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Boundary layers and time oscillations in rotating fluids

J.-Y. Chemin [*], B. Desjardins [†], I. Gallagher [‡], E. Grenier [§]

Introduction

We are interested in fast rotating viscous fluids between two horizontal plates with Dirichlet boundary conditions. More precisely, we shall study the limit when ε goes to 0 of the following system:

$$
(NSC_\varepsilon)\quad \left\{ \begin{array}{rcl} \partial_t u^\varepsilon + \operatorname{div}(u^\varepsilon \otimes u^\varepsilon) - \nu\Delta_h u^\varepsilon - \beta\varepsilon\partial_3^2 u^\varepsilon + \dfrac{e_3 \times u^\varepsilon}{\varepsilon} & = & -\nabla p^\varepsilon \\ \operatorname{div} u^\varepsilon & = & 0 \\ u^\varepsilon_{|\partial\Omega} & = & 0 \\ u^\varepsilon_{|t=0} & = & u_0 \in L^2(\Omega) \end{array} \right.
$$

where $\Omega = \mathbf{R}^2 \times]0,1[$. We shall use the following notation: if u is a vector on $\mathbf{R}^3$ we state $u = (u^1, u^2, u^3) = (u^h, u^3)$, and we will note $\Delta_h = \partial_1^2 + \partial_2^2$. Moreover, if f is a function on Ω, $\mathcal{F}f$ (and also $\widehat{f}$) will denote the Fourier transform with respect to the horizontal variable x_h.

These equations arise in physical contexts when one studies oceanic or atmospheric motions. Basically under high rotation, a three dimensional fluid tends to behave like a two dimensional one, and to become invariant in the direction of the rotation (see for instance the monographs [8],[11]). Moreover, keeping in mind this anisotropy, it is usual to consider an anisotropic viscosity like in (NSC_ε), the horizontal "turbulent" diffusion being larger than the vertical one. In the periodic case we refer to [1],[7],[9] for mathematical studies. For previous studies in the Dirichlet case, we refer to [6] and [10]. Our goal here is to present the results obtained in [5].

In the sequel, we shall assume that $u_0 \cdot n = u_0^3 = 0$ on $\partial\Omega$, and $\operatorname{div} u_0 = 0$. That implies that the vertical mean value of the horizontal part on the vector field is divergence free as a vector field on $\mathbf{R}^2$.

First of all we shall recall results concerning the linear problem in the well prepared case, which means that the function u_0 does not depend on the vertical variable x_3. Then we shall investigate the more delicate case when the initial data u_0 does actually depend on the vertical variable x_3. The result we shall present here is the following.

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Theorem 1 *Let u_0 be a divergence free vector field in L^2 such that $u^3_{0|\partial\Omega} = 0$. Let u^ε be a family of weak solutions of (NSC_ε) associated with u_0. Denoting by $\overline{u}$ the global solution of the two–dimensional Navier–Stokes equations*

$$\partial_t \overline{u} + \mathrm{div}_h(\overline{u}\otimes\overline{u}) - \nu\Delta_h \overline{u} + \nabla^h p + \sqrt{2\beta}\,\overline{u} = 0 \quad \textit{in} \quad \mathcal{D}'(\mathbf{R}^+\times\mathbf{R}^2),$$

$$\mathrm{div}_h\, \overline{u} = 0, \quad \textit{and} \quad \overline{u}_{|t=0} = \int_0^1 u_0(x_h, x_3)dx_3,$$

we have

$$\|u^\varepsilon - (\overline{u},0)\|_{L^\infty(\mathbf{R}_+;L^2_{loc}(\mathbf{R}^2\times]0,1[))} + \|\nabla^h\left(u^\varepsilon - (\overline{u},0)\right)\|_{L^2(\mathbf{R}_+;L^2_{loc}(\mathbf{R}^2\times]0,1[))} \to 0 \quad \textit{when} \quad \varepsilon\to 0.$$

This result is specific to the domain $\mathbf{R}^2\times]0,1[$. The key point for the proof of this theorem is that the dispersive phenomenon studied in [3] and [4] is not affected by boundary layers.

Before explaining the proof of that result, let us state some definitions and notations. As the phenomenon studied here is obviously anisotropic, it is natural to introduce the spaces $H^{s,0}$ which are defined as the closure of smooth compactly supported functions on Ω for the (semi) norm

$$\|f\|_{H^{s,0}} \stackrel{\text{def}}{=} \left(\int_{\mathbf{R}^2\times]0,1[} |\xi_h|^{2s}|\mathcal{F}f(\xi_h,x_3)|^2 d\xi_h dx_3\right)^{1/2}.$$

We also introduce

$$\begin{aligned} E_T(v) &\stackrel{\text{def}}{=} \sup_{0\le t\le T}\Big\{\|v(t)\|^2_{L^2} + \int_0^t \|\nabla^h v(t')\|^2_{L^2}dt'\Big\} \quad \text{and}\\ \overline{E}_T(v) &\stackrel{\text{def}}{=} \sup_{0\le t\le T}\Big\{\|v(t)\|^2_{L^2} + \int_0^t \|\nabla^h v(t')\|^2_{L^2}dt' + \int_0^t \|v(t')\|^2_{L^2}dt'\Big\}. \end{aligned}$$

1 Recollection of results in the well prepared case

Let us briefly recall some results of [10] concerning the well prepared case in the linearized situation. The purpose is to have information on approximate solutions of

$$(FRF^\varepsilon)\begin{cases} \partial_t v^\varepsilon - \nu\Delta_h v^\varepsilon - \beta\varepsilon\partial_3^2 v^\varepsilon + \dfrac{e_3\times v^\varepsilon}{\varepsilon} &= -\nabla p^\varepsilon + f^\varepsilon \quad \text{in} \quad \Omega\\ \mathrm{div}\, v^\varepsilon &= 0 \quad \text{in} \quad \Omega\\ v^\varepsilon_{|t=0} &= v^\varepsilon_0 \quad \text{with} \quad v_0^{\varepsilon,3} = 0\\ v^\varepsilon_{|\partial\Omega} &= 0 \end{cases}$$

with $\lim_{\varepsilon\to 0} f^\varepsilon = \overline{f}$ in $L^2(\mathbf{R}_+; H^{-1,0})$ and $\lim_{\varepsilon\to 0} v_0^\varepsilon = \overline{v}_0$ in L^2, where $\overline{f}\in L^2(\mathbf{R}^+; H^{-1}(\mathbf{R}^2))$. In all that follows, we shall denote

$$L^\varepsilon v \stackrel{\text{def}}{=} \partial_t v - \nu\Delta_h v - \beta\varepsilon\partial_3^2 v + \frac{e_3\times v}{\varepsilon}\cdot$$

In [10], it is proved in particular that

$$\lim_{\varepsilon\to 0} v^\varepsilon = (\overline{v},0) \quad \text{in} \quad L^\infty(\mathbf{R}^+;L^2)\cap L^2(\mathbf{R}^+;H^{1,0}),$$

where $\overline{v}$ is the solution of

$$(LE)\begin{cases} \partial_t \overline{v} - \nu\Delta_h \overline{v} + \sqrt{2\beta}\,\overline{v} &= -\nabla^h \overline{p} + \overline{f} \quad \text{in} \quad \mathbf{R}^+ \times \mathbf{R}^2 \\ \operatorname{div}_h \overline{v} &= 0 \quad \text{in} \quad \mathbf{R}^+ \times \mathbf{R}^2 \\ \overline{v}_{|t=0} &= \overline{v}_0 \quad \text{with} \quad \overline{v}_0^3 = 0. \end{cases}$$

The precise result is the following.

Lemma 1 *Let T be in $\overline{\mathbf{R}}^+$ and let $\overline{v}$ be a divergence free vector field in $L^\infty([0,T];L^2(\mathbf{R}^2))$ whose gradient belongs to $L^2([0,T]\times\mathbf{R}^2)$. Let us assume that its Fourier transform is supported in the ball of center 0 and radius N. Then a family of smooth divergence free vector fields $(v^\varepsilon_{app})_{\varepsilon>0}$ whose value is 0 on the boundary of Ω exists such that*

$$L^\varepsilon v^\varepsilon_{app} = \partial_t \overline{v} - \nu\Delta_h \overline{v} + \sqrt{2\beta}\,\overline{v} + O_N(\varepsilon^{\frac{1}{2}}) \quad in \quad L^2([0,T];H^{-1,0}).$$

The vector field v^ε_{app} goes to $\overline{v}$ in the following sense: a constant C_N exists such that

$$E_T(v^\varepsilon_{app} - \overline{v}) \leq C_N \varepsilon^{\frac{1}{2}} E_T(\overline{v}).$$

Moreover the family (v^ε_{app}) satisfies the following estimates

$$\begin{aligned} \int_0^T \sup_{x_3\in]0,1[} \|\nabla^h v^\varepsilon_{app}(t,\cdot,x_3)\|^2_{L^2(\mathbf{R}^2)} dt &\leq C_N E_T(\overline{v}) \\ \forall p\in[2,\infty]\,,\ \int_0^T\int_0^1 d(x_3)\|\partial_3 v^\varepsilon_{app}(t,\cdot,x_3)\|^2_{L^p(\mathbf{R}^2)} dt dx_3 &\leq C_N \overline{E}_T(\overline{v}) \quad and \\ \int_0^T\int_0^1 d(x_3)^2\|\partial_3 v^\varepsilon_{app}(t,\cdot,x_3)\|^2_{L^\infty(\mathbf{R}^2)} dt dx_3 &\leq C_N \varepsilon \overline{E}_T(\overline{v}) \end{aligned}$$

where $d(x_3)$ denotes the distance from x_3 to the boundary of $]0,1[$.

To simplify notations, we shall note in the following $\overline{v} = (\overline{v}, 0)$.

2 The linear problem in the ill prepared case

The goal of this section is to construct approximate solutions to

$$(FRF^\varepsilon)\begin{cases} \partial_t v^\varepsilon - \nu\Delta_h v^\varepsilon - \beta\varepsilon\partial_3^2 v^\varepsilon + \dfrac{e_3\times v^\varepsilon}{\varepsilon} &= -\nabla p^\varepsilon \quad \text{in} \quad \Omega \\ \operatorname{div} v^\varepsilon &= 0 \quad \text{in} \quad \Omega \\ v^\varepsilon_{|t=0} &= v_0 \\ v^\varepsilon_{|\partial\Omega} &= 0. \end{cases}$$

First of all, we shall rewrite the system (FRF^ε) in terms of the Fourier transform of the horizontal divergence and vorticity. To do so, let us decompose the horizontal part of the initial data on the Hilbert basis $(\cos(k\pi x_3))_{k\in\mathbf{N}}$ of $L^2(]0,1[)$. We can write for any horizontal vector field v^h,

$$v^h(x_h,x_3) = \sum_{k\in\mathbf{N}} v^{k,h}(x_h)\cos(k\pi x_3). \tag{2.1}$$

Note that the fact that v is divergence free implies that

$$v_0^3(x_h, x_3) = -\sum_{k\geq 1} \frac{1}{k\pi} \operatorname{div}_h v_0^{k,h}(x_h) \sin(k\pi x_3). \tag{2.2}$$

The choice of the basis $(\cos(k\pi x_3))_{k\in\mathbf{N}}$ for the horizontal component ensures that the boundary condition $v^3{}_{|\partial\Omega} = 0$ is satisfied. For reasons which will appear clearly when we deal with dispersive phenomena, we need to avoid extreme horizontal frequencies. So we approximate any divergence free vector field v of L^2 by

$$v_N \stackrel{\text{def}}{=} \mathcal{F}^{-1} \sum_{k=0}^{N} \Big(\mathbf{1}_{\mathcal{C}_N}(\xi_h) \widehat{v}^{k,h}(\xi_h) \cos(k\pi x_3), -\frac{1}{k\pi} \mathbf{1}_{\mathcal{C}_N}(\xi_h) \mathcal{F} \operatorname{div}_h v_0^{k,h}(\xi_h) \sin(k\pi x_3) \Big)$$

where $\mathcal{C}_N$ denotes the set of all ξ_h in $\mathbf{R}^2$ such that $|\xi_h| \in [N^{-1}, N]$. Let us define the following notation:

$$\widehat{d}^h \stackrel{\text{def}}{=} \mathcal{F} \operatorname{div}_h v^h\,, \quad \widehat{\omega}^h \stackrel{\text{def}}{=} \mathcal{F} \operatorname{curl}_h v^h\,, \quad \widehat{p} \stackrel{\text{def}}{=} \mathcal{F} p \quad \text{and} \quad W \stackrel{\text{def}}{=} (\widehat{d}^h, \widehat{\omega}^h, \mathcal{F} v^3).$$

For the sake of simplicity in the notation, we drop the ε in $\widehat{d}^h$, $\widehat{\omega}^h$, $\widehat{p}^h$ and W. In the following Π_1 denotes the projection on the first coordinate. Then for vectors of the type

$$\Big(W^{k,h}(t) \cos(k\pi x_3), -\frac{1}{k\pi} \Pi_1 W^k(t) \sin(k\pi x_3), \Big)$$

and pressures of the type $\widehat{p}^k \cos(k\pi x_3)$ the rotating fluid system is equivalent to the following ordinary differential system

$$\left\{ \begin{array}{rcl} \frac{d}{dt} W^{k,1} + \nu|\xi_h|^2 W^{k,1} + \beta\varepsilon(k\pi)^2 W^{k,1} - \frac{1}{\varepsilon} W^{k,2} &=& |\xi_h|^2 \widehat{p}^k \\ \frac{d}{dt} W^{k,2} + \nu|\xi_h|^2 W^{k,2} + \beta\varepsilon(k\pi)^2 W^{k,2} + \frac{1}{\varepsilon} W^{k,1} &=& 0 \\ \frac{d}{dt} W^{k,3} + \nu|\xi_h|^2 W^{k,3} + \beta\varepsilon(k\pi)^2 W^{k,3} &=& k\pi \widehat{p}^k \\ W^{k,1} + k\pi W^{k,3} &=& 0 \\ W_{|t=0} &=& W_0. \end{array} \right.$$

The divergence free condition enables one to transform that system into the following one:

$$(FRF_k^\varepsilon) \left\{ \begin{array}{rcl} \frac{d}{dt} W^{k,h} + \nu|\xi_h|^2 W^{k,h} + \beta\varepsilon(k\pi)^2 W^{k,h} - \frac{1}{\varepsilon} R_k W^{k,h} &=& 0 \\ W^{k,h}{}_{|t=0} &=& W_0^{k,h} \end{array} \right.$$

with

$$R_k = \begin{pmatrix} 0 & -\lambda_k^2 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad \lambda_k \stackrel{\text{def}}{=} \left(\frac{(k\pi)^2}{|\xi_h|^2 + (k\pi)^2} \right)^{\frac{1}{2}}.$$

In order to find the interior solution at order zero, and to get rid of the ε^{-1} terms in the equation, let us write $W = \sum_{k=1}^{N} \Big(\mathcal{L}_k\big(\frac{t}{\varepsilon}\big) \widetilde{W}^k(t) \cos(k\pi x_3), -\frac{1}{k\pi} \Pi_1 \mathcal{L}_k\big(\frac{t}{\varepsilon}\big) \widetilde{W}^k(t) \sin(k\pi x_3) \Big)$ where $\mathcal{L}_k$

is a function from $\mathbf{R}^+$ into $\mathcal{L}(\mathbf{R}^2)$ and $\widetilde{W}^k$ is a function from $\mathbf{R}^+$ into $\mathbf{C}^2$, to be determined. Then looking at the terms of size ε^{-1} in the above equation (FRF_k^ε), we get

$$\dot{\mathcal{L}}_k + R_k \mathcal{L}_k = 0 \quad \text{with} \quad \mathcal{L}_k(0) = \text{Id},$$

so

$$\mathcal{L}_k(\tau) = \begin{pmatrix} \cos \tau_k & \lambda_k \sin \tau_k \\ -\dfrac{1}{\lambda_k} \sin \tau_k & \cos \tau_k \end{pmatrix} \tag{2.3}$$

with, as in all that follows, $\tau_k = \lambda_k \tau$. Let us remark that when $k = 0$, (this corresponds of course to the well prepared case recalled in the previous section), we get $\mathcal{L}_k = \text{Id}$ because in this case $\lambda_k = 0$.

To state the approximation lemma, we need to define the following family of operators $(\mathcal{L}(\tau))_{\tau \in \mathbf{R}}$: for any vector v of the form (2.1,2.2),

$$\begin{aligned}(\mathcal{L}(\tau)v)(x_h, x_3) \;&\stackrel{\text{def}}{=}\; (v^{0,h}(x_h), 0) + \mathcal{F}^{-1} \sum_{k=1}^{\infty} \Big(A(\xi_h)\mathcal{L}_k(\tau)A^{-1}(\xi_h)\widehat{v}^{k,h}(\xi_h)\cos(k\pi x_3), \\ &\qquad \frac{i}{k\pi}\xi_h \cdot A(\xi_h)\mathcal{L}_k(\tau)A^{-1}(\xi_h)\widehat{v}^{k,h}(\xi_h)\sin(k\pi x_3)\Big)\end{aligned}$$

where $\mathcal{L}_k$ is defined by formula (2.3) and with $A(\xi_h) \stackrel{\text{def}}{=} \begin{pmatrix} \xi_1|\xi_h|^{-2} & -\xi_2|\xi_h|^{-2} \\ \xi_2|\xi_h|^{-2} & \xi_1|\xi_h|^{-2} \end{pmatrix}$. Let us note that $(\mathcal{L}(\tau))_{\tau \in \mathbf{R}}$ is bounded in $\mathcal{L}(H^{s,0})$ for any real number s. This operator $\mathcal{L}(\tau)$ is the Rossby wave operator. Similarly we shall need the definition of the following "Ekman operator":

$$\begin{aligned}(\mathcal{E}v)(x_h, x_3) \;&\stackrel{\text{def}}{=}\; \sqrt{2\beta}\,(v^{0,h}(x_h), 0) + \mathcal{F}^{-1} \sum_{k=1}^{\infty} \Big(A(\xi_h)B_k A^{-1}(\xi_h)\widehat{v}^{k,h}(\xi_h)\cos(k\pi x_3), \\ &\qquad \frac{i}{k\pi}\xi_h \cdot A(\xi_h)B_k A^{-1}(\xi_h)\widehat{v}^{k,h}(\xi_h)\sin k\pi x_3\Big)\end{aligned}$$

with

$$B_k \stackrel{\text{def}}{=} \frac{(1-\lambda_k^2)\lambda_k}{4} \begin{pmatrix} \gamma_k^- - \gamma_k^+ & -\lambda_k(\gamma_k^+ + \gamma_k^-) \\ \dfrac{\gamma_k^+ + \gamma_k^-}{\lambda_k} & \gamma_k^- - \gamma_k^+ \end{pmatrix} \quad \text{and} \quad \gamma_k^\pm \stackrel{\text{def}}{=} \left(1 \mp \frac{1}{\lambda_k}\right)\sqrt{\frac{2\beta}{1 \pm \lambda_k}}\,.$$

Lemma 2 *Let $(v_N)_{N \in \mathbf{N}}$ be a bounded sequence in the space $L^\infty([0,T];L^2) \cap L^2([0,T];H^{1,0})$ of divergence free vector fields of the form*

$$v_N \stackrel{\text{def}}{=} \mathcal{F}^{-1} \sum_{k=0}^{N} \Big(\widehat{v}_N^{k,h}(\xi_h)\cos(k\pi x_3), -\frac{1}{ik\pi}\xi_h \cdot \widehat{v}_N^{k,h}(\xi_h)\sin(k\pi x_3)\Big)$$

with Supp $\widehat{v}_N^{\varepsilon,k,h}(\xi_h) \subset \mathcal{C}_N$. A sequence of families $(v^\varepsilon_{app,N})_{N \in \mathbf{N}}$ of smooth divergence free vector fields whose value on the boundary of Ω is 0 exists, such that

$$L^\varepsilon v^\varepsilon_{app,N} = \mathcal{L}\left(\frac{t}{\varepsilon}\right)\Big(\partial_t v_N - \nu\Delta_h v_N + \mathcal{E}v_N\Big) - \nabla p^\varepsilon_{app,N} + R^\varepsilon_N \quad \textit{where}$$

$$\forall \eta\,,\ \exists N_1\,,\ \forall N > N_1\,,\ \exists \varepsilon_0 \,/\, \forall \varepsilon < \varepsilon_0\,,\ \|R_N^\varepsilon\|_{L^\infty(\mathbf{R}^+;H^{-1,0})\cap L^2(\mathbf{R}^+;L^2)} < \eta. \tag{2.4}$$

The vector field $v^\varepsilon_{app,N}$ *converges to* $\mathcal{L}\left(\frac{t}{\varepsilon}\right) v_N$ *in the following sense: a constant* C_N *exists such that*

$$E_T\left(v^\varepsilon_{app,N} - \mathcal{L}\Big(\frac{t}{\varepsilon}\Big)v_N\right) \leq C_N \varepsilon E_T(v_N).$$

Moreover the family $(v^\varepsilon_{app,N})$ *satisfies the following estimates*

$$\begin{aligned}
\int_0^T \sup_{x_3\in]0,1[} \|\nabla^h v^\varepsilon_{app,N}(t,\cdot,x_3)\|^2_{L^2(\mathbf{R}^2)} dt &\leq C_N E_T(v_N)\\
\forall p \in [2,\infty]\,,\ \int_0^T\int_0^1 d(x_3)\|\partial_3 v^\varepsilon_{app,N}(t,\cdot,x_3)\|^2_{L^p(\mathbf{R}^2)} dt dx_3 &\leq C_N \overline{E}_T(v_N) \quad \text{and}\\
\int_0^T\int_0^1 d(x_3)^2 \left\|\partial_3\left(v^\varepsilon_{app,N} - \mathcal{L}\Big(\frac{t}{\varepsilon}\Big)v_N\right)\right\|^2_{L^\infty(\mathbf{R}^2)} dt dx_3 &\leq C_N \varepsilon \overline{E}_T(v_N).
\end{aligned}$$

Remark. In the proof, we shall forget the part associated with $\widehat{v}_N^{0,h}$ because this case is nothing but the well-prepared case.

Let us give an idea of the proof of this lemma, which is one of the key points of [5]. In horizontal divergence and curl formulation, the system (FRF^ε) becomes

$$(\widetilde{FRF}^\varepsilon)\begin{cases}
\partial_t W^1 + \nu|\xi_h|^2 W^1 - \beta\varepsilon\partial_3^2 W^1 - \frac{1}{\varepsilon}W^2 &= |\xi_h|^2\widehat{p}\\
\partial_t W^2 + \nu|\xi_h|^2 W^2 - \beta\varepsilon\partial_3^2 W^2 + \frac{1}{\varepsilon}W^1 &= 0\\
\partial_t W^3 + \nu|\xi_h|^2 W^3 - \beta\varepsilon\partial_3^2 W^3 &= -\partial_3\widehat{p}\\
W^1 + \partial_3 W^3 &= 0\\
W_{|t=0} &= W_0\\
W_{|\partial\Omega} &= 0.
\end{cases}$$

From now on, we shall only consider the above system. Let us search for an approximate solution of the form

$$\begin{aligned}
W &= W_{0,int} + W_{0,BL} + \varepsilon W_{1,int} + \varepsilon W_{1,BL} + \cdots \quad \text{and}\\
\widehat{p} &= \frac{1}{\varepsilon}\widehat{p}_{-1,int} + \frac{1}{\varepsilon}\widehat{p}_{-1,BL} + \widehat{p}_{0,int} + \widehat{p}_{0,BL} + \cdots
\end{aligned}$$

where each component of $(v_{j,int}, p_{j,int})$ is a function of the form $f\Big(\frac{t}{\varepsilon}, t, x_3\Big)$ and each component of $(v_{j,BL}, p_{j,BL})$ is a function of the form $g\Big(\frac{t}{\varepsilon}, t, \frac{x_3}{\varepsilon}\Big) + h\Big(\frac{t}{\varepsilon}, t, \frac{1-x_3}{\varepsilon}\Big)\cdot$ In all that follows, we shall denote $\tau = t/\varepsilon$. First we have to determine the form of $W_{0,int}$ and $\widehat{p}_{-1,int}$. Considering the decomposition of the initial data, we look for $W_{0,int}$ of the form

$$\sum_{k=1}^N \left(\mathcal{L}_k\Big(\frac{t}{\varepsilon}\Big)\widetilde{W}^k_{0,int}(t)\cos(k\pi x_3), -\frac{1}{k\pi}\Pi_1\mathcal{L}_k\Big(\frac{t}{\varepsilon}\Big)\widetilde{W}^k_{0,int}(t)\sin(k\pi x_3)\right).$$

Now let us write

$$W_{0,int,N} \stackrel{\text{def}}{=} \sum_{k=1}^{N} \Big(\mathcal{L}_k\Big(\frac{t}{\varepsilon}\Big)\widetilde{W}^k_{0,int}\cos(k\pi x_3), -\frac{1}{k\pi}\Pi^1\mathcal{L}_k\Big(\frac{t}{\varepsilon}\Big)\widetilde{W}^k_{0,int}\sin(k\pi x_3)\Big). \tag{2.5}$$

We have to determine the boundary layer of size ε^0. As the third component of the interior solution at order 0 is identically 0, then the vertical component of the boundary layer of size ε^0 vanishes. This implies that $\partial_3 \widehat{p}^k_{-1,BL} = 0$. We recover the well known fact that the pressure does not vary in the boundary layer. Now let us study the term of size ε^{-1} for the horizontal component of the boundary layer. As $\cos(k\pi) = (-1)^k$ we look for the boundary layer of the form

$$W^{k,h}_{0,BL} \stackrel{\text{def}}{=} M_k\Big(\frac{x_3}{\varepsilon}\Big)\mathcal{L}_k\Big(\frac{t}{\varepsilon}\Big)\widetilde{W}^k_{0,int} + (-1)^k M_k\Big(\frac{1-x_3}{\varepsilon}\Big)\mathcal{L}_k\Big(\frac{t}{\varepsilon}\Big)\widetilde{W}^k_{0,int}.$$

The term of size ε^{-1} of $\partial_t W^{k,h}_{0,BL} - \beta\varepsilon\partial_3^2 W^{k,h}_{0,BL} + \frac{1}{\varepsilon}RW^{k,h}_{0,BL}$ must be equal to 0. We infer

$$M_k\dot{\mathcal{L}}_k - \beta M_k''\mathcal{L}_k + RM_k\mathcal{L}_k = 0.$$

In the case when $k=0$, we have $\mathcal{L}_k = \mathrm{Id}$. Now let us assume that $k \geq 1$. As $\dot{\mathcal{L}}_k = -R_k\mathcal{L}_k$, it turns out that the equation on the boundary layer is

$$\left\{\begin{array}{rcl} -\beta M_k'' &=& M_kR_k - RM_k \\ M_k(0) &=& -\mathrm{Id} \\ M_k(+\infty) &=& 0. \end{array}\right.$$

We infer

$$M_k(\zeta) = -\sum_{\pm}\frac{1}{2}\mu_k^{\pm}\exp(-\zeta_k^{\pm})M_k^{\pm}(\zeta_k^{\pm}) \quad \text{with}$$

$$M_k^{\pm}(\theta) \stackrel{\text{def}}{=} \begin{pmatrix} \cos\theta & \mp\lambda_k\sin\theta \\ -\sin\theta & \mp\lambda_k\cos\theta \end{pmatrix}, \quad \zeta_k^{\pm} \stackrel{\text{def}}{=} \frac{\zeta}{\sqrt{2\beta_k^{\pm}}}, \quad \beta_k^{\pm} \stackrel{\text{def}}{=} \frac{\beta}{1\pm\lambda_k} \quad \text{and} \quad \mu_k^{\pm} \stackrel{\text{def}}{=} 1 \mp \frac{1}{\lambda_k}.$$

So stating $E_k^{\pm} \stackrel{\text{def}}{=} 2\varepsilon^2\beta_k^{\pm}$, we infer by definition of $\mathcal{L}_k$ that

$$\begin{aligned} W^{k,h}_{0,BL} = -\frac{1}{2}\sum_{\pm}\mu_k^{\pm}\exp\Big(-\frac{x_3}{\sqrt{E_k^{\pm}}}\Big)M_k^{\pm}\Big(\frac{x_3}{\sqrt{E_k^{\pm}}} \mp \frac{\lambda_k t}{\varepsilon}\Big)\widetilde{W}^k_{0,int} \\ -(-1)^k\frac{1}{2}\sum_{\pm}\mu_k^{\pm}\exp\Big(-\frac{1-x_3}{\sqrt{E_k^{\pm}}}\Big)M_k^{\pm}\Big(\frac{1-x_3}{\sqrt{E_k^{\pm}}} \mp \frac{\lambda_k t}{\varepsilon}\Big)\widetilde{W}^k_{0,int}. \end{aligned}$$

The fact that the boundary layer must be divergence free implies that we have to introduce a vertical component of the boundary layer of size ε. It is given by the following formula:

$$\varepsilon\partial_3 W^{k,3}_{1,BL} = -\Pi_1 W^{k,h}_{0,BL}.$$

So with the notation $\mathrm{cs}^\pm \stackrel{\text{def}}{=} \cos \pm \sin$ and $\gamma_k^\pm \stackrel{\text{def}}{=} \mu_k^\pm \sqrt{2\beta_k^\pm}$, we get after integration

$$\begin{aligned}
W_{1,BL}^{k,3} \quad = \quad & -\frac{1}{4}\sum_\pm \gamma_k^\pm \exp\Big(-\frac{x_3}{\sqrt{E_k^\pm}}\Big)\Big(\mathrm{cs}^-\Big(\frac{x_3}{\sqrt{E_k^\pm}} \mp \frac{\lambda_k t}{\varepsilon}\Big)\widetilde{W}_{0,int}^{k,1} \\
& \qquad \mp \lambda_k \mathrm{cs}^+\Big(\frac{x_3}{\sqrt{E_k^\pm}} \mp \frac{\lambda_k t}{\varepsilon}\Big)\widetilde{W}_{0,int}^{k,2}\Big) \\
& +(-1)^k\frac{1}{4}\sum_\pm \gamma_k^\pm \exp\Big(-\frac{1-x_3}{\sqrt{E_k^\pm}}\Big)\Big(\mathrm{cs}^-\Big(\frac{1-x_3}{\sqrt{E_k^\pm}} \mp \frac{\lambda_k t}{\varepsilon}\Big)\widetilde{W}_{0,int}^{k,1} \\
& \qquad \mp \lambda_k \mathrm{cs}^+\Big(\frac{1-x_3}{\sqrt{E_k^\pm}} \mp \frac{\lambda_k t}{\varepsilon}\Big)\widetilde{W}_{0,int}^{k,2}\Big).
\end{aligned}$$

It is obvious that those two functions do not vanish respectively in $x_3 = 0$ and $x_3 = 1$. More precisely, up to exponentially small terms, we have

$$W_{1,BL}^{k,3}{}_{|x_3=0} = -(-1)^k W_{1,BL}^{k,3}{}_{|x_3=1} = -f_k\Big(\frac{t}{\varepsilon}\Big)$$

with

$$f_k(\tau) \stackrel{\text{def}}{=} \sum_\pm \frac{\gamma_k^\pm}{4}\Big(\mathrm{cs}^\pm(\tau_k)\widetilde{W}_{0,int}^{k,1} \mp \lambda_k \mathrm{cs}^\mp(\tau_k)\widetilde{W}_{0,int}^{k,2}\Big).$$

Now let us have a look at the terms of size ε^0 in the interior system. The system of equations is the following.

$$\left\{\begin{aligned}
(\partial_t + \nu|\xi_h|^2)W_{0,int,N}^h + \partial_\tau W_{1,int,N}^h + RW_{1,int,N}^h \quad &= \quad \begin{pmatrix} |\xi_h|^2\widehat{p}_{0,int,N} \\ 0 \end{pmatrix} \\
(\partial_t + \nu|\xi_h|^2)W_{0,int,N}^3 + \partial_\tau W_{1,int,N}^3 \quad &= \quad -\partial_3\widehat{p}_{0,int,N} \\
W_{1,int,N}^1 + \partial_3 W_{1,int,N}^3 \quad &= \quad 0 \\
W_{1,int,N}^3{}_{|\partial\Omega} \quad &= \quad -W_{1,BL}^{k,3}{}_{|\partial\Omega}.
\end{aligned}\right.$$

We are going to reduce this problem to a problem with a homogeneous Dirichlet boundary condition. To do so, let us first define the function r_ℓ where

$$r_\ell(x_3) = 1 \quad \text{when } \ell \text{ is odd}, \quad r_\ell(x_3) = 1 - 2x_3 \quad \text{when } \ell \text{ is even}.$$

Then let us state

$$\underline{W}_{1,int,N} \stackrel{\text{def}}{=} \sum_{\ell=1}^N f_\ell(\tau)\Big(\delta_\ell, 0, r_\ell(x_3)\Big) \quad \text{with} \quad \delta_\ell = 1 + (-1)^\ell$$

and let us look for $W_{1,int,N}$ of the form $W_{1,int,N} = \overline{W}_{1,int,N} + \underline{W}_{1,int,N}$. The above system can be written

$$\left\{\begin{aligned}
(\partial_t + \nu|\xi_h|^2)W_{0,int,N}^h + (\partial_\tau + R)\overline{W}_{1,int,N}^h \quad &= \quad \begin{pmatrix} |\xi_h|^2\widehat{p}_{0,int,N} \\ 0 \end{pmatrix} - (\partial_\tau + R)\underline{W}_{1,int,N}^h \\
(\partial_t + \nu|\xi_h|^2)W_{0,int,N}^3 + \partial_\tau \overline{W}_{1,int,N}^3 \quad &= \quad -\partial_3\widehat{p}_{0,int,N} - \partial_\tau \underline{W}_{1,int,N}^3 \\
\overline{W}_{1,int,N}^1 + \partial_3 \overline{W}_{1,int,N}^3 \quad &= \quad 0 \\
\overline{W}_{1,int,N}^3{}_{|\partial\Omega} \quad &= \quad 0.
\end{aligned}\right.$$

The original boundary condition appears through a forcing term. Considering this boundary condition on $\overline{W}_{1,int,N}$, it is natural to look for $\overline{W}_{1,int,N}$ and $\widehat{p}_{0,int,N}$ of the form

$$\begin{aligned}\overline{W}_{1,int,N} &= \Big(\sum_{k=0}^{N} W^{k,h}_{1,int,N}\cos(k\pi x_3), -\sum_{k=1}^{N}\frac{1}{k\pi}W^{k,1}_{1,int,N}\sin(k\pi x_3)\Big) \quad \text{and}\\ \widehat{p}_{0,int,N} &= \sum_{k=0}^{N} p^k_{0,int}\cos(k\pi x_3).\end{aligned}$$

Now let us decompose the forcing term in a low and a high vertical frequency part. In $L^2(]0,1[)$, we have

$$r_\ell(x_3) = \sum_{k\geq 1} r_{\ell,k}\sin(k\pi x_3) \quad \text{with} \quad r_{\ell,k} \stackrel{\text{def}}{=} \frac{1}{k\pi}(1+(-1)^{k+\ell}). \tag{2.6}$$

So we write $\underline{W}_{1,int,N} = F_N + R^\varepsilon_N$ with $F_N \stackrel{\text{def}}{=} \sum_{\ell=1}^{N} f_\ell(\tau)\Big(\delta_\ell, 0, \sum_{k=1}^{N} r_{\ell,k}\sin(k\pi x_3)\Big)$. Obviously, R^ε_N satisfies the property (2.4). Now let us project on cosine and sine functions, which yields

$$\begin{aligned}\Big(\frac{d}{d\tau}+R\Big)\overline{W}^{0,h}_{1,int,N} &= \sum_{\ell=1}^{N}\Big\{-\frac{df_\ell}{d\tau}(\delta_\ell,0) + f_\ell(\tau)(0,\delta_\ell)\Big\}\\ \frac{d}{d\tau}\overline{W}^{0,3}_{1,int,N} &= 0\end{aligned}$$

and, when $k\neq 0$, considering the form of $W_{0,int,N}$ given by (2.5)

$$\begin{aligned}\mathcal{L}_k(\tau)\Big(\frac{d}{dt}+\nu|\xi_h|^2\Big)\widetilde{W}^k_{0,int} + \Big(\frac{d}{d\tau}+R\Big)\overline{W}^{k,h}_{1,int,N} &= \begin{pmatrix} |\xi_h|^2\widehat{p}^k_{0,int}\\ 0\end{pmatrix}\\ -\Pi_1\frac{1}{k\pi}\mathcal{L}_k(\tau)\Big(\frac{d}{dt}+\nu|\xi_h|^2\Big)\widetilde{W}^k_{0,int} + \frac{d}{d\tau}\overline{W}^{k,3}_{1,int,N} &= -\sum_{\ell\leq N} r_{\ell,k}\frac{df_\ell}{d\tau} + k\pi\widehat{p}^k_{0,int}.\end{aligned}$$

Let us solve first the system when $k=0$. As $\overline{W}^{0,3}_{1,int,N} = 0$ the divergence free condition implies that $\overline{W}^{0,1}_{1,int,N} = 0$, so we get

$$W^0_{1,int,N} = \Big(0, \sum_{\ell=1}^{N}\sum_{\pm}\frac{\gamma^\pm_\ell}{4\lambda_\ell}\Big(\mp\text{cs}^\mp(\tau_\ell)\widetilde{W}^{\ell,1}_{0,int} + \lambda_\ell\text{cs}^\pm(\tau_\ell)\widetilde{W}^{\ell,2}_{0,int}\Big), 0\Big).$$

We also get an explicit formula for $p^0_{0,int,N}$ whose computation is left to the reader. Now let us study the case when $k\neq 0$. As usual, the divergence free condition determines the pressure and the system becomes

$$\mathcal{L}_k(\tau)\Big(\frac{d}{dt}+\nu|\xi_h|^2\Big)\widetilde{W}^k_{0,int} + \Big(\frac{d}{d\tau}+R_k\Big)\overline{W}^{k,h}_{1,int,N} - \sum_{\ell\leq N}\begin{pmatrix}\dfrac{k\pi r_{\ell,k}|\xi_h|^2}{|\xi_h|^2+(k\pi)^2}\dfrac{df_\ell}{d\tau}\\ 0\end{pmatrix} = 0. \tag{2.7}$$

The proof of the following lemma is left to the reader.

Lemma 3 *We have the following identity:*

$$-\sum_{\ell\leq N}\begin{pmatrix}\dfrac{k\pi r_{\ell,k}|\xi_h|^2}{|\xi_h|^2+(k\pi)^2}\dfrac{df_\ell}{d\tau}\\ 0\end{pmatrix}=\mathcal{L}_k(\tau)\Big(B_k+\sum_{\ell=1}^{N}B_{k,\ell}(\tau)\Big)\widetilde{W}^k_{0,int}$$

where $B_{k,\ell}(\tau)$ are matrices whose coefficients are cosine or sine functions of $(\lambda_k\pm\lambda_\ell)\tau$ for $\ell\neq k$ and of $\lambda_k\tau$ when $\ell=k$ and where

$$B_k=\frac{(1-\lambda_k^2)\lambda_k}{4}\begin{pmatrix}\gamma_k^- -\gamma_k^+ & -\lambda_k(\gamma_k^+ +\gamma_k^-)\\ \dfrac{\gamma_k^+ +\gamma_k^-}{\lambda_k} & \gamma_k^- -\gamma_k^+\end{pmatrix}.$$

Immediately we infer

$$\widetilde{W}^k_{0,int}(t)=\exp(-\nu|\xi_h|^2t-P_kt)\begin{pmatrix}\cos\delta_kt & \lambda_k\sin\delta_kt\\ -\dfrac{1}{\lambda_k}\sin\delta_kt & \cos\delta_kt\end{pmatrix}\widetilde{W}^k_{0,int}(0)\quad\text{with}$$

$$\delta_k\stackrel{\text{def}}{=}\frac{(1-\lambda_k^2)\lambda_k}{4}(\gamma_k^+ +\gamma_k^-)\,,\ P_k\stackrel{\text{def}}{=}\frac{(1-\lambda_k^2)\lambda_k}{4}(\gamma_k^- -\gamma_k^+)\quad\text{and}$$

$$W^{k,h}_{1,int,N}=\mathcal{L}_k\Big(\frac{t}{\varepsilon}\Big)\sum_{\ell=1}^{N}C_{k,\ell}\Big(\frac{t}{\varepsilon}\Big)\widetilde{W}^k_{0,int}(t)$$

where $C_{k,\ell}$ are (2×2 valued) smooth bounded functions of τ whose derivatives are the $B_{k,\ell}$. So applying the usual procedure for the higher order terms for boundary layers, we find the complete expression of the approximate solution and the lemma is obtained, up to the proof of the three last estimates for which we refer to [5].

3 The non linear estimates

This section consists in proving Theorem 1. We shall skip most of the details and simply give the steps of the proof. Let us first define

$$\begin{aligned}\overline{u}_0(x_h) &\stackrel{\text{def}}{=} \int_0^1 u_0(x_h,x_3)dx_3\,,\\ u_0^{k,h}(x_h) &\stackrel{\text{def}}{=} \frac{1}{2}\int_0^1 u_0(x_h,x_3)\cos(k\pi x_3)dx_3\,,\\ \overline{u}_{0,N} &\stackrel{\text{def}}{=} \mathcal{F}^{-1}(\mathbf{1}_{\mathcal{C}_N}\widehat{\overline{u}}_0)\quad\text{and}\\ u_{0,N}^{k,h} &\stackrel{\text{def}}{=} \mathcal{F}^{-1}(\mathbf{1}_{\mathcal{C}_N}\widehat{u}_0^{k,h}).\end{aligned}$$

For any positive real number η, an integer N_0 exists, depending of course on η and on the initial data u_0 such that

$$\begin{aligned}\|u_0-\overline{u}_{0,N}-\widetilde{u}_{0,N}\|_{L^2} &< \frac{\eta}{4}\quad\text{with}\\ \widetilde{u}_{0,N} &= \sum_{k=1}^{N}\Big(u_{0,N}^{k,h}(x_h)\cos(k\pi x_3),-\frac{1}{k\pi}\operatorname{div}_h u_{0,N}^{k,h}(x_h)\sin(k\pi x_3)\Big).\end{aligned}$$

Let us define $\overline{u}_N$ as the solution of

$$(LE_{\nu,\beta})\begin{cases}\partial_t\overline{u}_N-\nu\Delta_h\,\overline{u}_N+\sqrt{2\beta}\,\overline{u}_N &= -\nabla\overline{p}_N-\mathcal{F}^{-1}(\mathbf{1}_{\mathcal{C}_N}\mathcal{F}(\mathrm{div}_h(\overline{u}\otimes\overline{u})))\\ \mathrm{div}_h\,\overline{u}_N &= 0\\ \overline{u}_{N\,|t=0} &= \overline{u}_{0,N}.\end{cases}$$

A basic energy estimate implies that for any positive η, an integer N_0 exists such that, for any t in $\mathbf{R}^+$ and any N greater than or equal to N_0, we have

$$\|\overline{u}(t)-\overline{u}_N(t)\|_{L^2}^2+2\int_0^t\Big(\nu\|\nabla^h(\overline{u}(t')-\overline{u}_N(t'))\|_{L^2}^2+\sqrt{2\beta}\,\|\overline{u}(t')-\overline{u}_N(t')\|_{L^2}^2\Big)dt'<\frac{\eta^2}{16}\cdot \tag{3.1}$$

Sobolev embeddings also imply that $\overline{u}$ belongs to $L^4(\mathbf{R}^+\times\mathbf{R}^2)$ and $\overline{u}_N$ converges to $\overline{u}$ in the space $L^4(\mathbf{R}^+\times\mathbf{R}^2)$. Thus we have

$$\lim_{N\to\infty}\mathcal{F}^{-1}(\mathbf{1}_{\mathcal{C}_N}\mathcal{F}(\mathrm{div}_h(\overline{u}\otimes\overline{u})))=\lim_{N\to\infty}\mathrm{div}_h(\overline{u}_N\otimes\overline{u}_N)=\mathrm{div}_h(\overline{u}\otimes\overline{u}) \tag{3.2}$$

in the space $L^2(\mathbf{R}^+;H^{-1}(\mathbf{R}^2))$.

Now we shall use Lemmas 1 and 2 to define the (sequence of) approximate solutions of the system. Let us define $(u^\varepsilon_{wp,N})_{\varepsilon>0}$ as the families given by Lemma 1 applied with $v_N=\overline{u}_N$ solution of $(LE_{\nu,\beta})$; it represents the "well prepared" part of the solution. The so–called "ill prepared" part is defined as $(u^\varepsilon_{ip,N})$, the families given by Lemma 2 applied with v_N equal to the solution of the linear problem

$$\begin{cases}\partial_t v_N-\nu\Delta_h v_N+\mathcal{E}v_N &= -\nabla p_N\\ \mathrm{div}\,v_N &= 0\\ v_{N\,|t=0} &= \widetilde{u}_{0,N}.\end{cases}$$

Of course, we state $u^\varepsilon_{app,N}\stackrel{\mathrm{def}}{=}u^\varepsilon_{wp,N}+u^\varepsilon_{ip,N}$. Let us derive the equation satisfied by $u^\varepsilon_{app,N}$. Using Lemmas 1, 2 and (3.2), we get that

$$L^\varepsilon u^\varepsilon_{app,N}=R^\varepsilon_N+\nabla p_\varepsilon+\mathrm{div}_h(\overline{u}_N\otimes\overline{u}_N). \tag{3.3}$$

Using energy estimates, we get that

$$\begin{aligned}\|u^\varepsilon_{app,N}(t)\|_{L^2}^2+2\nu\int_0^t\|\nabla^h u^\varepsilon_{app,N}(t')\|_{L^2}^2dt'+2\beta\varepsilon\int_0^t\|\partial_3 u^\varepsilon_{app,N}(t')\|_{L^2}^2dt'\\ \leq\|u_0\|_{L^2}^2-\int_0^t(\mathrm{div}_h(\overline{u}_N\otimes\overline{u}_N)(t')|u^\varepsilon_{app,N}(t'))_{L^2}dt'+\rho^\varepsilon_N\end{aligned}$$

where, as in all that follows, ρ^ε_N denotes generically a scalar quantity such that

$$\forall\eta\,,\ \exists N_0\,,\ \forall N>N_0\,,\ \exists\varepsilon_0\ /\ \forall\varepsilon<\varepsilon_0\,,\ \rho^\varepsilon_N<\eta.$$

Thanks to Lemma 2 we have

$$\|u^\varepsilon_{app,N}-\mathcal{L}\left(\frac{t}{\varepsilon}\right)v_N\|_{L^2(\mathbf{R}^+;H^{1,0})}\leq C_N\varepsilon^{\frac12}\|u_0\|_{L^2}.$$

So, as $\mathrm{div}_h(\overline{u}_N \otimes \overline{u}_N)$ belongs to $L^2(\mathbf{R}^+ \times \Omega)$, we have

$$\int_0^t (\mathrm{div}_h(\overline{u}_N \otimes \overline{u}_N)(t')|u^\varepsilon_{app,N}(t'))_{L^2} dt' = \int_0^t (\mathrm{div}_h(\overline{u}_N \otimes \overline{u}_N)(t')|\overline{u}_N(t'))_{L^2} dt' + \int_0^t \Big(\mathrm{div}_h(\overline{u}_N \otimes \overline{u}_N)(t')|\mathcal{L}\Big(\frac{t'}{\varepsilon}\Big)v_N(t')\Big)_{L^2} dt' + \rho^\varepsilon_N,$$

and an easy computation shows that

$$\int_0^t (\mathrm{div}_h(\overline{u}_N \otimes \overline{u}_N)(t')|u^\varepsilon_{app,N}(t'))_{L^2} dt' = \rho^\varepsilon_N.$$

As an immediate consequence, it turns out that

$$\|u^\varepsilon_{app,N}(t)\|^2_{L^2} + 2\nu \int_0^t \|\nabla^h u^\varepsilon_{app,N}(t')\|^2_{L^2} dt' + 2\beta\varepsilon \int_0^t \|\partial_3 u^\varepsilon_{app,N}(t')\|^2_{L^2} dt' \leq \|u_0\|^2_{L^2} + \rho^\varepsilon_N. \quad (3.4)$$

Equation (3.3) can be rewritten as

$$L^\varepsilon u^\varepsilon_{app,N} + u^\varepsilon_{app,N} \cdot \nabla u^\varepsilon_{app,N} = R^\varepsilon_N + \nabla p_\varepsilon + F^\varepsilon_N$$

with

$$F^\varepsilon_N \stackrel{\text{def}}{=} u^\varepsilon_{app,N} \cdot \nabla u^\varepsilon_{app,N} - \overline{u}_N \cdot \nabla \overline{u}_N.$$

Now we use the classical method to prove weak–strong type estimates. We are exactly in this situation because we consider a weak solution u^ε without any additionnal regularity and a regular (approximate) solution $u^\varepsilon_{app,N}$. Let us denote by δ^ε the difference $u^\varepsilon - u^\varepsilon_{app,N}$, we write that

$$\begin{aligned} E^\varepsilon(t) &\stackrel{\text{def}}{=} \|\delta^\varepsilon(t)\|^2_{L^2} + 2\nu \int_0^t \|\nabla^h \delta^\varepsilon(t')\|^2_{L^2} dt' + 2\beta\varepsilon \int_0^t \|\partial_3 \delta^\varepsilon(t')\|^2_{L^2} dt' \\ &= \|u^\varepsilon(t)\|^2_{L^2} + 2\nu \int_0^t \|\nabla^h u^\varepsilon(t')\|^2_{L^2} dt' + 2\beta\varepsilon \int_0^t \|\partial_3 u^\varepsilon(t')\|^2_{L^2} dt' \\ &\quad + \|u^\varepsilon_{app,N}(t)\|^2_{L^2} + 2\nu \int_0^t \|\nabla^h u^\varepsilon_{app,N}(t')\|^2_{L^2} dt' + 2\beta\varepsilon \int_0^t \|\partial_3 u^\varepsilon_{app,N}(t')\|^2_{L^2} dt' \\ &\quad - 2(u^\varepsilon(t)|u^\varepsilon_{app,N}(t))_{L^2} - 4\nu \int_0^t (\nabla^h u^\varepsilon(t')|\nabla^h u^\varepsilon_{app,N}(t'))_{L^2} dt' \\ &\quad - 4\beta\varepsilon \int_0^t (\partial_3 u^\varepsilon(t')|\partial_3 u^\varepsilon_{app,N}(t'))_{L^2} dt'. \end{aligned}$$

As u^ε is a Leray solution of (RF_ε), it satisfies the energy inequality. So thanks to Inequality (3.4) we get

$$E^\varepsilon(t) \leq 2\|u_0\|^2_{L^2} - 2(u^\varepsilon|u^\varepsilon_{app,N})_{L^2} - 4\nu \int_0^t (\nabla^h u^\varepsilon(t')|\nabla^h u^\varepsilon_{app,N}(t'))_{L^2} dt' - 4\beta\varepsilon \int_0^t (\partial_3 u^\varepsilon(t')|\partial_3 u^\varepsilon_{app,N}(t'))_{L^2} dt' + \rho^\varepsilon_N.$$

The right–hand side is then transformed in a classical way (see [5]): using the fact that $u^\varepsilon_{app,N}$ is a smooth function vanishing at the boundary (and using also Lemmas 1 and 2), we get

$$\begin{aligned}(u^\varepsilon|u^\varepsilon_{app,N})_{L^2}(t) &= \|u_0\|^2_{L^2} + \rho^\varepsilon_N \\ &- 2\nu\int_0^t (\nabla^h u^\varepsilon(t')|\nabla^h u^\varepsilon_{app,N}(t'))_{L^2}dt' - 2\beta\varepsilon\int_0^t (\partial_3 u^\varepsilon(t')|\partial_3 u^\varepsilon_{app,N}(t'))_{L^2}dt' \\ &\quad - \int_0^t (\delta^\varepsilon(t')\cdot\nabla\delta^\varepsilon(t')|u^\varepsilon_{app,N}(t'))_{L^2}dt' + \int_0^t (F^\varepsilon_N(t')|u^\varepsilon(t'))_{L^2}dt',\end{aligned}$$

and we infer that

$$E^\varepsilon(t) \leq 2\int_0^t (\delta^\varepsilon(t')\cdot\nabla\delta^\varepsilon(t')|u^\varepsilon_{app,N}(t'))_{L^2}dt' + 2\int_0^t (F^\varepsilon_N(t')|u^\varepsilon(t'))_{L^2}dt' + \rho^\varepsilon_N.$$

The theorem will be obtained once we prove the following lemmas.

Lemma 4 *Let u (resp. v) be any vector field (resp. divergence free vector field) in $H^2(\Omega)$ (resp. $H^1_0(\Omega)$). If we define*

$$N(u)^2 \stackrel{\text{def}}{=} \sup_{x_3\in[0,1]} \|\nabla^h u(\cdot,x_3)\|^2_{L^2(\mathbf{R}^2)} + \int_0^1 d(x_3)\|\partial_3 u(\cdot,x_3)\|^2_{L^4(\mathbf{R}^2)}dx_3,$$

then we have

$$(v\cdot\nabla v|u)_{L^2} \leq \frac{\nu}{4}\|\nabla^h v\|^2_{L^2} + \frac{C}{\nu}N(u)^2\|v\|^2_{L^2}.$$

In the following lemma we note $O_N(\varepsilon^{\frac12})$ a quantity of the order of magnitude $\varepsilon^{\frac12}$, depending on N.

Lemma 5 *We have*

$$\int_0^t (F^\varepsilon_N(t')|u^\varepsilon(t'))_{L^2}dt' = O_N(\varepsilon^{\frac12}).$$

Before showing how to prove those results, let us finish the proof of the theorem. We get

$$E^\varepsilon(t) \leq \frac{\nu}{2}\int_0^t \|\nabla^h\delta^\varepsilon(t')\|^2_{L^2}dt' + \frac{C}{\nu}\int_0^t N(u^\varepsilon_{app,N}(t'))^2\|\delta^\varepsilon(t')\|^2_{L^2}dt' + \rho^\varepsilon_N.$$

But Lemmas 1 and 2 imply in particular that there is a constant C_N such that for any ε,

$$\int_0^\infty N(u^\varepsilon_{app,N}(t))^2dt \leq C_N\|u_0\|^2_{L^2}.$$

So a Gronwall lemma implies that

$$\sup_{t\geq0}\|u^\varepsilon(t) - u^\varepsilon_{app,N}(t)\|^2_{L^2} + 2\nu\int_0^\infty \|\nabla^h(u^\varepsilon(t') - u^\varepsilon_{app,N}(t'))\|^2_{L^2}dt' = \rho^\varepsilon_N,$$

and the result follows quite easily.

Now let us go back to the two technical lemmas stated above, starting with Lemma 4. We have to estimate

$$I_{j,k} \stackrel{\text{def}}{=} \int_\Omega v^j(x)v^k(x)\partial_k u^j(x)dx.$$

The case when $k \neq 3$ is simply a consequence of a 2D Gagliardo–Nirenberg inequality. In the case $k = 3$, we need the following Hardy–type lemma, whose proof is left out and yields quite directly the expected result.

Lemma 6 *Let v be a divergence free vector field in $H_0^1(\Omega)$. Then we have for almost every $x_h \in \mathbf{R}^2$*

$$\sup_{x_3\in]0,1[} \frac{|v^3(x_h,x_3)|}{d(x_3)^{\frac{1}{2}}} \leq \|\operatorname{div}_h v^h(x_h,\cdot)\|_{L^2(]0,1[)}.$$

The proof of Lemma 5 is much more delicate. To begin with let us decompose F_N^ε as

$$\begin{aligned} F_N^\varepsilon &= F_{N,1}^\varepsilon + F_{N,2}^\varepsilon \quad \text{with} \\ F_{N,1}^\varepsilon &\stackrel{\text{def}}{=} u_{app,N}^\varepsilon \cdot \nabla u_{app,N}^\varepsilon - u_{0,int,N}^\varepsilon \cdot \nabla u_{0,int,N}^\varepsilon \quad \text{and} \\ F_{N,2}^\varepsilon &\stackrel{\text{def}}{=} u_{0,int,N}^\varepsilon \cdot \nabla u_{0,int,N}^\varepsilon - \overline{u}_N \cdot \nabla \overline{u}_N. \end{aligned}$$

Those two terms are estimated in two different ways. The first estimate is achieved in a slightly tedious manner, but it only requires Hardy or Gagliardo–Nirenberg–type techniques and we refer to [5] for details. The second estimate requires the following Strichartz–type inequality (and hence is only true in the $\mathbf{R}^2 \times (0,1)$ case).

Lemma 7 *Let $p \in [1,+\infty]$ be given, and let us define $u_{0,int,N}^\varepsilon \stackrel{\text{def}}{=} \mathcal{L}\left(\frac{t}{\varepsilon}\right) u_N$ where u_N is the solution of*

$$\partial_t u_N - \nu\Delta_h u_N + \mathcal{E} u_N = 0$$

with the initial data $u_{0,N}$. Then we have

$$\|u_{0,int,N}^\varepsilon - \overline{u}_N\|_{L^p(\mathbf{R}^+;L^\infty(\Omega))} \leq C_N \varepsilon^{\frac{1}{2}-\frac{1}{4p}} \|u_0\|_{L^2}.$$

The following corollary is the result of an easy interpolation.

Corollary 8 *For any $p \in [1,+\infty]$, any $\alpha > 0$ and any $q \in]2,+\infty]$, we have*

$$\|\partial^\alpha(u_{0,int,N}^\varepsilon - \overline{u}_N)\|_{L^p(\mathbf{R}^+;L^q(\Omega))} = \rho_N^\varepsilon.$$

We refer once again to [5] for the proof Lemma 7. Note that this Strichartz–type estimate, in the case of rotating fluids, has been investigated in [3] in various situations.

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