 \le p_1 < \frac{2d}{d-2} < p_2 \le \infty$, $\frac{1}{p_i} + \frac{1}{p_i'} = 1$, $i = 1, 2$. Obviously, the same estimate is valid for $K_j(t,\tau)$:

$$\| \langle x - x_0\rangle^{-\nu_0} K_j(t,\tau) f\|_2 \le W(\hat{\mathbb{M}}) \frac{(\|f\|_{p_1'} + \|f\|_{p_2'})}{|t-\tau|^{d(\frac{1}{2}-\frac{1}{p_1})}\langle t-\tau\rangle^{-d(\frac{1}{p_1}-\frac{1}{p_2})}}. \tag{2.23}$$

The key point of our analysis is the following lemma that is essentially lemma 3.6 of [6].

Lemma 2.2 *Introduce the operators $T_{jki}(t,\tau)$, $j,k,i=1,\dots,N$, $i\neq k$*

$$T_{jki}(t,\tau)=A_j(t)K_k(t,\tau)A_i(\tau),$$

where $A_j(t)$ is the multiplication by $<x-b_j(t)>^{-\nu}$. Then, for $0\le t\le t_1$

$$\int_0^t d\tau\|T_{jki}(t,\tau)\|\le W(\hat{\mathbb{M}})(\epsilon_{ik}^{\nu_1}+\mathbb{M}_0(t)),$$

with some $\nu_1>0$. The norm $\|\cdot\|$ here stands for the $L_2\to L_2$ operator norm.

2.5 Estimates of the nonlinear terms

Here we derive the necessary estimates of D, D_j. Write D as the sum:

$$D=D^0+D^1+D^2,$$

where

$$D^0=N_{00}+\sum_j\left((\mathcal{V}_j-\tilde{\mathcal{V}}_j)\vec{\chi}+\tilde{\mathcal{V}}_j\vec{k}_j+e^{i\Phi_j\sigma_3}l(\sigma_j)\vec{\xi}_0(\cdot-b_j,E_j)\right),$$

$$N_{00}=F(|\psi_s|^2)\begin{pmatrix}\psi_s\\-\bar{\psi}_s\end{pmatrix}-\sum_j F(|w_j|^2)\begin{pmatrix}w_j\\-\bar{w}_j\end{pmatrix}+\mathcal{V}(\psi_s)\vec{\chi}-\sum_j\mathcal{V}_j\vec{\chi},$$

$$D^1=F(|\psi_s+\chi|^2)\begin{pmatrix}\psi_s+\chi\\-\bar{\psi}_s-\bar{\chi}\end{pmatrix}-F(|\psi_s|^2)\begin{pmatrix}\psi_s\\-\bar{\psi}_s\end{pmatrix}-\mathcal{V}(\psi_s)\vec{\chi}-F(|\chi|^2)\begin{pmatrix}\chi\\-\bar{\chi}\end{pmatrix},$$

$$D^2=F(|\chi|^2)\begin{pmatrix}\chi\\-\bar{\chi}\end{pmatrix}.$$

In a similar way,

$$D_j=D_j^0+D^1+D^2,\quad j=1,\dots N,$$

where

$$D_j^0=N_{00}+(\mathcal{V}_j-\tilde{\mathcal{V}}_j)\vec{\chi}-R_j\vec{\chi}+\sum_k e^{i\Phi_k\sigma_3}l(\sigma_k)\vec{\xi}_0(\cdot-b_k,E_k).$$

The direct calculations give

$$\|D^0\|_{L_1\cap L_2},\ \|D_j^0\|_{L_1\cap L_2}\le W(\mathbb{M})[e^{-c|b_{jk}(t)|}+(\mathbb{M}_0\mathbb{M}_1+\mathbb{M}_1^2+\mathbb{M}_2^2)\rho^{\mu_1}(t)], \tag{2.24}$$

$$\|D^1+D^2\|_{L_1\cap L_{m'}}\le W(\mathbb{M})[\mathbb{M}_1^2+\mathbb{M}_1^{2-\frac{1}{p}}\mathbb{M}_2^{\frac{1}{p}}+\mathbb{M}_2^{1+\frac{1}{p}}]\rho^{\mu_1}(t),\quad \text{if } d=3,$$

$$\|D^1\|_{L_1\cap L_{m'}}+\|D^2\|_{L_{r'}\cap L_{m'}}$$

$$\le W(\mathbb{M})[\mathbb{M}_1^2+\mathbb{M}_1^{2-\frac{1}{p}}\mathbb{M}_2^{\frac{1}{p}}+\mathbb{M}_2^{1+p}]\rho^{\mu_1}(t),\ \text{if } \frac{1}{2}<p<1 \tag{2.25}$$

$$\|D^1+D^2\|_{L_{r'}\cap L_{m'}}\le W(\mathbb{M})\mathbb{M}_2^{1+p}\rho^{\mu_1}(t),\quad \text{if } p\le\frac{1}{2}.$$

Here $r'=\frac{2}{1+p}$.

2.6 Estimates of χ

To estimate $M_1(t)$ we use representation (2.10). By (2.22), for the first term (I) one has

$$\| < y_j >^{-\nu} (\mathrm{I})\|_2 \le W(\mathbb{M})\mathcal{N} < t >^{-d/2} . \tag{2.26}$$

Consider expression (II). By lemmas 2.1, 2.2,

$$\| < y_j >^{-\nu} (\mathrm{II})\|_2 \le W(\hat{\mathbb{M}})(\mathbb{M}_0^\theta + \epsilon_{ik}^{\nu_2})\mathbb{M}_1(t)\rho^{\mu_1}(t), \tag{2.27}$$

where $0 < \theta = 1 - \frac{2\mu_1}{d}$, $\nu_2 = \theta\nu_1$.

Consider the two last terms in the r.h.s. of (2.10). Using (2.23), (2.24), (2.14), (2.15), (2.25), one can get

$$\| < y_j >^{-\nu} (\mathrm{III})\|_2, \| < y_j >^{-\nu} (\mathrm{IV})\|_2 \le W(\hat{\mathbb{M}})[\epsilon + \mathbb{M}_0\mathbb{M}_1 + \mathbb{M}_1^2 + \mathbb{M}_2^{r_1}]\rho^{\mu_1}(t), \tag{2.28}$$

where $r_1 = 1 + \min\{p, p^{-1}\}$.

Combining (2.26), (2.27), (2.28), one obtains

$$\mathbb{M}_1 \le W(\hat{\mathbb{M}})[\mathcal{N} + \epsilon^{\nu_2} + \mathbb{M}_0^\theta\mathbb{M}_1 + \mathbb{M}_1^2 + \mathbb{M}_2^{r_1}].$$

Changing if necessary the coefficient function W one can simplify this inequality:

$$\mathbb{M}_1 \le W(\hat{\mathbb{M}})[\mathcal{N} + \epsilon^{\nu_2} + \mathbb{M}_2^{r_1}]. \tag{2.29}$$

To estimate L_m - norm of χ we use representation (2.7). It is not difficult to check that

$$\|N\|_{m'} \le W(\mathbb{M})[\sum_{\substack{k,i\\k\ne i}} e^{-c|b_{ik}(t)|} + \mathbb{M}_1^2 + \mathbb{M}_2^{r_1})\rho^{\mu_1}(t)].$$

As a consequence,

$$\mathbb{M}_2 \le W(\hat{\mathbb{M}})[\mathcal{N} + \epsilon^{1-\mu_2} + \mathbb{M}_1]. \tag{2.30}$$

2.7 Estimates of majorants

Combining (2.18), (2.29), (2.30) one gets

$$\hat{\mathbb{M}}_1, \hat{\mathbb{M}}_2 \le W(\hat{\mathbb{M}})(\mathcal{N} + \epsilon^{\nu_3}), \quad \hat{\mathbb{M}}_0 \le W(\hat{\mathbb{M}})(\mathcal{N}^2 + \epsilon^{2\nu_3}), \tag{2.31}$$

$\nu_3 = \min\{\frac{1}{2}, \nu_2, 1 - \mu_2\} > 0$, the coefficient functions $W(\mathbb{M})$ being independent of t_1. These inequalities mean that for $\mathcal{N}$ and ϵ sufficiently small $\hat{\mathbb{M}}$ can belong either to a small neighborhood of zero or to some domain whose distance from zero is bounded from below uniformly with respect to $\mathcal{N}$, ϵ. Since $\hat{\mathbb{M}}_l$ are continuous functions of t_1 and for $t_1 = 0$ are small only the first possibility can be realized. This means that for $\mathcal{N}$ and ϵ in some finite vicinity of zero,

$$\mathbb{M}_1(t), \mathbb{M}_2(t) \le c(\mathcal{N} + \epsilon^{\nu_3}), \quad \mathbb{M}_0(t) \le c(\mathcal{N}^2 + \epsilon^{2\nu_3}), \quad 0 \le t \le t_1.$$

The constant c here is independent t_1. Since t_1 is arbitrary these estimates are valid, in fact, for all $t \geq 0$. More precisely, one has

$$M_0(t) \leq c(\mathcal{N}^2 + \epsilon^{2\nu_3}),\ M_1(t) \leq c(\mathcal{N} + \epsilon^{\nu_3})\rho_\infty^{\mu_1}(t),\ M_2(t) \leq c(\mathcal{N} + \epsilon^{\nu_3})\rho_\infty^{\mu_2}(t), \tag{2.32}$$

where $\rho_\infty(t)$ is the weight function corresponding to $t_1 = \infty$:

$$\rho_\infty(t) =< t >^{-1} + \sum_{\substack{j,k \\ j\neq k}} < t - t_{jk}^\infty >^{-1},$$

$t_{jk}^\infty = 0$ if $t_{jk}^0 \leq 0$, and

$$\int_0^{t_{jk}^\infty} ds \frac{v_{jk}(s) \cdot v_{jk}^0}{|v_{jk}^0|^2} = t_{jk}^0,$$

if $t_{jk}^0 > 0$.

By (2.15), (2.16), estimates (2.32) imply the existence of the limit trajectories $\sigma_{+j}(t) = (\beta_{+j}(t), E_{+j}, b_{+j}(t), v_{+j})$, $j = 1, \ldots, N$,

$$b_{+j}(t) = v_{+j}t + b_{+j}, \quad v_{+j} = v_{0j} + \int_0^\infty ds v_j'(s),$$

$$b_{+j} = b_{0j} + \int_0^\infty ds(c_j'(s) + v_j(s) - v_{+j}),$$

$$\beta_{+j}(t) = (E_{+j} - \frac{|v_{+j}|^2}{4})t + \beta_{+j}, \quad E_{+j} = E_{0j} + \int_0^\infty ds E_j'(s),$$

$$\beta_{+j} = \beta_{0j} + \int_0^\infty ds\big(E_j - E_{+j} + \frac{|v_j - v_{+j}|^2}{4} + \gamma_j' - \frac{1}{2}v_j' \cdot c_j\big).$$

Obviously, as $t \to +\infty$,

$$|E_j(t) - E_{+j}|,\ |v_j(t) - v_{+j}| = 0(t^{-2\mu_1+1}),$$

$$|b_j(t) - b_{+j}(t)|,\ |\beta_j(t) - \beta_{+j}(t)| = O(t^{-2\mu_1+2}).$$

References

[1] Berestycki, H.; Lions, P.-L. Nonlinear scalar field equations, I, II, Arch. Rat. Mech. Anal. **1983**, *82* (4), 313-375.

[2] Buslaev V.S.; Perelman, G.S. Scattering for the nonlinear Schrödinger equation: states close to a soliton. St. Petersburg Math. J. **1993**, *4* (6),1111-1143.

[3] Cuccagna, S. Stabilization of solutions to nonlinear Schrödinger equation, Comm. Pure Appl. Math. **2001** *54*, 1110-1145.

[4] Ginibre, J.; Velo G. On a class of nonlinear Schrödinger equations I, II. J.Func.Anal. **1979**, *32*, 1-71.

[5] Ginibre, J.; Velo G. On a class of nonlinear Schrödinger equations III. Ann. Inst. H.Poincare -Phys. Theor. **1978**, *28* (3), 287-316.

[6] Hagedorn, G. Asymptotic completeness for the impact parameter approximation to three particle scattering. Ann. Inst. Henri Poincaré. **1982**, *36* (1), 19-40.

[7] McLeod, K. Uniqueness of positive radial solutions of $\triangle u + f(u) = 0$ in $\mathbb{R}^n$. Trans. Amer. Math. Soc. **1993**, *339* (2), 495-505.

[8] Nier, F.; Soffer, A. Dispersion and Strichartz estimates for some finite rank perturbations of the Laplace operator. J. of Func. Analysis, to appear.

[9] Perelman, G. Some results on the scattering of weakly interacting solitons for nonlinear Schrödinger equation. In: Spectral Theory, Microlocal Analysis, Singular Manifolds, M.Demuth et al., eds., Math. Top. 14, Berlin, Akademie Verlag, 1997, pp. 78-137.

[10] Perelman, G. Asymptotic stability of solitons for nonlinear Schrödinger equations, preprint.

[11] Soffer A.; Weinstein, M.I. Multichannel nonlinear scattering theory for nonintegrable equations I. Commun. Math. Phys. **1990**, *133* (1), 119-146.

[12] Soffer A.; Weinstein, M.I. Multichannel nonlinear scattering theory for nonintegrable equations II. J. Diff. Eq. **1992**, *98* (2), 376-390.

[13] Weinstein, M.I. Modulation stability of ground states of nonlinear Schrödinger equations. SIAM J. Math. Anal. **1985**, *16* (3), 472-491.

[14] Weinstein, M.I. Lyapunov stability of ground states of nonlinear dispersive evolution equations. Comm. Pure Appl. Math. **1986**, *39* (1), 51-68.