Centre de Mathématiques Laurent Schwartz
Ecole Polytechnique

SEMINAIRE

Equations aux Dérivées Partielles

2002-2003

François Germinet

Quantum Dynamics and generalized fractal dimensions: an introduction

Séminaire É. D. P. (2002-2003), Exposé n° XVIII, 14 p.

<http://sedp.cedram.org/item?id=SEDP_2002-2003____A18_0>

U.M.R. 7640 du C.N.R.S.
F-91128 PALAISEAU CEDEX
Fax : 33 (0)1 69 33 49 49
Tél : 33 (0)1 69 33 49 99

Quantum Dynamics and generalized fractal dimensions: an introduction

François Germinet
UMR 8524 CNRS, UFR de Mathématiques,
Université de Lille 1,
F-59655 Villeneuve d'Ascq Cédex, France
E-mail: germinet@agat.univ-lille1.fr

Abstract

We review some recent results on quantum motion analysis, and in particular lower bounds for moments in quantum dynamics. The goal of the present exposition is to stress the role played by quantities we shall call *Transport Integrals* and by the so called *generalized dimensions* of the spectral measure in the analysis of quantum motion. We start with very simple derivations that illustrate how these quantities naturally enter the game. Then, gradually, we present successive improvements, up to most recent result.

1 Introduction to the subject

Let H be a self-adjoint operator acting on a Hilbert space $\mathcal{H}$, and $U(t) = e^{-iHt}$ the strongly continuous one parameter unitary group that H generates. We denote by $\mathcal{D}(H) \subset \mathcal{H}$ the domain of H. The example to keep in mind is H a Schrödinger operator acting either on $\ell^2(\mathbb{Z}^d)$ or $\mathrm{L}^2(\mathbb{R}^d)$. The goal is to study the time behaviour of the solution ψ_t of the Schrödinger equation $-i\partial_t\psi_t = H\psi_t$, with initial condition $\psi_{t=0} = \psi \in \mathcal{D}(H)$ of norm 1. The solution is given by $\psi_t = U(t)\psi$. Our basic tool from operator theory is the spectral theorem, which reads here: there exists a positive Borel measure μ_ψ of total mass 1, called the spectral measure associated to ψ, s.t. for all bounded Borelian function f,

$$\langle \psi, f(H)\psi\rangle = \int_{\mathbb{R}} f(\lambda)\mu_\psi(\mathrm{d}\lambda). \tag{1.1}$$

Replacing f by $e^{-i\lambda t}$ leads to $\langle \psi, U(t)\psi\rangle = \widehat{\mu_\psi}(t)$, the Fourier transform of μ_ψ. So that looking for the behaviour of $\langle \psi, U(t)\psi\rangle$ as $t \to \infty$ reduces to the understanding of the asymptotic behaviour of $\widehat{\mu_\psi}(t)$. It is thus quite clear that the regularity of μ_ψ plays an important role in the understanding of the solution ψ_t. The first remark is that, by the Riemann-Lebesgue Lemma, one has $\langle \psi, U(t)\psi\rangle \to 0$ as soon as μ_ψ is absolutely continuous with respect to Lebesgue. Note that, in some vague sense, $\langle \psi, U(t)\psi\rangle \to 0$ means that the solution ψ_t dissociates itself from its initial state, which can be read as a weak notion of *delocalization* or *quantum transport*. On the opposite side one expects singular measures (say atomic) to be associated to *localization* or *absence of quantum* *transport*. Here are further signs that reinforce this picture. Consider the correlation coefficient

$$C(T)=\frac{1}{T}\int_0^T |\langle\psi,U(t)\psi\rangle|^2\mathrm{d}t=\frac{1}{T}\int_0^T |\widehat{\mu_\psi}(t)|^2\mathrm{d}t. \tag{1.2}$$

Decompose μ_ψ into its atomic and continuous parts: $\mu_\psi=\sum_n a_n\delta_{\lambda_n}+\nu_\psi$, where δ_x is the Dirac mass. It is a rather immediate consequence of (1.1) and the Lebesgue Dominated Convergence Theorem that $\lim_{T\to\infty}C(T)=\sum_n|a_n|^2$, a result called the Wiener Theorem. In particular solutions ψ_t corresponding to continuous spectral measures exhibit some *delocalization* in the sense that $\lim_{T\to\infty}C(T)=0$, while atoms prevent ψ_t from being totally disconnected from ψ, in the sense that $\lim_{T\to\infty}C(T)\neq 0$. Next step is to understand how fast $C(T)$ vanishes if it does. Here again, better regularity of μ_ψ will yield faster delocalization. Imagine $\mu_\psi=f\mathrm{d}\lambda$, with $f\in\mathrm{L}^2(\mathbb{R})$, then Parseval formula implies $(C(T)+C(-T))\sim\|f\|^2_{\mathrm{L}^2}T^{-1}$. A more refined statement is due to Strichartz [St]: Assume that μ_ψ is Uniformly α Hölder continuous (UαH), $\alpha\in]0,1]$, namely,

$$\exists\varepsilon_0>0,\ \exists C_\alpha<+\infty,\ \forall\varepsilon\in]0,\varepsilon_0[,\ \text{for } \mu \text{ a.e. } x\in\mathrm{supp}\mu:\ \mu_\psi(B(x,\varepsilon))\leq C_\alpha\varepsilon^\alpha\,, \tag{1.3}$$

where $B(x,\varepsilon)=(x-\varepsilon,x+\varepsilon)$. Then $C(T)\leq C'T^{-\alpha}$. In particular

$$\liminf_{T\to\infty}\frac{\log C(T)}{\log 1/T}\geq\alpha\,. \tag{1.4}$$

We shall recover this result Line (1.10). In other terms, in the sense of correlation coefficient, the more the spectral measure is regular, the stronger is the delocalization effect. This picture will be confirmed in the sense of time behaviour of *moments* $\mathbb{M}(p,\psi,T)$ defined in (1.17) (see (3.8) and below).

However, during the 90's, thanks to some concrete models, it became clear that taking only into account the regularity of spectral measures cannot explain totally quantum transport. Fast delocalization (in terms of moments $\mathbb{M}(p,\psi,T)$) has been shown to occur even in presence of poor regularity of the spectral measure [La, BCM, DBF, Ma, DRJLS] (or more recently [JSBS, DT, DST, GKT, Tc2]).

Therefore, thiner properties of measures rather than just regularity, should play an important role in a theory of quantum transport. In this context, the goal of the present note is to stress the importance of a family of integrals that we shall call *Transport Integrals*, as well as their normalized growth exponents (we note that these integrals already appeared in the context of quantum dynamics in [SBB]).

Definition 1.1 (Transport Integrals.). *Let μ be a probability measure on $\mathbb{R}$. For all $q\in\mathbb{R}$ and $\varepsilon\in(0,1)$, set*

$$I_\mu(q,\varepsilon)=\int_{\mathrm{supp}\mu}\mu(x-\varepsilon,x+\varepsilon)^{q-1}\mathrm{d}\mu(x)\ \in\ [0,+\infty]. \tag{1.5}$$

For $q\neq 1$, we further define the lower and upper Hentschel-Procaccia dimensions

$$D_\mu^-(q)=\liminf_{\varepsilon\downarrow 0}\frac{\log I_\mu(q,\varepsilon)}{(q-1)\log\varepsilon}\ ,\quad D_\mu^+(q)=\limsup_{\varepsilon\downarrow 0}\frac{\log I_\mu(q,\varepsilon)}{(q-1)\log\varepsilon}\ , \tag{1.6}$$

taking values in $[0,+\infty]$.

These dimensions $D_\mu^\pm(q)$ have many other names: generalized dimensions, generalized fractal dimensions, multifractal dimensions, generalized entropy dimensions.

A simple way to be convinced that these integrals and dimensions play a role in quantum transport is to do the following calculation:

$$
\begin{aligned}
C(T) &\leq c\iint \mu_\psi(\mathrm{d}x)\mu_\psi(\mathrm{d}y)\int_{-\infty}^{+\infty} e^{-(\frac{t}{T})^2}e^{-i(x-y)t}\frac{\mathrm{d}t}{T} && (1.7)\\
&= c'\int \mu_\psi(\mathrm{d}x)\left(\int \mu_\psi(\mathrm{d}y)\exp(-((x-y)T)^2)\right) && (1.8)\\
&\sim \int \mu_\psi(\mathrm{d}x)\left(\int \mu_\psi(\mathrm{d}y)\chi_{[-1,1]}((x-y)T)\right) && (1.9)\\
&= \int \mu_\psi(\mathrm{d}x)\mu_\psi(x-\varepsilon,x+\varepsilon) = I_{\mu_\psi}(2,\varepsilon), \qquad \varepsilon=\frac{1}{T}\,. && (1.10)
\end{aligned}
$$

The equivalence sign in (1.9) means: $f \sim g$ iff there exists $C > 0$ finite s.t. $C^{-1}g \leq f \leq Cg$. It is a consequence of a more general result proved in [BGT3, Tc1]. On the lower bound side, note that

$$
\begin{aligned}
C(T) &= c\iint \frac{\sin((x-y)T)}{(x-y)T}\mu_\psi(\mathrm{d}x)\mu_\psi(\mathrm{d}y) && (1.11)\\
&\geq c\iint_{|x-y|\leq\frac{1}{T}} \frac{\sin((x-y)T)}{(x-y)T}\mu_\psi(\mathrm{d}x)\mu_\psi(\mathrm{d}y) = c'I_{\mu_\psi}(2,\frac{1}{T}), && (1.12)
\end{aligned}
$$

since $\frac{\sin u}{u} \geq \sin 1$ for $u \in]0,1]$. As a consequence (a result that goes back, at least, to [SBB, BCM])

$$
\liminf_{T\to\infty}\frac{\log C(T)}{\log 1/T} = D^-_{\mu_\psi}(2)\,, \qquad \limsup_{T\to\infty}\frac{\log C(T)}{\log 1/T} = D^+_{\mu_\psi}(2)\,. \tag{1.13}
$$

Compare to (1.4), and note that $D^-_{\mu_\psi}(2)$ can be significantly greater than α. For instance consider $\mu^{(a)}(\mathrm{d}x) = \chi_{[0,1)}(x)x^{-a}\mathrm{d}x$, $a \in [0,1[$. Then short calculations show that $\mu^{(a)}$ is uniformly $(1-a)$-continuous (and not better), while $D^-_{\mu^{(a)}}(2) = 2(1-a)$ if $a \in [\frac{1}{2},1]$ and $D^-_{\mu^{(a)}}(2) = 1$ if $a \in]0,\frac{1}{2}]$ (use the equivalence with the Rényi sums – see next section – or look at [BGT3, Section 6, Example 5]). In Proposition 2.1 we shall actually see to which dimension UαH continuity is related.

What makes the difference is that integrals $I_\mu(q,\varepsilon)$ not only exploit local regularities of the measure μ, but they also take advantage of statistical effects of these regularities. This will be of crucial importance when we shall observe quantum transport in presence of pure point spectrum. It will be sufficient that, ε being given, there are enough points of local regularity (i.e. with $\mu(B(x,\varepsilon)) = C_x\varepsilon^\alpha$) to give large values to the generalized dimensions. See discussion below Proposition 2.1.

We turn to the main dynamical quantities we want to study: moments. To do that, we first need to specify a bite more what our setting is. While it is very possible to investigate moments with respect to any orthonormal basis of the Hilbert space $\mathcal{H}$, we are mostly interested in operators coming from quantum mechanics, and more precisely, in Schrödinger operators acting on $\mathcal{H} = \ell^2(\mathbb{Z}^d)$ or $\mathcal{H} = \mathrm{L}^2(\mathbb{R}^d)$. Thus moments should refer to "physical position operators" $\langle X\rangle^p$ (namely: multiplication by $\langle x\rangle^p$), rather than to an abstract basis[1], where for $x \in \mathbb{Z}^d$ or $\mathbb{R}^d$ we use the following notations

$$\langle x\rangle = \sqrt{1+|x|^2} \quad \text{and} \quad (\langle X\rangle\psi)(x) = \langle x\rangle\psi(x),\ \psi \in \mathcal{H}. \tag{1.14}$$

In the continuous case $\mathcal{H} = \mathrm{L}^2(\mathbb{R}^d)$ we assume that the potential satisfies to the following regularity property:

$$V = V^{(1)} + V^{(2)}, \quad \text{with} \quad 0 \le V^{(1)} \in \mathrm{L}^1_{\mathrm{loc}}(\mathbb{R}, \mathrm{d}x), \tag{1.15}$$

and $V^{(2)}$ is relatively $-\Delta$ form-bounded with relative bound < 1. To fix notations we thus require the existence of two constants $\Theta_1 < 1$ and Θ_2 such that

$$\left|\langle \psi, V^{(2)}\psi\rangle\right| \le \Theta_1\|\nabla\psi\| + \Theta_2\|\psi\|^2, \text{ for all } \psi \in \mathcal{D}(\nabla). \tag{1.16}$$

Definition 1.2 (Moments and transport exponents). *For $p \in]0,+\infty[$, we define the (time-averaged) moment of order p, with initial state ψ and at time $T > 0$, $\mathbb{M}(p,\psi,T)$, and the corresponding (normalized) lower and upper transport exponent, $\beta^{\pm}(p,\psi)$, as*

$$\mathbb{M}(p,\psi,T) = \frac{1}{T}\int_0^T \langle \psi_t, \langle X\rangle^p \psi_t\rangle \,\mathrm{d}t\,, \quad \beta^{\pm}(p,\psi) = \lim_{T\uparrow\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\log \mathbb{M}(p,\psi,T)}{p\log T}\,. \tag{1.17}$$

Note that we implicitely assume that ψ_t belongs to the domain of $\langle X\rangle^p$ for all time t.

As a general result,

Proposition 1.1 ([GK]). *Let $f \in \mathcal{C}_c^\infty(\mathbb{R})$, and $\psi = f(H)\chi_0$. Then ψ_t belongs to the domain of $\langle X\rangle^p$ for all $p > 0$ and $t \in \mathbb{R}$. And one has*
(i) $\beta^{\pm}(p,\psi)$ are increasing functions of p.
(ii) $\beta^{\pm}(p,\psi) \in [0,1]$, for all p.

The central result we want to present and to discuss here is the

Theorem 1.1 ([BGT1]). *Let $f \in \mathcal{C}_c^\infty(\mathbb{R})$, and $\psi = f(H)\chi_0$. For all $p > 0$, there exists a finite constant $C_p > 0$, such that*

$$\mathbb{M}(p,\psi,T) \ge \left(\frac{C_p}{\log T} I_{\mu_\psi}(q, T^{-1})\right)^{\frac{1}{q}}, \qquad q = \frac{1}{1+p/d}\,. \tag{1.18}$$

As a consequence, for all $p > 0$,

$$\beta^{\pm}(p,\psi) \ge \frac{1}{d} D^{\pm}_{\mu_\psi}(q)\,, \qquad q = \frac{1}{1+p/d}\,. \tag{1.19}$$

A similar result but under stronger hypotheses is proved in [GSB2]. For previous works on lower bounds, see [G, C, La, BCM, GSB1, BT, KL] and the discussion made in Section 3. A generalized version of Theorem 1.1 that takes into account the space behaviour of generalized eigenfunctions is obtained in [Tc1]. We also refer the reader

[1] In the discrete case $\mathcal{H} = \ell^2(\mathbb{Z}^d)$, these two points of view coincide if one considers the orthonormal basis $(\delta_n)_{n\in\mathbb{Z}^d}$. Then Position operators are given by $\langle X\rangle^p = \sum_{n\in\mathbb{Z}^d} \langle n\rangle^p \langle \delta_n, \cdot\rangle\delta_n$.

to [BGSB] for another interesting extension of Theorem 1.1 in the context of self-dual operators.

The rest of the discussion is organized as follows.

In Section 2 we go over some basic properties of these transport integrals, and in particular the equivalence with other definitions that will turn out to be quite crucial. We also get immediate lower bounds on these integrals, first using only regularity properties of the measure, and second, using the statistical effect of these integrals.

In Section 3 we present the derivation of Theorem 1.1, following Guarneri's idea but showing how transport integrals enter the game.

In Section 4 we review the analysis of wave-packets performed in [GKT], which leads to a characterization of transport exponents for both small and large values of the order p.

In Section 5 we present a general lower bound for moments in one dimension. This result is a continuation of Theorem 1.1 and Theorem 2.1 using transfer matrices. It will allow to get the first examples of Schrödinger operators with non trivial generalized dimensions, and thereby the first applications of Theorem 1.1 to Schrödinger operators.

2 Generalized dimensions: equivalent definitions and properties

We start with the

Definition 2.1 (Generalized Rényi Dimensions.). *Let μ be a probability measure on $\mathbb{R}$, and $q>0$. The associated generalized Rényi sums, resp. integrals, are given by*

$$S_\mu(q,\varepsilon)=\sum_{j\in\mathbb{Z}}\mu([j\varepsilon,(j+1)\varepsilon[)^q\ ,\quad \textit{resp.}\quad L_\mu(q,\varepsilon)=\frac{1}{\varepsilon}\int_{\mathbb{R}}\mu(B(x,\varepsilon))^q\,\mathrm{d}x\ . \tag{2.1}$$

The generalized lower and upper Rényi dimensions, resp. integral Rényi dimensions, are defined, for $q\neq 1$, as:

$$RD^{\pm}_\mu(q)=\lim_{\varepsilon\downarrow 0}\begin{matrix}\sup\\ \inf\end{matrix}\frac{\log S_\mu(q,\varepsilon)}{(q-1)\log\varepsilon}\ ,\quad IRD^{\pm}_\mu(q)=\lim_{\varepsilon\downarrow 0}\begin{matrix}\sup\\ \inf\end{matrix}\frac{\log L_\mu(q,\varepsilon)}{(q-1)\log\varepsilon}\ ,$$

taking values in $[0,+\infty]$.

To our best knowledge, sums $S_\mu(q,\varepsilon)$ have been introduced by Rényi in the 50's [Re] when he developed Information Theory, generalizing the Shannon entropy. They have been shown to play a role in hyperbolic dynamical systems [Pe, TV].

That the above definitions are actually equivalent is a well-known fact for $q>1$, and the proof is very simple. If $q\in]0,1[$, it is a different story. If the measure is "doubling" then the proof for $q\in]0,1[$ reduces to the proof for $q>1$ (see [Ol, Pe]). But in full generality, this equivalence for $q\in]0,1[$ is proved in [BGT3] and it is the content of Theorem 2.1 below. For $q<0$, the above Rényi dimensions do not make sense. They are replaced by discrete sums over packings or coverings of suppμ. Their equivalence to integrals $I_\mu(q,\varepsilon)$ is treated rigorously in [GT].

Theorem 2.1 ([BGT3]). *Let μ be a probability measure on $\mathbb{R}$, and $q \in]0,1[\cup]1,+\infty[$. Transport integrals, Rényi sums and Rényi integrals, either all diverge, or are equivalent if they converge: there exists constants $c_i(q) > 0$, $i = 1,2,3$, s.t. for all $\varepsilon \in (0,1)$,*

$$I_\mu(q,\varepsilon) \le c_1(q) S_\mu(q,\varepsilon) \le c_2(q) L_\mu(q,\varepsilon) \le c_3(q) I_\mu(q,\varepsilon)\,. \tag{2.2}$$

As a consequence for all $q \in]0,1[\cup]1,+\infty[$,

$$D_\mu^-(q) = RD_\mu^-(q) = IRD_\mu^-(q) \;\; \text{and} \;\; D_\mu^+(q) = RD_\mu^+(q) = IRD_\mu^+(q)\,. \tag{2.3}$$

Following Theorem 2.1, and for $q \in]0,1[\cup]1,+\infty[$, we shall refer to any of these dimensions as to *generalized dimensions*, and use the notation $D_\mu^\pm(q)$. One has the following basic properties.

Theorem 2.2 ([BGT3]). *Let μ be a probability measure on $\mathbb{R}$. Set*

$$q_\mu^* = \inf\{q \in \mathbb{R} \setminus \{1\},\; D_\mu^+(q) < \infty\}. \tag{2.4}$$

Then
(i) $q_\mu^* \le 1$ and for all $q > 1$, $D_\mu^\pm(q) \in [0,1]$.
(ii) Dimensions $D_\mu^\pm(q)$ are decreasing functions of q on $\mathbb{R}\setminus\{1\}$, and continuous on $q \in]q_\mu^*, 1[\cup]1,+\infty[$.
(iii) If μ is compactly supported, then $q_\mu^* \le 0$ and one has $D_\mu^\pm(q) \in [0,1]$, for $q > q_\mu^*$.
(iv) For all $q < 1$ and $q' > 1$, one has

$$D_\mu^-(q) \ge \dim_H(\mu) \ge D_\mu^-(q'),\; \text{and}\; D_\mu^+(q) \ge \dim_P(\mu) \ge D_\mu^+(q'),$$

where $\dim_H(\mu)$ and $\dim_H(\mu)$ stand for, respectively, the Hausdorff and Packing dimensions of μ (see e.g. [SBB, BGT3]).

Relations between Uniformly Hölder continuous measure and the generalized dimensions are given by the following Proposition. It says that the best rate of UαH continuity is actually given by $D_\mu^-(+\infty)$.

Proposition 2.1. *Set $D_\mu^\pm(+\infty) = \lim_{q\to+\infty} D_\mu^\pm(q)$.*
(i) Suppose that μ is UαH continous for some $\alpha \in]0,1]$, then $\forall\, q > 1$, $D_\mu^-(q) \ge \alpha$, and thus $D_\mu^-(+\infty) \ge \alpha$.
(ii) Assume that for some $q > 1$, $D_\mu^-(q) > 0$. Then μ is UαH continous with $\alpha = D_\mu^-(q)(1-\frac{1}{q}) - \nu > 0$, for any $\nu > 0$ small enough. Moreover one has $D_\mu^\pm(+\infty) = \sup_{q>1}\{D_\mu^\pm(q)(1-\frac{1}{q})\}$. As a consequence $D_\mu^-(+\infty) > 0$ and μ is UαH continous with $\alpha = D_\mu^-(+\infty) - \nu > 0$, for any $\nu > 0$.
(iii) More generally, one has

$$D_\mu^\pm(+\infty) = \lim_{\varepsilon\downarrow 0} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\log(\sup_x \mu(B(x,\varepsilon)))}{\log\varepsilon}\,. \tag{2.5}$$

And it is nonzero iff $D^\pm(q) > 0$ for some $q > 1$.

Remark 2.1. *The case $q = 2$ has already been observed in [La, Theorem 3.1].*

Proof: (i) is immediate. For (ii), notice that for $q > 1$ and all $x \in \mathrm{supp}\mu$, $I_\mu(q,\varepsilon) \geq \int_{x-\frac{\varepsilon}{2}}^{x+\frac{\varepsilon}{2}} \mu(B(y,\varepsilon))^{q-1}\mathrm{d}\mu(y) \geq \mu(B(x,\frac{\varepsilon}{2}))^q$. On the other hand for all $\nu > 0$ and ε small enough, $I_\mu(q,\varepsilon) \leq \varepsilon^{(D_\mu^-(q)-\nu)(q-1)}$. It follows, recalling (1.3) that μ is UαH continuous with $\alpha = D_\mu^-(q)(1-\frac{1}{q}) - \nu$ for all $q > 1$ and $\nu > 0$. It is a consequence of [BGT3, Proposition 3.4 (iii)], that $D_\mu^\pm(+\infty) = \sup_{q>1}\{D_\mu^\pm(q)(1-\frac{1}{q})\}$. Thus for all $q > 1$, $D_\mu^-(q) \geq D_\mu^-(+\infty) > 0$. As a consequence, μ is UαH continuous with $\alpha = \sup_{q>1}\{D_\mu^-(q)(1-\frac{1}{q})\} - \nu = D_\mu^-(+\infty) - \nu$ for all $\nu > 0$. And (iii) is a simple generalization of (i) and (ii). $\square$

We now show how to obtain lower bounds on the transport integrals $I_\mu(q,\varepsilon)$ and thus on the dimensions $D_\mu^\pm(q)$, $q \in]0,1[$, by taking advantage of the equivalence given by Theorem 2.1.

Of course if μ is UαH continuous (recall (1.3)), then, with $q \in]0,1[$: $I_\mu(q,\varepsilon) \geq C_\alpha \varepsilon^{(q-1)\alpha}$, and thus $D_\mu^-(q) \geq \alpha$. But this is of little interest for the best such α is related to the smallest possible dimensions, i.e. $D_\mu^-(\infty)$ (Proposition 2.1). Let us have a look now to what the Rényi sums (or integrals) give, and how statistics can play a role. Assume that there are $\mathcal{O}(\frac{1}{\varepsilon^s})$ disjoints intervals $I_j = [j\varepsilon,(j+1)\varepsilon[$ such that $\mu(I_j) \approx c\varepsilon^a$. Then, as an immediate consequence $S_\mu(q,\varepsilon) \geq c\varepsilon^{aq-s}$, and $D_\mu^-(q) \geq \frac{s-qa}{1-q}$, a bound that gets better as q goes to zero. Note that the latter situation is only possible with $a \geq s$ (since for $q = 1$, $S_\mu(q,\varepsilon) = 1$). In other words, in the regime $q \in]0,1[$, a sufficiently large number intervals of very small weight gives a better contribution.

Looking at $L_\mu(q,\varepsilon)$, the natural condition to ask is that

$$\exists\varepsilon_0 > 0,\ \exists C_a > 0,\ \forall\varepsilon \in]0,\varepsilon_0[,\ \text{for } \textbf{Lebesgue} \text{ a.e. } x \in \mathrm{supp}\mu:\ \mu(B(x,\varepsilon)) \geq C_a\varepsilon^a. \tag{2.6}$$

(We implicitely assume that $|\mathrm{supp}\mu| > 0$). Compare (2.6) to (1.3). The following can then easily be derived.

Theorem 2.3. *Define*

$$D_\mu^\pm(-\infty) = \lim_{q\to-\infty} D_\mu^\pm(q),\ \textit{and}\ g_\mu^\pm = \lim_{\varepsilon\downarrow 0} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\log(\inf_x \mu(B(x,\varepsilon)))}{\log\varepsilon}. \tag{2.7}$$

Assume that $|\mathrm{supp}\mu| > 0$ *and that* $g_\mu^+ < \infty$. *Then*

$$D_\mu^\pm(q) \geq \frac{1-qg_\mu^\mp}{1-q}. \tag{2.8}$$

Remark 2.2. *The quantity* g_μ^+ *turns out to be of special importance. Its finiteness is crucial to get finite generalized dimensions for* $q < 0$. *It is indeed proved in [GT] for compactly supported measures:*
(i) $g_\mu^+ < \infty \Longleftrightarrow \forall q < 0\ D_\mu^\pm(q) < \infty \Longleftrightarrow q_\mu^* = -\infty \Longleftrightarrow D_\mu^\pm(-\infty) < \infty$.
(ii) If $g_\mu^+ < \infty$ *then* $D_\mu^\pm(-\infty) = g_\mu^\pm$.
It also follows that if $g_\mu^+ < \infty$ *then dimensions* $D_\mu^\pm(q)$ *are continuous on* $\mathbb{R}\setminus\{1\}$, *and* $D_\mu^\pm(0) = \dim_B^\pm(\mathrm{supp}\mu)$, *where* $\dim_B^\pm$ *denotes the lower and upper box counting dimensions. Then one checks that (2.8) can actually be improved to* $D_\mu^\pm(q) \geq \frac{D_\mu^\pm(0)-qg_\mu^\mp}{1-q}$ *(see [Tc1]).*

Compare Theorem 2.3 and Remark 2.2 (ii) to Proposition 2.1.

3 Bounding moments from below: following Guarneri's approach

Let us first slightly extend the simple calculation (1.7). Pick $\psi \in \mathcal{H}$ and $\varphi = f(H)$, $\phi = g(H)$, $f, g \in \mathrm{L}^2(\mathbb{R}, \mathrm{d}\mu_\psi)$. Set $A_{\varphi,\phi} = \langle \varphi, \cdot \rangle \phi$. If $h \in \mathcal{C}_0^\infty(\mathbb{R})$ is s.t. $|\hat{h}|$ decays faster than any polynomial:

$$\left| \int h(t/T) \langle U(t)\psi, A_{\varphi,\phi} U(t)\psi \rangle \frac{\mathrm{d}t}{T} \right| \tag{3.1}$$

$$= \left| \int \int f(x) \overline{g(y)}\, \hat{h}((x-y)T) \mu_\psi(\mathrm{d}x) \mu_\psi(\mathrm{d}y) \right| \tag{3.2}$$

$$\leq \|f\|_{\mathrm{L}^2(\mathrm{d}\mu_\psi)} \|g\|_{\mathrm{L}^2(\mathrm{d}\mu_\psi)} \left(\int \mu_\psi(\mathrm{d}x) \left(\int \mu_\psi(\mathrm{d}y) |\hat{h}|^2((x-y)T) \right) \right)^{\frac{1}{2}} \tag{3.3}$$

$$\leq c \|\varphi\|_{\mathcal{H}} \|\phi\|_{\mathcal{H}} \left(I_{\mu_\psi}(2, \frac{1}{T}) \right)^{\frac{1}{2}}. \tag{3.4}$$

Note that $\|A_{\varphi,\phi}\|_{\mathrm{HS}} = \|\varphi\|_{\mathcal{H}} \|\phi\|_{\mathcal{H}}$. And the above computation can indeed be extended from rank one operators to Hilbert-Schmidt ones[2]:

$$\left| \int h(t/T) \langle U(t)\psi, A U(t)\psi \rangle \frac{\mathrm{d}t}{T} \right| \leq \|A\|_{\mathrm{HS}} \left(I_{\mu_\psi}(2, \frac{1}{T}) \right)^{\frac{1}{2}}. \tag{3.5}$$

This is enough to understand Guarneri's way to bound from below moments of the dynamics. Assume that for some compact energy interval I, $\psi = E_H(I)\psi$. One has, for $N > 0$ and $h \in \mathcal{C}_0^\infty([0,1])$ as above,

$$\mathbb{M}(p, \psi, T) \geq N^p \left(\|\psi\|^2 - \left| \int h(t/T) \langle \psi_t, P_N E_H(I) \psi_t \rangle \frac{\mathrm{d}t}{T} \right| \right) \tag{3.6}$$

$$\geq N^p \left(\|\psi\|^2 - \|P_N E_H(I)\|_{HS} \left(I_{\mu_\psi}(2, T^{-1}) \right)^{\frac{1}{2}} \right), \tag{3.7}$$

where we used (3.5). That $\|P_N E_H(I)\|_{HS} \leq C N^{\frac{d}{2}}$ is immediate if $\mathcal{H} = \ell^2(\mathbb{Z}^d)$. In the continous case, under hypotheses (1.15)-(1.16), the latter follows from [KKS] (see [GKT]). Choosing the biggest N such that $C N^{\frac{d}{2}} \left(I_{\mu_\psi}(2, T^{-1}) \right)^{\frac{1}{2}} \leq \frac{1}{2} \|\psi\|^2$ leads to

$$\mathbb{M}(p, \psi, T) \geq C \|\psi\|^2 \left(\frac{\|\psi\|^4}{I_{\mu_\psi}(2, T^{-1})} \right)^{p/d}. \tag{3.8}$$

As a consequence $\beta^\pm(p, \psi) \geq \frac{1}{d} D^\pm_{\mu_\psi}(2)$. It implies Guarneri's initial result [G]: $\beta^-(p,\psi) \geq \frac{1}{d}\alpha$ if μ_ψ is UαH continuous. However it is not as good as what we announced in Theorem 1.1. It is not even sufficient to imply results of [La, BCM] and [GSB1],

[2] To have an idea, consider the special case $\mathcal{H} = \ell^2(\mathbb{Z}^d)$, $P_N = \sum_{|n| \leq N} \langle \delta_n, \cdot \rangle \delta_n$. According to the spectral theorem, $\langle \psi_t, \delta_n \rangle = \int \mathrm{e}^{-ixt} g_n(x) \mathrm{d}\mu_\psi(x)$, with $\|g_n\|_{L^2(\mathrm{d}\mu_\psi)} = \|P_\psi \delta\|_{\ell^2} \leq 1$, where P_ψ is the projection onto the cyclic subspace of ψ. Then the l.h.s. of (3.5), is bounded, after a Cauchy-Schwarz, by $\|S_N\|_{L^2(\mathrm{d}\mu_\psi \times \mathrm{d}\mu_\psi)} I_{\mu_\psi}(2, \frac{1}{T})^{\frac{1}{2}}$, where $S_N(x,y) = \sum_{|n| \leq N} \overline{g_n(x)} g_n(y)$. Writing $|S_N|^2 = S_N \overline{S_N}$ as a double sum, and noting that $\int \overline{g_n(x)} g_m(x) \mathrm{d}\mu_\psi(x) = \langle P_\psi \delta_n, \delta_m \rangle$, leads to $\|S_N\|^2_{L^2(\mathrm{d}\mu_\psi \times \mathrm{d}\mu_\psi)} \leq \sum_{|n|,|m| \leq N} |\langle P_\psi \delta_n, \delta_m \rangle|^2 \leq \sum_{|n| \leq N} \|P_\psi \delta_n\|^2 \leq c_d N^d$.

which asserts, respectively, $\beta^-(p,\psi) \geq \frac{1}{d}\dim_H(\mu_\psi)$ and $\beta^+(p,\psi) \geq \frac{1}{d}\dim_P(\mu_\psi)$ (recall Proposition 2.2 (iv)). In particular (3.8) is still too rough to enable one to extract transport in presence of pure point measure; since atomic measures satisfy $\dim_H(\mu) = \dim_P(\mu) = D_\mu^\pm(q) = 0$, $q > 1$. However *almost* all the conceptual tools we shall need to reach Theorem 1.1 are already in place: bounding $\mathbb{M}(p,\psi,T)$ by cutting out a ball of radius N, use transport integrals, take the best N. It is now mostly a matter of being more clever and more careful.

Introduce $\psi = \varphi + \chi$, $\varphi \perp \chi$, into the expression of $\mathbb{M}(p,\psi,T)$. Rewrite χ as $\chi = \psi - \varphi$ in the crossed term. It leads to the following quantity that we bound as follows:

$$\left| \int h(t/T) \langle U(t)\varphi, AU(t)\psi \rangle \frac{\mathrm{d}t}{T} \right| \leq \|A\|_{\mathrm{HS}} \left(I^h_{\varphi,\psi}(2, T^{-1}) \right)^{\frac{1}{2}}, \tag{3.9}$$

where we set $I^h_{\varphi,\psi}(q,\varepsilon) = \int \mu_\varphi(\mathrm{d}x) b^h_\psi(x,\varepsilon)^{q-1}$, with $b^h_\psi(x,\varepsilon) = \int \mathrm{d}\mu_\psi(y) |\hat{h}|^2((x-y)\varepsilon^{-1})$. Compare to (3.5). The proof is however trickier, and we refer to [BGT1, Theorem 3.2]. A derivation in the same spirit as before leads to

$$\mathbb{M}(p,\psi,T) \geq C \sup_\varphi \|\varphi\|^2 \left(\frac{\|\varphi\|^4}{I^h_{\varphi,\psi}(2,T^{-1})} \right)^{p/d}. \tag{3.10}$$

As noticed in [BT], one may now choose the best φ for each time T. It is a new degree of freedom that will enable us to extract from ψ_t, at time T, the best possible part of the wave-packet, i.e. the part with best statistics. The result, which clearly implies Theorem 1.1, is that

$$\left(\frac{C}{\log T} I_{\mu_\psi}(q, T^{-1}) \right)^{\frac{1}{q}} \leq \sup_\varphi \|\varphi\|^2 \left(\frac{\|\varphi\|^4}{I^h_{\varphi,\psi}(2,T^{-1})} \right)^{p/d} \leq C' I_{\mu_\psi}(q, T^{-1})^{\frac{1}{q}}, \tag{3.11}$$

with $q = (1+p/d)^{-1}$. The upper bound is Hölder inequality and equivalence of transport integrals (in the spirit of (1.8)-(1.9)). The lower bound is obtained by choosing $\varphi = \chi_{\Omega_r}(H)\psi$, with $\Omega_r = \{x, b^h_\psi(x,T^{-1}) \in [T^{-r-\frac{c}{\log T}}, T^{-r})\}$, and such that $I_{\varphi,\psi}(q,T^{-1}) \geq \frac{C}{\log T} I_{\mu_\psi}(q,T^{-1})$ for all $T > 0$[3].

4 Characterization of the transport exponents at low and large order

Varying the order p of moments from 0 to $+\infty$ should favorize different parts of the wave-packet. Since *normalized* transport exponents $\beta^\pm(p,\psi)$ increase with p, it seems natural to expect that fastest parts of the wave-packet will play a bigger role as p gets

[3]Here is an alternative derivation of the latter which is due to S. Tcheremchantsev (in the spirit of [Tc1]): for $s \in]0,1[$ set $\varphi = \chi_{\Omega_s}(H)\psi$, with $\Omega_s = \{x, b^h_\psi(x,T^{-1}) \leq s\}$. The bound (3.10) thus implies that $\mathbb{M}(p,\psi,T) \geq (C \sup_s \mu_\psi(\Omega_s) s^{-\frac{p/d}{1+p/d}})^{1+p/d}$. But notice that $\sup_s f(s) \geq \int_0^1 f(s) s^{-1+\nu} \mathrm{d}s$. As a consequence, and after Fubini, with $b = \mu_\psi(B(x,T^{-1}))$: $\sup_s \mu_\psi(\Omega_s) s^{-\frac{p/d}{1+p/d}} \geq \int \mathrm{d}\mu(x) \int_b^1 s^{-\frac{p/d}{1+p/d}-1+\nu} \mathrm{d}s \geq \int \mathrm{d}\mu(x) \int_b^{2b} s^{-\frac{p/d}{1+p/d}-1+\nu} \mathrm{d}s \geq c \int \mathrm{d}\mu(x) b^{-\frac{p/d}{1+p/d}+\nu}$. It implies that $\beta^\pm(p,\psi) \geq \frac{1}{d} D^\pm_{\mu_\psi}(\frac{1}{1+p/d} + \nu)$ for all $\nu > 0$, and thus $\beta^\pm(p,\psi) \geq \frac{1}{d} D^\pm_{\mu_\psi}(\frac{1}{1+p/d})$ by continuity of the generalized dimensions

large, while a slow but essential part of the wave-packet should have more influence on moments behaviour as p goes to zero. In this section, we want to give a precise contain to this rough intuition.

Set, for $\alpha \in [0, +\infty[$,

$$P(\alpha, T) = \frac{1}{T} \int_0^T \left\| Q_{(T^\alpha - 2)} \, \mathrm{e}^{-iHt} \psi \right\|_{\mathcal{H}}^2 \mathrm{d}t, \tag{4.1}$$

where Q_N is the spatial projection *outside* the ball of radius N and center the origin. Define $S^{\pm}(\alpha)$ the growth exponents of $P(\alpha, T)$:

$$S_\psi^-(\alpha) = - \liminf_{T \to +\infty} \frac{\log P_\psi(\alpha, T)}{\mathrm{logT}}, \quad S_\psi^+(\alpha) = - \limsup_{T \to +\infty} \frac{\log P_\psi(\alpha, T)}{\mathrm{logT}}. \tag{4.2}$$

If $P_\psi(\alpha, T) = 0$ for some $\alpha > 0$ starting from $T \geq T_0$, we set $S_\psi^{\pm}(\alpha) = +\infty$. Definitions are so that $0 \leq S_\psi^+(\alpha) \leq S_\psi^-(\alpha) \leq +\infty$. Then, natural quantities to define are

$$\alpha_l^{\pm} = \sup\{\alpha \geq 0 \mid S_\psi^{\pm}(\alpha) = 0\}, \quad \alpha_u^{\pm} = \sup\{\alpha \geq 0 \mid S_\psi^{\pm}(\alpha) < +\infty\}. \tag{4.3}$$

Since $S_\psi^{\pm}(\alpha)$ are non decreasing functions, $0 \leq \alpha_l^{\pm} \leq \alpha_u^{\pm} \leq +\infty$. One can interpret $\alpha_l^{\pm}$ as the (lower and upper) rates of propagation of the *essential* part of the wave packet, and $\alpha_u^{\pm}$ as the rates of propagation of the *fastest* (polynomially small) part of the wave packet. Indeed, if $S_\psi(\alpha) = 0$ then roughly $P_\psi(\alpha, T) \approx T^{-S_\psi(\alpha)} = \mathcal{O}(1)$: most of the wave-packet escapes the ball of radius T^α, meaning that the essential part of the wave-packet travels faster than T^α; while if $S_\psi(\alpha) = +\infty$ then $P_\psi(\alpha, T) = \mathcal{O}(T^{-\infty})$: parts of the wave-packet that may escape from the ball of radius T^α are negligeable.

One proves

Theorem 4.1 ([GKT]). *Assume that $\beta^+(\infty, \psi) \leq \xi$ (for example, under assumptions of Proposition 1.1, $\xi = 1$). Then $0 \leq \alpha_l^{\pm} \leq \alpha_u^{\pm} \leq \xi$, and*

$$\frac{1}{p} (S_\psi^{\pm})^\sharp(p) \leq \beta^{\pm}(p, \psi) \leq \inf_{\alpha \in (\alpha_l^{\pm}, \alpha_u^{\pm})} \max \left(\alpha, \alpha_u^{\pm} - \frac{S_\psi^{\pm}(\alpha)}{p} \right). \tag{4.4}$$

where $g^\sharp$ denotes the Legendre transform of g: $g^\sharp(p) = \sup_\alpha (p\alpha - g(\alpha))$. As a consequence,

$$\beta^{\pm}(0 + 0, \psi) = \alpha_l^{\pm}, \quad \beta^{\pm}(+\infty, \psi) = \alpha_u^{\pm}. \tag{4.5}$$

5 A lower bound in dimension 1

In this section, we assume the potential V to be polynomially bounded: there exists $a, b > 0$ such that

$$|V(x)| \leq a \langle x \rangle^b, \tag{5.1}$$

for all $x \in \mathbb{Z}^+$ in the discrete case, and $x \in \mathbb{R}^+$ in the continuous case.

For a given operator H on $\ell^2([1, +\infty))$, resp. $\mathrm{L}^2([0, +\infty))$, we define the transfer matrices $T(E, x, y)$ between sites y and x as:

$$T(E, x, y) = \begin{pmatrix} u_0(E, x+1) & u_{\pi/2}(E, x+1) \\ u_0(E, x) & u_{\pi/2}(E, x) \end{pmatrix}, \quad \text{resp.} \quad \begin{pmatrix} u_0'(E, x) & u_{\pi/2}'(E, x) \\ u_0(E, x) & u_{\pi/2}(E, x) \end{pmatrix}, \tag{5.2}$$

where $u_\theta(E,x)$ denotes the solution of $Hu = Eu$, $E \in \mathbb{R}$, satisfying $u_\theta(E,y) = \sin\theta$, $u_\theta(E,y+1) = \cos\theta$, resp. $u_\theta(E,y) = \sin\theta$, $u'_\theta(E,y) = \cos\theta$ (note that $T(E,x,x) = \mathrm{Id}$). It follows from the definitions that if u is a solution of the eigenvalue equation $Hu = Eu$, $E \in \mathbb{R}$, then

$$\begin{pmatrix} u(E,x+1) \\ u(E,x) \end{pmatrix} = T(E,x,y)\begin{pmatrix} u(E,y+1) \\ u(E,y) \end{pmatrix}, \quad \text{resp.} \quad \begin{pmatrix} u'(E,x) \\ u(E,x) \end{pmatrix} = T(E,x,y)\begin{pmatrix} u'(E,y) \\ u(E,y) \end{pmatrix}. \tag{5.3}$$

Boundedness of transfer matrices is known to be related to a.c. spectrum [Si1]. By exploiting this fact in a spirit close to [CM], we show that on a spectral scale ε (i.e. on a time scale $T = \varepsilon^{-1}$), it is enough to know the behaviour of the transfer matrices up to a length scale $N = \varepsilon^{-(1+\nu)}$, $\nu > 0$, in order to get a useful control on the spectral measure.

Proposition 5.1 ([GKT]). *Let $H = -\Delta + V$ where V satisfies (5.1), and in addition (1.15)-(1.16) in the continuous case. Let $f \in \mathcal{C}_c^\infty(\mathbb{R})$, $\psi = f(H)\chi_0$, and let I be a compact interval. There exist a universal constant C_1 and for all $M > 0$ and $\sigma > 0$, a constant $C_{I,M,\sigma,a,b}$, $i = 1,2$, such that for all $\varepsilon \in]0,1[$ and all $\lambda \in I$, one has (setting $N = [\varepsilon^{-1-\sigma}]$ is the discrete case and $N = \varepsilon^{-1-\sigma}$ in the continuum)*

$$\mu_{\psi_0}(\lambda-\varepsilon,\lambda+\varepsilon) \geq C_1 \int_{\lambda-\frac{\varepsilon}{2}}^{\lambda+\frac{\varepsilon}{2}} \frac{k(E)\,\mathrm{d}E}{\|T(E,N,0)\|^2} - C_{I,M,\sigma,a,b}\,\varepsilon^M \; ; \tag{5.4}$$

$k(E)$ is a finite constant, positive for Lebesgue a.e. E, given by

$$\begin{cases} k(E) = 1 & \text{on } \mathcal{H} = \ell^2([1,+\infty)) \\ k(E) = K_{\Theta_1,\Theta_2}\frac{|\langle u_0(E),\chi_0\rangle|^2}{1+|E|} & \text{on } \mathcal{H} = \mathrm{L}^2([0,+\infty)), \end{cases} \tag{5.5}$$

where the constant $K_{\Theta_1,\Theta_2} > 0$ depends only on Θ_1, Θ_2.

Corollary 5.1. *Assume that $|\mathrm{supp}\mu_\psi| > 0$ and that for some $\gamma < \infty$, and for all $E \in \mathrm{supp}\mu$: $\|T(E,N,0)\| \leq C(E)N^\gamma$, where $C(E)$ is positive and finite for* Lebesgue *almost all $E \in \mathrm{supp}\mu_\psi$. Then $g^+_{\mu_\psi} \leq 1 + 2\gamma$. As a consequence,*
(i) $\mathbb{M}(p,f,T) \geq C_p T^{p-2\gamma}$ for all T, $\beta^-(p,\psi) \geq 1 - \frac{2\gamma}{p}$.
(ii) $\alpha_u^- = \beta^-(\infty,\psi) = 1$

Application of this corollary to sparse potential models and random decaying potentials are given in [GKT]. Examples include models with pure point spectrum, models with a spectral transition pp / sc spectrum, models for which the nature of the spectrum is not known. However it is not the application we would like to put the accent on in this introductive review. While Corollary 5.1 requires some strong and uniform statement on the polynomial behaviour of the $\|T(E,N,0)\|$'s, one can see from Proposition 5.1 that, at a given scale ε, it is enough to get a polynomial behaviour of $\|T(E,N,0)\|$ on a scale of order $N \approx \frac{1}{\varepsilon}$, and not for *all* N. Moreover, it is enough to get such informations on $\|T(E,N,0)\|$ on sets of positive Lebesgue measure that may tend to zero as ε does. Both these possibilities are contained in the following general statement. And both these possibilities will be useful to re-visit a by now well-known "pathological" example that is due to Last [La] and Del Rio, Jitomirskaya, Last, Simon [DRJLS].

Theorem 5.1 ([GKT]). *Let H be as in Proposition 5.1. Let S be a set of positive Lebesgue measure. Pick $f \in \mathcal{C}^\infty_{c,+}(\mathbb{R})$, $f \geq 0$ and $f = 1$ on S. For any $q \in (0,1)$ and $\sigma > 0$ there exists constants $C_q > 0$ and $C_{q,f,\sigma,a,b} < \infty$ (depending only on the indicated parameters) such that for all $\varepsilon \in (0,1)$,*

$$I_{\mu_{f(H)\psi_0}}(q,\varepsilon) \geq C_q\, \varepsilon^{q-1} \int_S \frac{k(E)\,\mathrm{d}E}{\|T(E,N,0)\|^{2q}} - C_{q,f,\sigma,a,b}\, \varepsilon \ , \tag{5.6}$$

where $N = [\varepsilon^{-(1+\sigma)}]$ in the discrete case and $N = \varepsilon^{-(1+\sigma)}$ in the continuous case, and where $k(E)$ is given in Theorem 5.1 Eq. (5.5). As a consequence, for any $p > 0$ and $T > 0$,

$$\mathbb{M}(p,f,T) \geq C_p\, T^p \left(\frac{1}{\log T} \int_S \frac{k(E)\,\mathrm{d}E}{\|T(E,N,0)\|^{\frac{2}{p+1}}} \right)^{p+1} - C_{p,f,\sigma,a,b} \ , \tag{5.7}$$

with $N = [T^{1+\sigma}]$ in the discrete case and $N = T^{1+\sigma}$ in the continuous case. The constants $C_p > 0$ and $C_{p,f,\sigma,a,b} < \infty$ in (5.7) depend also on Θ_1, Θ_2 in the continuous case.

We turn to the following application:

$$H_{\theta,\alpha,\lambda} = -\Delta + \Lambda \cos(\pi\alpha n + \theta) + \lambda(\delta_1,\cdot)\delta_1 \,. \tag{5.8}$$

Here we take α irrational and $\Lambda > 2$ so that the Lyapunov exponent is positive everywhere: as a consequence the spectrum $H_{\theta,\alpha,0}$ is purely singular for a.e. θ [He, CFKS], and so is the one of $H_{\theta,\alpha,\lambda}$ for all given λ [RS].

The result is the following

Theorem 5.2 ([GKT]). *There exists a dense G_δ set of irrationals Ω such that for any $\alpha \in \Omega$, for all $\theta \in [0, 2\pi[$ and $\lambda \in [0,1]$, for any $q \in (0,1)$, there exist a constant C_q and a sequence $\varepsilon_k \to 0$ such that*

$$I_{\mu_{\delta_1}}(q,\varepsilon_k) \geq \frac{C_q}{\log |\varepsilon_k^{-1}|} \left(\frac{1}{\varepsilon_k}\right)^{1-q} . \tag{5.9}$$

As a consequence $D^+_{\mu_{\delta_1}}(q) = 1$ for $q \in (0,1)$, and thus $\beta^+(p,\delta_1) = 1$.

To prove Theorem 5.2 we proceed by induction as in [La, DRJLS], and rely on Theorem 5.1. One constructs periodic approximants of $H_{\theta,\alpha,\lambda}$. The spectrum of these approximants operators is then a.c. and transfer matrices are bounded. By perturbation, it implies the following for $H_{\theta,\alpha,\lambda}$: Fix $\sigma > 0$, then there exists a sequence of $\varepsilon_k \in]0,1[$ and of bounded sets S_k such that

$$\sup_{E \in S_k} \ \sup_{N \leq (\frac{1}{\varepsilon_k})^{1+\sigma}} \|T_{\theta,\alpha,\lambda}(E,N,0)\|^2 \leq \log |\varepsilon_k^{-1}| \,, \tag{5.10}$$

and

$$|S_k| \geq \frac{1}{\log |\varepsilon_k^{-1}|} \,. \tag{5.11}$$

The result follows from Theorem 5.1.

Acknowledgements: For so many enjoyable discussions, it is a great pleasure to thank S. Tcheremchantsev, as well as my two other my co-authors on this subject matter J.M. Barbaroux and A. Kiselev. I also would like to thank warmly J.M. Combes, S. De Bièvre and A. Klein for their support.

References

[BCM] J.-M. Barbaroux, J. M. Combes, R. Montcho, *Remarks on the relation between quantum dynamics and fractal spectra,* J. Math. Anal. Appl. **213** , 698-722 (1997)

[BGT1] J.-M. Barbaroux, F. Germinet, S. Tcheremchantsev, *Fractal dimensions and the phenomenon of intermittency in quantum dynamics,* Duke Math. J. **110**, 161-193 (2001).

[BGT2] J.-M. Barbaroux, F. Germinet, S. Tcheremchantsev, *Quantum diffusion and generalized fractal dimensions: the* $\mathrm{L}^2(\mathbb{R}^d)$ *case*, Actes des journées EDP de Nantes (2000).

[BGT3] J.-M. Barbaroux, F. Germinet, S. Tcheremchantsev, *Generalized fractal dimensions: equivalence and basic properties,* J. Math. Pure et Appl. **80**, 977-1012 (2001)

[BT] J.-M. Barbaroux, S. Tcheremchantsev, *Universal lower bounds for quantum diffusion,* J. Funct. Anal. **168**, 327-354 (1999).

[BGSB] J. Bellissard, I. Guarneri, H. Schulz-Baldes, *Phase-averaged transport for quasi-periodic Hamiltonians*, Commun. Math. Phys. **227**, 515-539 (2002).

[C] J.-M. Combes, *Connection between quantum dynamics and spectral properties of time evolution operators,* in "Differential Equations and Applications in Mathematical Physics", Eds. W.F. Ames, E.M. Harrel, J.V. Herod (Academic Press 1993), 59-69.

[CM] J.M. Combes, G. Mantica, *Fractal Dimensions and Quantum Evolution Associated with Sparse Potential Jacobi Matrices,* Long time behaviour of classical and quantum systems (Bologna, 1999), 107-123, Ser. Concr. Appl. Math., **1**, World Sci. Publishing, River Edge, NJ, 2001.

[CFKS] H. Cycon, R. Froese, W. Kirsch and B. Simon, *Schrödinger Operators,* Springer-Verlag, 1987.

[DT] D. Damanik and S. Tcheremchantsev, *Power-law bounds on transfer matrices and quantum dynamics in one dimension,* Commun. Math. Phys. **236**, 513-534 (2003).

[DST] D. Damanik, A. Suto, S. Tcheremchantsev, *Power-law Bounds on Transfer Matrices and Quantum Dynamics in One Dimension II,* Preprint.

[DBF] S. De Bièvre, G. Forni, *Transport properties of kicked and quasiperiodic Hamiltonians*, J. Statist. Phys. **90**, 1201-1223 (1998).

[DRJLS] R. Del Rio, S. Jitomirskaya, Y. Last, and B. Simon, *Operators with singular continuous spectrum. IV. Hausdorff dimensions, rank one perturbations, and localization,* J. Anal. Math. **69**, 153-200 (1996).

[GKT] F. Germinet, A. Kiselev, S. Tcheremchantsev, *Transfer matrices and transport for 1D Schrödinger operators with singular spectrum*, Preprint.

[GK] F. Germinet, A. Klein, *A characterization of the Anderson metal-insulator transport transition*, Preprint.

[GT] F. Germinet, S. Tcheremchantsev, *Generalized fractal dimensions on the negative axis for compactly supported measures*, Preprint.

[G] I. Guarneri, *Spectral properties of quantum diffusion on discrete lattices,* Europhys. Lett. **10** (1989), 95-100; *On an estimate concerning quantum diffusion in the presence of a fractal spectrum,* Europhys. Lett. **21**, 729-733 (1993).

[GSB1] I. Guarneri and H. Schulz-Baldes, *Lower bounds on wave packet propagation by packing dimensions of spectral measures,* Math. Phys. Elec. J. **5**, paper 1, (1999).

[GSB2] I. Guarneri and H. Schulz-Baldes, *Intermittent lower bound on quantum diffusion,* Lett. Math. Phys. **49** (1999), 317-324.

[He] M. Hermann, *Une méthode pour minorer les exposants de Lyapunov et quelques exemples montrant le caractère local d'un théorème d'Arnod et de Moser sur le tore en dimension 2,* Comment. Math. Helv. **58**, 453-502 (1983).

[JSBS] S. Jitomirskaya, H. Schulz-Baldes, G. Stolz, *Delocalization in random polymer models* Commun. Math. Phys. **233**, 27-48 (2003).

[KL] A. Kiselev and Y. Last, *Solutions, spectrum, and dynamics for Schrödinger operators on infinite domains,* Duke Math. J. **102**, 125-150 (2000).

[KKS] A. Klein, A. Koines, M. Seifert, *Generalized Eigenfunctions for Waves in Inhomogeneous Media,* J. Funct. Anal **190**, 255-291 (2002).

[La] Y. Last, *Quantum dynamics and decomposition of singular continuous spectrum,* J. Funct. Anal. **142**, 406-445 (1996).

[Ma] G. Mantica, *Quantum intermittency in almost periodic systems derived from their spectral properties*, Physica D **103**, 576-589 (1997); *Wave propagation in almost-periodic structures*, Physica D **109**, 113-127 (1997).

[Ol] L. Olsen, *A multifractal formalism*, Adv. Math. **116**, 82-195 (1995).

[Pe] Y. Pesin, *Dimension Theory in Dynamical Systems: Contemporary Views and Applications*, Univ. Chicago Press, (1996).

[RS] M. Reed, B. Simon, *Methods of Modern Mathematical Physics III, Scattering theory,* Academic London 1979.

[Re] A. Rényi, *Probability Theory*, Amsterdam (1970); *Calcul des Probabilités, avec un appendice sur la théorie de l'information*, Dunod 1966.

[SBB] H. Schulz-Baldes, J. Bellissard, *Anomalous transport: a mathematical framework*, Rev. Math. Phys. **10**, 1-46 (1998).

[Si1] B. Simon, *Bounded eigenfunctions and absolutely continuous spectra for one-dimensional Schrödinger operators,* Proc. AMS **124** (1996), 3361-3369.

[St] R. Strichartz, *Fourier asymptotics of fractal measures.* J. Funct. Anal. **89**, 154–187 (1990).

[TV] F. Takens, E. Verbitski, *Multifractal analysis of local entropies for expansive homeomorphisms with specification,* Commun. Math. Phys. **203**, 593-612 (1999).

[Tc1] S. Tcheremchantsev, *Mixed lower bounds in quantum transport,* J. Funct. Anal. **197** (2003), 247-282.

[Tc2] S. Tcheremchantsev, *Dynamical analysis of Schrödinger operators with sparse potentials*, in preparation.