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François Murat

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U.M.R. 7640 du C.N.R.S.
F-91128 PALAISEAU CEDEX
Fax : 33 (0)1 69 33 49 49
Tél : 33 (0)1 69 33 49 99

Existence of a solution to $-\operatorname{div} a(x, Du) = f$ with $a(x,\xi)$ a maximal monotone graph in ξ for every x given

François Murat
Laboratoire Jacques-Louis Lions
Université Pierre et Marie Curie (Paris 6)

In this lecture I report on recent joint work [2] with Gilles Francfort and Luc Tartar.

Consider the problem of finding u such that

$$\begin{cases} u \in W_0^{1,p}(\Omega), \\ -\operatorname{div} a(x, Du) = f \quad \text{in } \mathcal{D}'(\Omega), \end{cases} \tag{1}$$

for a given $f \in W^{-1,p'}(\Omega)$. Here Ω is a bounded open set of $\mathbb{R}^N$, p is a real number with $1 < p < +\infty$, and $p' = p/(p-1)$. The usual setting is the case where $a : (x,\xi) \in \Omega \times \mathbb{R}^N \to a(x,\xi) \in \mathbb{R}^N$ is a Carathéodory function (i.e. a single-valued function, continuous in ξ for almost every $x \in \Omega$, and measurable in x for every $\xi \in \mathbb{R}^N$), which is monotone, i.e. satisfies, for almost every $x \in \Omega$ and for every $\xi_1 \in \mathbb{R}^N$ and $\xi_2 \in \mathbb{R}^N$

$$(a(x,\xi_1) - a(x,\xi_2))(\xi_1 - \xi_2) \geq 0, \tag{2}$$

and which further satisfies for almost every $x \in \Omega$ and for every $\xi \in \mathbb{R}^N$

$$\begin{cases} a(x,\xi)\xi \geq \alpha|\xi|^p - |a(x)|, \\ a(x,\xi)\xi \geq \beta|a(x,\xi)|^{p'} - |b(x)|, \\ a(x,0) = 0, \end{cases} \tag{3}$$

for some $\alpha > 0$, $\beta > 0$, $a \in L^1(\Omega)$ and $b \in L^1(\Omega)$ (the second assertion of (3) is another way of stating the classical growth condition $|a(x,\xi)| \leq \gamma|\xi|^{p-1} + |h(x)|$ for some h in $L^{p'}(\Omega)$). In such a setting it is well known that there exists a solution to (1).

In our work we investigate the case where for $x \in \Omega$ given, $\xi \to a(x,\xi)$ is no more a single-valued monotone continuous function from $\mathbb{R}^N$ into $\mathbb{R}^N$, but actually a multi-valued maximal monotone graph of $\mathbb{R}^N \times \mathbb{R}^N$. In such a case, the measurability properties of the graph which replaces $a(x,\xi)$ and the choice of a proper approximation procedure of this graph are the two main obstacles that should be overcome.

To our knowledge, the only paper in this direction is that of V. Chiadò Piat, G. Dal Maso & A. Defranceschi [1], in which delicate measurability assumptions are made in the definition of the graph and equally delicate measurability selection theorems are used in the proofs. In our paper we provide a simpler framework – which can be proved to be equivalent to theirs – and only make use of "classical" theorems in the proofs.

Specifically, we prove the existence of a function u and of a vector field d such that

$$(4)\qquad \begin{cases} u \in W_0^{1,p}(\Omega),\ d \in L^{p'}(\Omega)^N, \\ e = Du,\ -\operatorname{div} d = f \quad \text{in } \mathcal{D}'(\Omega), \\ (e,d) \in \mathcal{A}, \end{cases}$$

whenever the graph $\mathcal{A} \subset L^p(\Omega)^N \times L^{p'}(\Omega)^N$ satisfies $(0,0) \in \mathcal{A}$, is such that, for some $\alpha > 0$, $\beta > 0$, $a \in L^1(\Omega)$ and $b \in L^1(\Omega)$ and every $(e,d) \in \mathcal{A}$,

$$(5)\qquad \begin{cases} d(x)e(x) \geq \alpha |e(x)|^p - |a(x)|, \\ \\ d(x)e(x) \geq \beta |d(x)|^{p'} - |b(x)|, \end{cases} \qquad \text{for a.e. } x \in \Omega,$$

and further satisfies one of the the two following equivalent conditions.

– *First condition.*
Let $\phi : (x,\lambda) \in \Omega \times \mathbb{R}^N \to \phi(x,\lambda) \in \mathbb{R}^N$ be a (single-valued) Carathéodory function which is defined on the whole of $\Omega \times \mathbb{R}^N$ and satisfies

$$(6)\qquad \begin{cases} x \to \phi(x,\lambda) \text{ is measurable on } \Omega \text{ for every } \lambda \in \mathbb{R}^N, \\ |\phi(x,\lambda_1) - \phi_(x,\lambda_2)| \leq |\lambda_1 - \lambda_2|, \quad \text{a.e. } x \in \Omega,\ \forall \lambda_1, \lambda_2 \in \mathbb{R}^N. \end{cases}$$

Then $\mathcal{A}$ is defined by

$$(7)\qquad (e,d) \in \mathcal{A} \quad \Longleftrightarrow \quad d(x) - e(x) = \phi(x, d(x) + e(x)), \ \text{ a.e. } x \in \Omega.$$

– *Second condition.*
Let $j : \mathbb{R}^N \to \mathbb{R}^N$ be the function defined by $j(\lambda) = |\lambda|^{p'-2}\lambda$. Then $\mathcal{A} \subset L^p(\Omega)^N \times L^{p'}(\Omega)^N$ is pointwise monotone, i.e. satisfies for every $(e_1,d_1), (e_2,d_2) \in \mathcal{A}$

$$(8)\qquad (d_1(x) - d_2(x))(e_1(x) - e_2(x)) \geq 0, \quad \text{a.e. } x \in \Omega,$$

and is such that for every $g \in L^p(\Omega)^N$ and for every $t > 0$, there exists a unique (e,d) such that

$$(9)\qquad \begin{cases} (e,d) \in \mathcal{A}, \\ e(x) + tj(d(x)) = g(x), \quad \text{a.e. } x \in \Omega. \end{cases}$$

We prove that the two conditions above are equivalent in the following sense. If the graph $\mathcal{A}$ is defined by (7) for some function ϕ defined on the whole of $\Omega \times \mathbb{R}^N$ and satisfying (6), then $\mathcal{A}$ satisfies (8) and (9). Conversely, if the graph $\mathcal{A}$ satisfies (8) and (9), then there exists a fonction ϕ which is defined on the whole of $\Omega \times \mathbb{R}^N$ and which satisfies (6), such that $\mathcal{A}$ is defined by (7).

The two equivalent conditions provide a convenient definition of a "multi-valued maximal monotone graph in $\mathbb{R}^N \times \mathbb{R}^N$ depending on x". The justification of this assertion is twofold. On the one hand, if $\mathcal{A}$ satisfies (8) and (9), it is easily established that $\mathcal{A}$ is a maximal monotone graph of $L^p(\Omega)^N \times L^{p'}(\Omega)^N$. The converse is also true, but this is a result of abstract maximal monotone graphs theory which is more difficult to prove. On the other hand, consider ϕ independent of x; if ϕ is defined on the whole of $\mathbb{R}^N$ and satisfies (6), it is easy to prove that $\mathcal{A}$ defined by (7) is a maximal monotone graph of $\mathbb{R}^N \times \mathbb{R}^N$. The converse is also true, but the proof uses Kirzbraun's theorem.

If the graph $\mathcal{A}$ satisfies (5) and one of the two equivalent conditions given above, we prove the existence of a solution to (4). We give two different proofs of this existence result, using in each proof a different approximation procedure which relies on one of the two conditions.

In a forthcoming paper [3], we will prove a compactness result with respect to H-convergence (or in other words an homogenization result) for this class of graphs. Indeed, from every sequence ε of graphs $\mathcal{A}_\varepsilon$ which satisfy (5) uniformly (i.e. for the same $\alpha > 0$, $\beta > 0$, $a \in L^1(\Omega)$ and $b \in L^1(\Omega)$), with $(0,0) \in \mathcal{A}_\varepsilon$ and which are such that one of the two equivalent conditions given above is satisfied, one can extract a subsequence, still denoted by ε, and there exists a graph $\mathcal{A}_0$ in the same class, such that for every $f \in W^{-1,p'}(\Omega)$, any accumulation point (u,d) (in the weak topology of $W_0^{1,p}(\Omega) \times L^{p'}(\Omega)^N$) of the solutions $(u_\varepsilon, d_\varepsilon)$ to

$$(10_\varepsilon) \qquad \begin{cases} u_\varepsilon \in W_0^{1,p}(\Omega),\ d_\varepsilon \in L^{p'}(\Omega)^N, \\ e_\varepsilon = Du_\varepsilon,\ -\operatorname{div} d_\varepsilon = f \quad \text{in } \mathcal{D}'(\Omega), \\ (e_\varepsilon, d_\varepsilon) \in \mathcal{A}_\varepsilon, \end{cases}$$

is a solution (u_0, d_0) to (10_0). (Observe that u_ε is bounded in $W_0^{1,p}(\Omega)$ and that d_ε is bounded in $L^{p'}(\Omega)^N$). This provides a new (and in our opinion simpler) proof of the analogous result proved twelve years ago by V. Chiadò Piat, G. Dal Maso & A. Defranceschi in [1].

[1] Valeria Chiadò Piat, Gianni Dal Maso & Anneliese Defranceschi,
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