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Pseudospectrum for differential operators

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Résumé. Dans cet exposé nous décrivons des résultats obtenus avec N. Dencker et M. Zworski sur le comportement du pseudospectre des opérateurs différentiels dans la limite semiclassique. Ces résultats sont des adaptations de résultats connus dans d'autres domaines des équations aux dérivées partielles et de l'analyse semiclassique. Nous décrivons aussi un résultat récent avec M. Hager sur l'asymptotique de Weyl des valeurs propres de certains opérateurs perturbés.

0. Introduction.

Non-selfadjoint operators appear naturally in many problems:

Scattering theory and the study of scattering poles (resonances),

The damped wave equation,

Other problems, like linearizations of certain non-linear problems, Fokker-Planck equations.

A typical difficulty is that the norm of the resolvent may be large even far from the spectrum, and hence violating the estimate $\|(z-P)^{-1}\| \leq (\mathrm{dist}\,(z, \sigma(P)))^{-1}$ which holds when P is a selfadjoint (or more generally normal) operator, acting in some complex Hilbert space. Clearly this is a substantial difficulty for instance if we want to study the operator e^{-itP}, and while we may still believe that the spectrum of P is the most relevant quantity for the asymptotics when $t \to \infty$, it is far from clear how to study these evolution operators when $|t|$ is very large but confined to some bounded interval.

In recent years there has been a trend pursued by people like N. Trefethen [Tr], E.B. Davies [Da2,Da] and M. Zworski [Zw], to recognize the region where the resolvent is large as a set of interest in its own right. This set, called somewhat vaguely the pseudospectrum, might even be more important than the spectrum in certain situations.

In this talk we first review (in Section 1) some general facts about the pseudospectrum, following a recent survey of E.B. Davies [Da]. The main part of this talk (Section 2) is a description of some results from a joint paper with N. Dencker and M. Zworski [DeSjZw]. We conclude (in Section 3) by describing a recent result with M. Hager about Weyl asymptotics for certain perturbed operators in dimension 1.

1. Pseudospectrum

In this section we follow basically the survey [Da] and refer to that work for further references. Let $\mathcal{H}$ be a complex Hilbert space and let $A : \mathcal{H} \to \mathcal{H}$ be a closed densely defined operator. Let $\rho(A) \subset \mathcal{H}$ be the resolvent set and let $\sigma(A) = \mathbf{C} \setminus \rho(A)$ be the spectrum of A. Recall that if A is selfadjoint or more generally normal, we have

$$\|(z-A)^{-1}\| \leq \frac{1}{\mathrm{dist}\,(z, \sigma(A))}, \ z \in \rho(A), \tag{1.1}$$

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where dist denotes the standard distance on the complex plane.

Definition 1.1. Let $\epsilon > 0$. Then the ϵ-pseudospectrum is defined by

$$\sigma_\epsilon(A) = \sigma(A) \cup \{z \in \rho(A); \; \|(z - A)^{-1}\| \geq \frac{1}{\epsilon}\}. \tag{1.2}$$

Notice that unlike the spectrum itself, the pseudospectrum is not independent of the choice of norm on $\mathcal{H}$; if we replace the given norm on $\mathcal{H}$ by some equivalent norm, then in general, $\sigma_\epsilon(A)$ changes. We know of course (for instance from the theory of resonances) that the choice of norm is important for many problems, and the modification of norms is usually done in such a way that the pseudospectrum decreases in some given region of interest. This idea is used in the proof of Theorem 2.3 below.

The pseudospectrum is related to spectral instability:

Theorem 1.2. *(Roch–Silberman)*

$$\sigma_\epsilon(A) = \overline{\bigcup_{B \in \mathcal{L}(\mathcal{H}); \; \|B\| \leq \epsilon} \sigma(A + B)}.$$

We can also consider the numerical range

$$\mathrm{Num}\,(A) = \overline{\{(Au|u); \; u \in \mathcal{D}(A), \; \|u\| = 1\}}.$$

Here $\mathcal{D}(A)$ is the domain of A and $(u|v)$ denotes the scalar product on $\mathcal{H}$. This is a convex set as shown by Davies. If $\mathbf{C} \setminus \mathrm{Num}\,(A)$ is connected and has a non-empty intersection with the resolvent set, then

$$\sigma_\epsilon(A) \subset \{z \in \mathbf{C}; \; \mathrm{dist}\,(z, \mathrm{Num}\,(A)) \leq \epsilon\}.$$

Example 1. Let

$$A_\delta = \begin{pmatrix} 0 & 1 & 0 & .. & .. & 0 \\ 0 & 0 & 1 & 0 & .. & 0 \\ .. & .. & .. & .. & .. & .. \\ .. & .. & .. & .. & .. & .. \\ 0 & .. & .. & .. & 0 & 1 \\ \delta & 0 & .. & .. & 0 & 0 \end{pmatrix} : \mathbf{C}^n \to \mathbf{C}^n,$$

with $n \gg 1$. Then $\sigma(A_0) = \{0\}$ and we check that for $z \neq 0$, we have $|z|^{-n-1} \leq \|(z - A_0)^{-1}\|$, while for $|z| > 1$: $\|(z - A)^{-1}\| \leq (|z| - 1)^{-1}$. This implies that for $0 < \epsilon < 1$:

$$D(0, \epsilon^{1/(n+1)}) \subset \sigma_\epsilon(A_0) \subset \overline{D(0, 1 + \epsilon)}, \tag{1.3}$$

where $D(z, r)$ denotes the open disc of center z_0 and radius r. We also verify that

$$D(0, \frac{n-1}{n}) \subset \mathrm{Num}\,(A_0) \subset \overline{D(0, 1)}. \tag{1.4}$$

Now switch on the perturbation. An easy calculation gives

$$\sigma(A_\delta) = \{\delta^{1/n} e^{ij2\pi/n}; \ j = 0, 1, ..., n-1\}.$$

If $\delta > 0$ is fixed, we see that $\sigma(A_\delta)$ is in an arbitrarily small neighborhood of the boundary of $\sigma_\epsilon(A_\delta)$, if we first choose $\epsilon > 0$ small enough and then let $n \to \infty$.

Example 2. The non-selfadjoint harmonic oscillator has recently been explored by Davies and L. Boulton. Let $c \in \mathbf{C}$ with $\operatorname{Re} c, \operatorname{Im} c > 0$. Consider

$$Hu(x) = -u''(x) + cx^2 u(x) \text{ on } L^2(\mathbf{R}),$$

with domain $\mathcal{D}(\mathcal{H}) = \{u \in L^2(\mathbf{R}); \ u'', x^2 u \in L^2\}$. It is well-known, and easy to check using complex scaling, that

$$\sigma(H) = \{(2k+1)c^{1/2}; \ k = 0, 1, 2, ...\}.$$

Boulton showed, using the uncertainty relation, that

$$\operatorname{Num}(H) = \{s + tc; \ s, t > 0, \ st \geq \frac{1}{4}\},$$

and that

$$\|(z - H)^{-1}\| = \mathcal{O}(1), \text{ when } \operatorname{Re} z \to +\infty, \ |\operatorname{Im} z| \leq \mathcal{O}(1).$$

Davies showed that $\|(z - H)^{-1}\| \to \infty$ if $z = re^{i\theta}$ and $\theta \in]0, \arg c[$ is fixed, while $r \to +\infty$. The last result was complemented by Boulton, who obtained the same conclusion for $\operatorname{Im} z = (\operatorname{Re} z)^\alpha$ with $\operatorname{Re} z \to +\infty$ and α fixed in the interval $]1/3, 1]$. Numerically, Boulton and M. Embree have found that the exponent $1/3$ is optimal and a preliminary result of N. Kaidi indicates that this exponent is indeed optimal. Notice that the study of this operator can be viewed as a semiclassical problem, since

$$H - z = |z|(-(h\partial_y)^2 + cy^2 - \frac{z}{|z|}), \text{ with } h = \frac{1}{|z|}, \ x = |z|^{1/2} y.$$

2. Results in the semiclassical case.

This is the main section of this exposé and we here describe some results from the joint work with N. Dencker and M. Zworski [DeSjZw].

a. Quasi-modes and pseudospectrum. Let $1 \leq m(X) \in C^\infty(\mathbf{R}^{2n})$ be a weight function satisfying

$$m(X) \leq C_0 \langle x - y \rangle^{N_0} m(Y), \ X, Y \in \mathbf{R}^{2n}.$$

We will consider in parallel the C^∞-case and the analytic case with the following more precise assumptions:

C^∞-*case.* We let $p \in S(m)$, i.e. we assume that $p \in C^\infty(\mathbf{R}^{2n})$ and that

$$\partial_X^\alpha p(X) = \mathcal{O}_\alpha(m(X)). \tag{2.1}$$

Analytic case. We assume that p is holomorphic in a tubular neighborhood of $\mathbf{R}^{2n}$ in $\mathbf{C}^{2n}$ satisfying: $p(X) = \mathcal{O}(m(\mathrm{Re}\,X))$. By the Cauchy inequalities, we see that (the restriction of p to $\mathbf{R}^{2n}$) belongs to $S(m)$.

If $h > 0$ is small enough, we can consider $P = p^w(x, hD)$ as a closed densely defined operator: $L^2 \to L^2$ whose domain is equal to $H(m)$, the naturally defined Sobolev space associated to the weight function m.

Theorem 2.1. *([DeSjZw]) Consider the C^∞-case. Let $(x_0, \xi_0) \in \mathbf{R}^{2n}$ and assume that*

$$\frac{1}{i}\{p, \overline{p}\}(x_0, \xi_0) > 0. \tag{2.2}$$

Then there exists a family $u = u_h \in \mathcal{S}(\mathbf{R}^n)$ with $\|u_h\|_{L^2} = 1$, $0 < h \le h_0 > 0$, such that

$$\|(P - z)u\| = \mathcal{O}(h^\infty), \tag{2.3}$$

where $z = p(x_0, \xi_0)$. Moreover, $WF_h(u) = \{(x_0, \xi_0)\}$, where $WF_h(u)$ denotes the frequency set. $\{f, g\} = f'_\xi \cdot g'_x - f'_x \cdot g'_\xi$ is the Poisson bracket.

Notice that this result is an adaptation to the semiclassical frame-work of a classical non-hypoellipticity result of Hörmander subsequent to the classical work of H. Lewy on non-solvable operators (see [Hö,Hö2]).

Example 3. Let $P = -h^2\Delta + V(x)$ be a non-selfadjoint Schrödinger operator on $\mathbf{R}^n$ with smooth potential. Then $p(x, \xi) = \xi^2 + V(x)$ and $\frac{1}{i}\{p, \overline{p}\} = -4\xi \cdot \mathrm{Im}\,V'(x)$. For such operators in dimension 1, Davies [Da2] proved the above theorem, without any explicit reference to the Poisson bracket. Zworski [Zw], observed the link between the two results.

The proof of the theorem is also an easy adaptation of that of Hörmander's theorem. We try $u_h(x) = h^{-n/4}a(x; h)e^{i\phi(x)/h}$, where $a(x; h) \sim \sum_{\nu=0}^\infty a_\nu(x)h^\nu$, $a_0(x_0) \ne 0$, $p(x, \phi'(x)) - z = \mathcal{O}(|x - x_0|^\infty)$. This can be done locally in a neighborhood of x_0 in such a way that $\phi'(x_0) = \xi_0$ and $\mathrm{Im}\,\phi''(x_0) > 0$, and after we have found ϕ we construct a_0 by solving a sequence of transport equations to ∞-order at $x = x_0$.   #

Theorem 2.2. *([DeSjZW]) In the analytic case we have the same conclusion with $\mathcal{O}(h^\infty)$ replaced $\mathcal{O}(e^{-1/(C_0 h)})$ for some fixed $C_0 > 0$, and with $WF_h(u)$ replaced by the corresponding analytic frequency set.*

This is an adaptation of a classical result of Sato–Kawai–Kashiwara [SaKaKa]. A similar result for finite difference operators on the circle was recently obtained by Trefethen and S.J. Chapman [TrCh].

Notice that the sets of points z in the theorems are open. The last theorem can be applied to the non-selfadjoint harmonic oscillator and provides an improvement of the result of Davies that we mentioned earlier. Indeed, it suffices to apply the reduction to a semiclassical situation explained at the end of the previous section, and we see that we have a family of functions $u_z \in \mathcal{S}(\mathbf{R})$ with $\|u_z\| = 1$ such that

$$(H - z)u_z = \mathcal{O}(e^{-|z|/C_0}), \text{ for } \arg z \in]\epsilon_0, \arg a - \epsilon_0[,$$

for every fixed $\epsilon_0 > 0$ with a $C_0 > 0$ that depends on ϵ_0.

In [DeSjZw] we also give some elementary geometrical and topological arguments which show that the assumptions of our theorems are fulfilled "most of the time" and for "most values" of $z \in p(\mathbf{R}^{2n})$. Notice however that if $z \in \partial p(\mathbf{R}^{2n})$, then the assumptions are not fulfilled. We will in the following slightly abuse our terminology by calling $\Sigma(p) := \overline{p(\mathbf{R}^{2n})}$, the pseudospectrum of P. This is justified by the observation that if $z_0 \in \mathbf{C}$ and dist $(z_0, \Sigma(p)) < \epsilon$, then z_0 belongs to $\sigma_\epsilon(P)$ if $h > 0$ is small enough and if dist $(z_0, \Sigma(p)) > \epsilon$, and the assumption (2.4) below holds, then $z_0 \notin \sigma_\epsilon(P)$ when h is small enough.

b) Study of boundary points of the pseudospectrum. Let $z_0 \in \partial\Sigma(p)$ and assume that

$$|p(x, \xi) - z_0| > \frac{m(x, \xi)}{C}, \text{ for } |(x, \xi)| \geq C, \tag{2.4}$$

for some $C > 0$. Notice that if $z \in \text{neigh}(z_0)$, then $(z - P)^{-1}$ exists with $\|(z - P)^{-1}\|$ bounded when $h \to 0$, if and only if $z \notin \Sigma(p)$.

We also assume

$$\text{If } \rho \in \mathbf{R}^{2n} \text{ and } p(\rho) = z_0, \text{ then } dp(\rho) \neq 0, \tag{2.5}$$

$$\exists \epsilon_0 > 0, \theta_0 \in \mathbf{R}, \text{ such that } (z_0+]0, \epsilon_0[e^{i]\theta_0-\epsilon_0,\theta_0+\epsilon_0[}) \cap \Sigma(p) = \emptyset. \tag{2.6}$$

If we put $q = ie^{-i\theta_0}(p - z_0)$, then $d\text{Re}\,q \neq 0$ on $p^{-1}(z_0)$. Assume

No H_q-trajectory can remain in $p^{-1}(z_0)$ during an unbounded interval of time.
$$\tag{2.7}$$

Theorem 2.3 *([DeSjZw]). We make the assumptions above.*
a) In the C^∞-case, we have for every $C > 0$ that

$$D(z_0, Ch\log\frac{1}{h}) \cap \sigma(P) = \emptyset \text{ for } 0 < h \leq h_0(C) > 0.$$

b) In the analytic case, there exist $C_0 > 0$, $h_0 > 0$ such that

$$D(z_0, \frac{1}{C_0}) \cap \sigma(P) = \emptyset, \ 0 < h \leq h_0.$$

This result is analogous to results about absence of resonances when there are no trapped trajectories, due to Helffer-Sjöstrand (implicit in [HeSj]) in the analytic case and more recently to A. Martinez [Ma] in the smooth case . The idea of the proof is similar, namely to modify the norm by means of a weight $e^{\epsilon G(x,\xi)/h}$, where (in our case) $G \in C_0^\infty$ with $H_{\text{Re}\,q}G > 0$ on $q^{-1}(0)$. In the analytic case we can take $\epsilon > 0$ fixed and small and in the C^∞-case we choose ϵ of the order $h\log(1/h)$. In the latter case our proof is related to the approach to propagation of singularities by means of pseudodifferential operators of variable order that has been developed by J. and A.

Unterberger ([BoUn, Un]. The analytic case is also related to the propagation result for micro-hyperbolic operators of M. Kashiwara and T. Kawai [KaKa].

We shall next describe what happens if we replace (2.6), (2.7) by a basically stronger assumption. The assumptions (2.4), (2.5) remain. Let us say that $z_0 \in \partial\Lambda(p)$ is of finite type if for every $(x_0, \xi_0) \in p^{-1}(z_0)$ there exists $k \geq 1$ and $I \in \{1,2\}^k$ such that

$$p_I(x_0, \xi_0) \neq 0, \text{ where } p_I = H_{p_1} H_{p_2} \cdots H_{p_{i_{k-1}}} p_{i_k},$$

where $p_1 = \operatorname{Re} p$, $p_2 = \operatorname{Im} p$. The order of p at $w = (x_0, \xi_0)$ is then defined as $\max\{j \in \mathbf{N}; p_I(w) = 0, |I| \leq j\}$ and the order of z_0 is defined to be the maximum of the orders of p at all points $w \in p^{-1}(z_0)$.

Theorem 2.4 ([DeSjZw]) *We consider the C^∞-case and assume (2.4), (2.5). Also assume that z_0 is of finite type and of order $k \geq 1$. Then k is even and for $0 < h \leq h_0 > 0$ we have:*

$$\|(P - z_0)^{-1}\| \leq C h^{-\frac{k}{k+1}}. \tag{2.8}$$

In particular, $\exists C > 0$ such that

$$D(z_0, \frac{h^{\frac{k}{k+1}}}{C}) \cap \sigma(P)) = \emptyset, \ 0 < h \leq h_0. \tag{2.9}$$

The proof is an adaptation of Hörmander's treatment of subelliptic operators of principal type (see [Hö]).

Example 4. We consider again the complex harmonic oscillator and recall the scaling at the end of Section 1:

$$H - \zeta = -\partial_x^2 + cx^2 - \zeta = |\zeta|(-(h\partial_y)^2 + cy^2 - \frac{\zeta}{|\zeta|}), \ h = \frac{1}{|\zeta|}.$$

Choose $z_0 = 1$. Then Theorem 2.4 applies with $k = 2$ and we find

$$\|(H - \zeta)^{-1}\| \leq \frac{\mathcal{O}(1)}{|\zeta|}(\frac{1}{|\zeta|})^{-\frac{2}{3}} = \mathcal{O}(1)|\zeta|^{-\frac{1}{3}},$$

for $|\zeta/|\zeta| - 1| \leq \mathcal{O}(1)^{-1}|\zeta|^{-2/3}$, i.e. for $|\zeta - |\zeta|| \leq |\zeta|^{1/3}/\mathcal{O}(1)$. (Compare with one of the results of Boulton that we mentioned in Section 1.)

Example 5. Let P be the complex harmonic oscillator in 2 dimensions, given by

$$P = (hD_{x_1})^2 + x_1^2 + (hD_{x_2})^2 + ix_2^2, \ D_{x_j} = -i\partial_{x_j}.$$

Then $\sigma(P) = \{(h(2n+1 + e^{i\pi/4}(2k+1)); k, n \in \mathbf{N}$ and $\Sigma(p)$ is the first quadrant. The assumptions of Theorem 2.3 are fulfilled at the point $z = i$, but not at the point $z = 1$.

3. Spectral instability.

Zworski ([Zw2]) considered the non-selfadjoint operator

$$P = (hD_x)^2 + i(hD_x) + x^2 = e^{x/2h} \circ ((hD_x)^2 + x^2 + \frac{1}{4}) \circ e^{-x/2h}.$$

Using the second representation it is easy to show that the spectrum of P is equal to $\{\frac{1}{4} + (2k+1)h; \ k = 0, 1, 2, \ldots\}$. When doing numerical calculations however, the computer ignores that P is a conjugation of a selfadjoint operator and due to instability (due to large resolvent norm), it makes larger and larger errors when $h \to 0$. In [Zw2] the numerical computations give the correct eigenvalues when $h = 0, 1$ but for $h = 0, 01$ we get a cloud of eigenvalues in the complex domain. For $h = 0, 001$ the numerically calculated eigenvalues seem to fill up the boundary of the set $\Sigma(p) = \{z \in \mathbf{C}; \operatorname{Re} z \geq (\operatorname{Im} z)^2\}$.

One may interpret this result by saying that the computer calculates the spectrum of a small perturbation of P and we are then led to the general mathematical problem of describing what happens with the spectrum in some given bounded domain of the complex plane if we switch on a small perturbation of the operator.

We end this exposé by describing a very recent joint result with M. Hager [Ha] about the distribution of eigenvalues for suitable perturbations of a simple differential operator. This result does not show accumulation of most of the eigenvalues to the boundary of the pseudospectrum, but rather a nice distribution according to a natural Weyl law.

Let $g(x)$ be an analytic function on $S^1 = \mathbf{R}/2\pi\mathbf{Z}$ and assume that $\operatorname{Im} g(x)$ has only two critical points, the point $x_{\max}$ of maximum and the point $x_{\min}$ of minimum. We consider

$$hD_x + g(x) \text{ on } L^2(S^1), \tag{3.1}$$

with domain $H^1(S^1)$. The symbol of this operator is

$$p(x, \xi) = \xi + g(x), \tag{3.2}$$

and

$$\Sigma(p) = p(T^*S^1) = \{z \in \mathbf{C}; \operatorname{Im} g(x_{\min}) \leq \operatorname{Im} z \leq \operatorname{Im} g(x_{\max})\}.$$

For every $z \in \operatorname{int}(\Sigma(p))$ there exist exactly two points $\rho_\pm = (x_\pm, \xi_\pm) \in T^*(S^1)$ with $p(\rho_\pm) = z$ and the Poisson bracket takes opposite signs at the two points:

$$\pm \frac{1}{i}\{p, \bar{p}\}(\rho_\pm) > 0.$$

More explicitly $x_\pm$ is determined by the condition

$$\operatorname{Im} g(x_\pm(z)) = \operatorname{Im} z, \ \mp(\operatorname{Im} g)'(x_\pm(z)) > 0,$$

and

$$\xi_\pm(z) = \operatorname{Re}(z - g(x_\pm(z))).$$

A simple computation gives the spectrum of P:

$$\sigma(P) = \{\langle g \rangle + kh;\ k \in \mathbf{Z}\},\ \text{where } \langle g \rangle := \frac{1}{2\pi} \int_0^{2\pi} g(x)dx. \tag{3.3}$$

Now consider a perturbation

$$P_\delta = P + \delta Q.$$

Here

$$\delta = e^{-\epsilon/h}, \tag{3.4}$$

where $\epsilon > 0$ is small but independent of h, and Q is an integral operator with kernel

$$K_Q(x,y) = \sum_{j=1}^{N(\epsilon)} e^{i\phi_j(x,y)/h} \chi(x - x_j)\chi(y - y_j), \tag{3.5}$$

where $\chi \in C_0^\infty(]-\pi, \pi[)$ is equal to 1 near $x = 0$ and independent of ϵ. Further,

$$\phi_j(x,y) = (x - x_j)\xi_j + \frac{i}{2}(x - x_j)^2 - (y - y_j)\eta_j + \frac{i}{2}(y - y_j)^2,$$

where $(x_j, \xi_j; y_j, \eta_j) \in (T^*S^1)^2$. Here χ is independent of ϵ, while $(x_j, \xi_j; y_j, \eta_j)$ and the number $N(\epsilon)$ will depend on ϵ. (We identify the x, y-variables on S^1 with suitable lifts to $\mathbf{R}$ in such a way that Q becomes well defined on $L^2(S^1)$.)

In order to avoid accidental cancellations for the restriction to the diagonal of a certain FBI-transform of K_Q, we assume,

$$(x_j, \xi_j; y_j, \eta_j) \notin \Gamma(x_k, \xi_k; y_k, \eta_k),\ j \neq k, \tag{3.6}$$

where $\Gamma(x, \xi; y, \eta) \subset (T^*S^1)^2$ is a certain analytic submanifold of dimension 2 which depends analytically on $(x, \xi; y, \eta)$. Notice that this is a generic assumption.

Let $\gamma \subset \text{int}\,(\Sigma)$ be a simple closed C^1-loop with $\dot{\gamma} \neq 0$, and assume, for $\epsilon \in]0, \epsilon_0[$, for some $\epsilon_0 > 0$:

$$\forall z \in \gamma, \exists j \in \{1, 2, \ldots, N(\epsilon)\},\ \text{such that dist}\,((x_j, \xi_j; y_j, \eta_j), (\rho_-(z), \rho_+(z))) \leq \frac{\sqrt{\epsilon}}{C_0}, \tag{3.7}$$

where $C_0 > 0$ is sufficiently large but independent of ϵ.

Theorem 3.1 *(M. Hager, J. Sjöstrand) Under the above assumptions, the number of eigenvalues of P_δ in the interior of γ is equal to*

$$\frac{1}{2\pi h}\left(\text{Vol}\,\Omega_+(\gamma) + \text{Vol}\,\Omega_-(\gamma) + \mathcal{O}(\sqrt{\epsilon})\right),\ h \to 0,$$

for $h > 0$ small enough depending on ϵ. Here $\Omega_\pm(\gamma) \subset T^*S^1$ is the bounded domain whose boundary is given by the image of $\rho_\pm \circ \gamma$. The $\mathcal{O}$-term is uniform with respect to h.

It is clear from the proof that this result can and will be considerably generalized to more general unperturbed operators P and more general perturbations. This will be treated in [Ha].

References

[BoUn] J. Bokobza, A. Unterberger, *Les opérateurs pseudodifférentiels d'ordre variable,* C.R.A.S. Paris 261(1965), 2271–2273.

[Da] E. B. Davies, *Non-selfadjoint differential operators,* Bull. London Math. Soc. 34(5)(2002), 513–532.

[Da2] E. B. Davies, *Semi-classical states for non-self-adjoint Schrödinger operators,* Comm. Math. Phys., 200(1999), 35–41.

[DeSjZw] N. Dencker, J. Sjöstrand, M. Zworski, *Pseudospectra of semi-classical (pseudo)differential operators,* Comm. P. Appl. Math., to appear.

[Ha] M. Hager, Thèse, In preparation.

[HeSj] B. Helffer, J. Sjöstrand, *Résonances en limite semi-classique,* Mémoires de la S.M.F. 114(3)(1986).

[Hö] L. Hörmander, *The analysis of linear partial differential operators, I–IV,* Springer verlag 1983–1985.

[Hö2] L. Hörmander, *Differential operators of principal type,* Math. Ann. 140(1960), 124–146; *Differential equations without solutions,* Math. Ann. 140(1960), 169–173.

[KaKa] M. Kashiwara, T. Kawai, *Microhyperbolic pseudo-differential operators, I,* J. Math. Soc. Japan 27(1975), 359–404.

[Ma] A. Martinez, *Resonance free domains for non globally analytic potentials,* Ann. Henri Poincaré 4(2002), 739–756.

[SaKaKa] M. Sato, T. Kawai, M. Kashiwara, *Microfunctions and pseudo-differential equations. Hyperfunctions and pseudo-differential equations* pp. 265–529. Lecture Notes in Math., Vol. 287, Springer, Berlin, 1973.

[TrCh] L.N. Trefethen, S.J. Chapman, *Wave packet pseudomodes of twisted Toeplitz matrices,* preprint (2002).

[Un] A. Unterberger, *Résolution d'équations aux dérivées partielles dans des espaces de distributions d'ordre variable,* Ann. Inst. Fourier, 21(2)(1971), 85–128.

[Zw] M. Zworski, *A remark on a paper by E.B. Davies,* Proc. A.M.S., 129(2001), 2955-2957.

[Zw2] M. Zworski, *Numerical linear algebra and solvability of partial differential equations,* Comm. Math. Phys. 229(2002), 293–307.