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On the stationary Boltzmann equation.*

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Abstract. For stationary kinetic equations, entropy dissipation can sometimes be used in existence proofs similarly to entropy in the time dependent situation. Recent results in this spirit obtained in collaboration with A. Nouri, are here presented for the nonlinear stationary Boltzmann equation in bounded domains of $I\!R^n$ with given indata and diffuse reflection on the boundary.

1 Preliminaries and results

In the approach to existence problems for the time-dependent Boltzmann equation introduced by R. DiPerna and P. L. Lions [7], conservation laws and entropy control are fundamental to obtain necessary à priori bounds and compactness properties. In the corresponding stationary problems, only the flows of such quantities are under control, and these are not by themselves enough to imply the desired results. However, energy control or similar properties are available from moment flows, and mass control may be forced onto the problem at a price. To replace an unavailable entropy bound, there is a weaker and more involved entropy dissipation control.

Using such devices, the last few years with A. Nouri, we have been developing an approach to stationary existence in an L^1- context for nonlinear Boltzmann related equations, also far from maxwellian equilibrium. Concerning the perturbation of a global maxwellian equilibrium on the other hand, that case has been systematically studied already from the late 1960ies onwards. Methods of a more general type can then be used, such as Hilbert space techniques and contraction mappings, the pioneers being H. Grad [9] and J. P. Guiraud [10], cf also [14], [12].

To introduce the observation behind the present large data approach, let us consider the stationary Boltzmann equation (cf [6]) in $\Omega \subset I\!R^n$,

$$v \cdot \nabla_x f(x,v) = Q(f,f), \quad x \in \Omega, v \in I\!R^n, \tag{1.1}$$

where Ω is a strictly convex, bounded domain with C^1- boundary. The nonnegative function f represents the density of a rarefied gas, x is the position, and v

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the velocity. The operator Q is the nonlinear Boltzmann collision operator,

$$Q(f,f)(v) = \int_{I\!R^n} \int_{S^{n-1}} B(v-v_*,\sigma)(f(x,v')f(x,v'_*) - f(x,v)f(x,v_*))dv_*d\sigma =$$

$$= Q^+(f,f) - f\nu(f).$$

The unit sphere in $I\!R^n$ is denoted by S^{n-1}, and the pre- and post-collisional velocities are connected by

$$\begin{aligned} v' = v'(v,v_*,\sigma) &:= \frac{v+v_*}{2} + \frac{\mid v-v_* \mid}{2}\sigma, \\ v'_* = v'_*(v,v_*,\sigma) &:= \frac{v+v_*}{2} - \frac{\mid v-v_* \mid}{2}\sigma. \end{aligned} \tag{1.2}$$

The function B is the kernel of the classical nonlinear Boltzmann operator,

$$\mid v-v_* \mid^\beta b(\sigma) \text{ with } -n<\beta<2, \quad b \in L^1_+(S^{n-1}), \quad b(\sigma) \geq c > 0, \ a.e..$$

The inward and outward boundaries in phase space are

$$\begin{aligned} \partial\Omega^+ &= \{(x,v) \in \partial\Omega \times I\!R^n; v \cdot n(x) > 0\}, \\ \partial\Omega^- &= \{(x,v) \in \partial\Omega \times I\!R^n; v \cdot n(x) < 0\}, \end{aligned}$$

where $n(x)$ denotes the inward normal on $\partial\Omega$.

Given a function $f_b > 0$ defined on $\partial\Omega^+$ and a diffuse reflection operator $\mathcal{R}$, solutions f to (1.1) are sought with boundary conditions

$$f(x,v) = (\theta\mathcal{R}f + (1-\theta)f_b)(x,v), \quad (x,v) \in \partial\Omega^+, \tag{1.3}$$

for some $0 \leq \theta \leq 1$, i.e. a mixture of diffuse reflection and a boundary source. Energy control and entropy dissipation control by boundary value integrals, follow from suitable applications of Green's identity. Here the entropy dissipation is

$$e(f) = \int_{\Omega \times I\!R^{2n} \times S^{n-1}} B(f^{\alpha'} f_*^{\alpha'} - f^\alpha f_*^\alpha) \ln \frac{f^{\alpha'} f_*^{\alpha'}}{f^\alpha f_*^\alpha} dx dv dv_* d\sigma,$$

The entropy dissipation is zero for a maxwellian approximating a Dirac measure, so in general it does not give any information about mass concentrations. But if on sufficiently large sets, $f_* > c_1 > 0$ together with $f', f'_* < c_2$, then $e(f)$ behaves similarly to $\int f \ln f$, and helps to control concentrations. That is often the case in stationary problems. It has turned out that this is enough to obtain existence for the Povzner [3], [13] and Enskog [11] equations in bounded domains in $I\!R^n$, and for the Boltzmann equation in a slab under no other restrictions than Grad's angular cut-off $b \in L^1_+(S^{n-1})$, and $n > 1$. A typical result in the slab case is the following ([2], [4]).

Theorem 1.1. *Consider the stationary Boltzmann equation in a slab,*

$$\xi \frac{\partial}{\partial x} f(x,v) = Q(f,f)(x,v), \; x \in [0,L], \; v \in I\!R^n.$$

For hard forces ($\beta \geq 0$) and given the β-moment $\int (1+|v|)^\beta f dx dv = M$, there is a weak solution to the Boltzmann equation with β-moment M, satisfying a Maxwellian diffuse reflection boundary condition ($\theta = 1$). Also with $\theta = 0$ and the indata f_b on the boundary satisfying some mild integrability conditions, there is a weak solution to the Boltzmann equation in the slab with moment M and boundary profile f_b, i.e. with $f = kf_b$ on the ingoing boundary for some $k > 0$.

The references [2], [4] also contain the same type of results for soft forces, but with renormalized solutions, as well as additional generalizations.

The basic compactness argument used to prove the above cases, is not fully available for the Boltzmann equation itself in more than one space dimension. However, in the spatially n-dimensional case the entropy dissipation estimate still allows different but weaker control mechanisms , which also lead to existence results (see [5]). In contrast to the earlier cases based on our original method, so far complete results are here only obtained when the velocities smaller than some $\eta > 0$ are eliminated. This is connected with the mass only being uniformly controlled by energy away from velocity zero, but not in a neighbourhood of zero. If we were to keep the small velocities, then a variant of the limiting procedure employed, would still work but, besides admitting the desired solution of the boundary value problem, would also allow the (unwanted) alternative of a total collapse of mass at velocity zero.

Mathematically, the imposed small velocity cutoff is a serious problem, but physically less so, if e.g. the velocities were removed only below a Planck scale. Physically more serious is the lack of interesting uniqueness results of any generality, a problem on the other hand shared with the time dependent theory in its present state.

The n-dimensional existence result of [5] may be stated as follows. For $\eta > 0$ given, introduce the cut-off for small velocities,

$$\chi_\eta(v, v_*, \sigma) = 0 \text{ if } |v| < \eta \text{ or } |v_*| < \eta \text{ or } |v'| < \eta \text{ or } |v'_*| < \eta,$$
$$\chi_\eta(v, v_*, \sigma) = 1 \text{ else.}$$

Theorem 1.2. *Suppose that $f_b > ae^{-dv^2}$ for some $a, d > 0$ and a.a. $(x,v) \in \partial\Omega^+$, and that*

$$\int_{(x,v)\in\partial\Omega^+} [v \cdot n(x)(1 + v^2 + \ln^+ f_b(x,v)) + 1] f_b(x,v) dx dv < \infty.$$

Then the Boltzmann equation boundary value problem (1.1), (1.3) with collision kernel $B\chi_\eta$ has an L^1-solution for $0 \leq \theta < 1$.

In the theorem, solutions are understood in renormalized sense or an equivalent one, such as mild, exponential, or iterated integral form (cf [7], [1]). The last two solution concepts in particular, are used in the present proof. Test functions φ are taken in $L^\infty(\bar\Omega \times I\!R^n)$ with compact support, with $v \cdot \nabla_x \varphi \in L^\infty(\Omega \times I\!R^n)$, continuously differentiable along characteristics, and vanishing on $\partial\Omega^-$.

The removal of the small velocity truncation in Theorem 1.2 seems to require fresh ideas, whereas the technical restrictions on B, f_b, and Ω may be considerably relaxed. Under the formulation above, however, it is possible to lay bare the new technical developments without distraction, whereas those ideas would have to be mixed with other heavy but familiar devices in the case of more general B, f_b, and Ω , making the proofs harder to penetrate.

2 On the proof of existence in the $I\!R^n$-case

Without loss of generality, the existence proof for Theorem 1.2 will now be discussed in the case $n = 3$, $\theta = 0$. The *first step in the proof* is to solve the equation with an extra absorption term αf in the equation. The weak form of the equation then becomes

$$\int_{\Omega\times I\!R^3}[-\alpha f^\alpha + f^\alpha v \cdot \nabla_x + Q(f^\alpha, f^\alpha)]\varphi(x,v)dxdv =$$
$$-\int_{\partial\Omega+} v.n(x) f_b \varphi(x,v)dxdv - \int_{\partial\Omega-} v.n(x) f^\alpha \varphi(x,v)dxdv.$$

The collision integral $\int Q(f,f)\varphi dv$ vanishes for $\varphi = 1, v, v^2$, and is non-positive for $\varphi = \ln f$. This leads to à priori α-dependent estimates of mass $\int f$, energy $\int f v^2$, and entropy $\int f \ln f$. Using fixed point arguments and devices related to techniques from the time-dependent case, they imply that the α-approximation has a non-negative solution f^α.

Again using the weak form, we can estimate outgoing mass flow à priori by ingoing mass flow independently of $\alpha > 0$. The exponential form of the equation is

$$f^\alpha(x,v) = f_b(x - s^+(x,v)v, v)e^{-\int_{-s^+(x,v)}^0(\alpha+\nu(f^\alpha)(x+sv,v))ds}$$
$$+\int_{-s^+(x,v)}^0 Q^+(f^\alpha, f^\alpha)(x-\tau v, v)e^{-\int_{-\tau}^0(\alpha+\nu(f^\alpha)(x+tv,v))dt}d\tau.$$

Here s^+ is the time it takes to reach the ingoing boundary point along the characteristic $(x - sv, v)$. It follows that

$$f_b e^{-\int\nu} \le f(x,v) \le f_{outgoing} e^{\int\nu},$$

and so the exponential form gives uniform estimates of f^α along characteristics outside a small set; given $\epsilon > 0$ there is a constant C_ϵ independent of α, so that outside a set (depending on α) of characteristics of measure ϵ, it holds that $f^\alpha < C_\epsilon$. For f_α replaced by zero outside the nicely bounded characteristics, the weak limit $f_\epsilon = w - \lim f^\alpha_{restr}$ increases with $1/\epsilon$.

With the final limit $f = s\text{-}\lim f_\epsilon$ of these approximate solutions as a candidate for a true solution, it remains to prove that it satisfies the desired problem. We use the iterated integral form of the equation, where it is easy to suppress the solution along whole characteristics, by setting the test function equal to zero along them,

$$\int_{\partial\Omega^+} (f_b\varphi)(x,v)v \cdot n(x)dxdv + \int_{\partial\Omega^-} (\int_{-s^+(x,v)}^{0} [-\alpha f\varphi \\ +Q(f,f)\varphi + fv \cdot \nabla_x\varphi](x+\sigma v, v)d\sigma) \mid v \cdot n(x) \mid dxdv = 0.$$

The iterated collision integral is well defined even when Q is not integrable. The replacement of the test functions by zero along certain characteristics is possible, since the test functions are in L^∞ and only required to be differentiable along characteristics.

One difficulty with the removal procedure just described, is the following. Consider the collision frequency $\nu = \int d\sigma \int Bf^\alpha_* dv_*$. It may happen at a point $x \in \Omega$ along a retained characteristic for f^α, that other characteristics through the point $x \in \Omega$ are not retained. This may decrease the collision frequency at x, which is an integral in the second velocity variable v_*. For an approximation of the present type to deliver the correct equation, the effect of that decrease should disappear in the final limit. The *second step in the proof* consists in a study of the interaction between what is retained and what is removed. Finally the *third step in the proof* is a check that the final L^1-limit of the approximations by itself solves the boundary value problem.

A key part of step two and the whole proof, is a lemma quantifying in what sense the contribution of the large f^α-values is small, from those space points that support a "non-negligible amount of good" characteristics. In a following lemma the influence from the large f^α-values of all remaining relevant space points, is then also shown to be negligible in the limit. It is a key observation in the study, that the possibly bad behaviour along a particular small set of characteristics, in the limit does not influence the behaviour based on the rest of phase space, in spite of the mixing non-linear character of the collision operator.

Now to a more detailed presentation of those lemmas. Let ζ_{xv} be a characteristic through $x \in \Omega$ in direction v, and let $\gamma = \frac{v}{|v|}$. Denote by $\mathbf{X}^\alpha_\mathbf{n}(\gamma)$ the subset of Ω consisting of those 'reasonably non-tangential' characteristics in direction γ for which f^α is 'reasonably bounded with the collision frequency integral along ζ_{xv} also reasonably bounded' and this for 'most $|v|$'. (Consult the complete proof in [5] for precise definitions here.) Fix $V >> \eta$, and restrict in velocity space to those v with $\eta \leq |v| \leq V$. Set

$\mathbf{f}^\alpha_\lambda := f^\alpha$ if $f^\alpha \geq \lambda$, $\quad f^\alpha_\lambda := 0$ else,
$\mathbf{a}_0(z) = \max\{1, \log z\}$ and inductively $\mathbf{a}_{i+1}(z) = \max\{1, \log a_i(z)\}$.
$\mathbf{O}_{\alpha,\lambda} := \{x \in \Omega; \int_{\eta \leq |v| \leq V} f^\alpha_\lambda(x,v)dv > 0\}$,
$\mathbf{O}_{\alpha,i,n,\lambda} := \{x \in O_{\alpha,\lambda}; \mathrm{meas}\{\mu \in S^2; x \in X^\alpha_n(\mu)\} > \frac{4\pi}{i}\}$.

Lemma 2.1. *Let V, i, n be given in $I\!N$ and sufficiently large. For λ large enough with respect to V, i, n, it holds that*

$$\int_{O_{\alpha,i,n,\lambda}} \int_{\eta \leq |v| \leq V} f^\alpha_\lambda(x,v)dvdx \leq g_1(i,n,\lambda), \tag{2.1}$$

where the function g_1 does not depend on α,

$$g_1(i,n,\lambda) := \frac{ci^9 n^2 a_{i^3}(\lambda)}{a_{i^3-1}(\lambda)},$$

with c not depending on V, i, n, λ, α.

The lemma holds for $\lambda = e^{e^{\cdot^{\cdot^{e^n}}}}$ with i^4 exponentials, and $n \geq e^{e^{e^i}}$. A discussion of the idea of proof can be found at the end of this paper.

For $\gamma \in S^2$, set

$$\mathbf{A}_{\gamma,\alpha,\mathbf{i},\mathbf{n},\lambda} := X^\alpha_n(\gamma) \cap (O_{\alpha,\lambda} \setminus O_{\alpha,i,n,\lambda}).$$

The contribution of the large f^α-values at the other relevant space points, namely those with "good characteristics but only a negligible amount" of them, is "mostly" small in the following simple way.

Lemma 2.2. *There is a subset $I_{\alpha,i,n,\lambda}$ of S^2 such that $\mid I^c_{\alpha,i,n,\lambda} \mid < \frac{c}{\sqrt{i}}$ and*

$$\int_{A_{\gamma,\alpha,i,n,\lambda}} \int f^\alpha_\lambda(x,v)dvdx < \frac{1}{\sqrt{i}}, \quad \gamma \in I_{\alpha,i,n,\lambda}.$$

Proof. Let χ_A denote the characteristic function of a set A.

$$\int_{S^2} \int_{A_{\gamma,\alpha,i,n,\lambda}} \int f^\alpha_\lambda(x,v)dvdxd\gamma = \int (\int f^\alpha_\lambda(x,v)dv \int_{S^2} \chi_{A_{\gamma,\alpha,i,n,\lambda}}(x)d\gamma)dx$$
$$\leq \frac{4\pi}{i} \int_\Omega \int f^\alpha_\lambda(x,v)dvdx \leq \frac{c}{i}.$$

Here the earlier mentioned energy estimate (uniform in α) is used for the last inequality. So for the inner integral,

$$\int_{A_{\gamma,\alpha,i,n,\lambda}} \int f^\alpha_\lambda(x,v)dvdx > \frac{1}{\sqrt{i}}$$

only holds for directions γ defining a set $I^c_{\alpha,i,n,\lambda}$ in S^2 with an area bounded by $\frac{c}{\sqrt{i}}$. For $\gamma \in I_{\alpha,i,n,\lambda}$, on the other hand

$$\int_{A_{\gamma,\alpha,i,n,\lambda}} \int f^\alpha_\lambda(x,v)dvdx \leq \frac{1}{\sqrt{i}}.$$

□

With these two lemmas, the tools are at hand to prove that the truncation limit f really satisfies the present boundary value problem for the Boltzmann equation, i.e. the third, concluding step in the proof of Theorem 1.2. In order to prove that f is a solution to (1.1), (1.3) in the hard force case, it is enough to prove that the absolute value of the left-hand side of

$$\int_{\partial\Omega^+} (f_b\varphi)(X,v) \mid v\cdot n(X) \mid dXdv + \int_{\partial\Omega^-} (\int_{-s^+(X,v)}^0 [Q(f,f)\varphi$$
$$+fv\cdot\bigtriangledown_x\varphi](X+\sigma v,v)d\sigma) \mid v\cdot n(X)\mid dXdv = 0$$

is smaller than ϵ for any $\epsilon > 0$. Start from the equation for $\chi^\alpha_{\bar{\bar{e}}_k} f^\alpha$, where $\bar{\bar{\mathbf{e}}}_{\mathbf{k}} = (i_k, k, \lambda_k)$ with the sequences (i_k) and (λ_k) increasing to infinity, and $\chi^\alpha_{\bar{\bar{e}}_k}$ is the characteristic function of the remaining phase space, when a suitable small set of characteristics given by $\bar{\bar{e}}_k$ are deleted using Lemma 2.1-2.

Since $\chi^\alpha_{\bar{\bar{e}}}$ commutes with $v\cdot\bigtriangledown_x$, the approximated problem in weak form gives

$$\int_{\partial\Omega^+} (\chi^\alpha_{\bar{\bar{e}}_k} f_b\varphi)(X,v)v\cdot n(X)dXdv$$
$$+\int_{\partial\Omega^-} (\int_{-s^+(X,v)}^0 e^{\alpha(\sigma+s^+(X,v))}[\chi^\alpha_{\bar{\bar{e}}_k} Q(f^\alpha,f^\alpha)\varphi$$
$$+\chi^\alpha_{\bar{\bar{e}}_k} f^\alpha v\cdot\bigtriangledown_x\varphi](X+\sigma v,v)d\sigma) \mid v\cdot n(X)\mid dXdv = 0.$$

By the construction the first term tends to

$$\int_{\partial\Omega^+} (f_b\varphi)(X,v) \mid v\cdot n(X)\mid dXdv,$$

when α (subsequence) tends to zero, and then k tends to infinity. Also the last term

$$\int_{\partial\Omega^-}\int_{-s^+(X,v)}^0 e^{\alpha(s+s^+(X,v))}\chi^\alpha_{\bar{\bar{e}}_k} f^\alpha v\cdot\bigtriangledown_x\varphi(X+\sigma v,v)d\sigma \mid v\cdot n(X)\mid dXdv$$

tends to

$$\int_{\partial\Omega^-}\int_{-s^+(X,v)}^0 fv\cdot\bigtriangledown_x\varphi(X+\sigma v,v)d\sigma \mid v\cdot n(X)\mid dXdv,$$

when $\alpha \to 0$, and then $k \to \infty$. The convergence of the collision term in an essential way depends on Lemma 2.1-2, and follows by an elaboration of the methods from the time-dependent case ([7], [8]). The soft force case is similarly proved.

Let us end with a discussion of the proof of Lemma 2.1. The properties in that lemma are deduced from local aspects of the entropy dissipation control.

A decomposition of the angular directions is needed. Take $i = 2^j$ for $j \in I\!N$. Split S^2 into i disjoint neighborhoods $S_1, ..., S_i$ with piecewise smooth boundaries,

$$\mid S_k \mid = \frac{4\pi}{i}, \quad \mathrm{diam}(S_k) \leq \frac{\bar{c}}{\sqrt{i}}, \quad \text{for} \quad 1 \leq k \leq i,$$
$$-S_k = S_l, \quad \text{for} \quad \text{some} \quad 1 \leq l \leq i,$$

where $\bar{c} \geq 4\pi$ is an i-independent constant. Consider $x \in O_{\alpha,i,n,\lambda}$. Take $1 \leq k \leq i$ such that $|I_x| \geq \frac{4\pi}{i^2}$, where

$$\mathbf{I_x} := S_k \cap \{\mu \in S^2; x \in X_n^\alpha(\mu)\},$$

By the underlying construction $-I_x = S_l \cap \{\mu \in S^2; x \in X_n^\alpha(\mu)\}$. Define

$$\mathbf{V_x} := \{v \in I\!R^3;\ \eta \leq \mid v \mid \leq V, \text{ where } f_\lambda^\alpha(x,v) \text{ is the largest, and}$$
$$\int_{V_x} f_\lambda^\alpha(x,v)dv = i^{-5} \int_{\eta \leq |v| \leq V} f_\lambda^\alpha(x,v)dv\}.$$

It holds that

$$\mid V_x \mid \leq \frac{4}{3}\pi(V^3 - \eta^3)i^{-5}.$$

Divide I_x into four quarters of equal area, and defined by two orthogonally intersecting geodesics in S^2. Let the direction Oz, in velocity space $I\!R^3$, be parallel to the element $\gamma_o \in S^2$ defining the intersection of those two orthogonal geodesics. For v in V_x, consider the plane in velocity space $I\!R^3$, defined by v and Oz. In this plane denote the (normalized) coordinate of v in the γ_o-direction by ζ and the orthogonal coordinate by ξ. We assume $V >> \eta$.

The proof of the estimate (2.1) is split into several cases, depending on the position of v in this plane. The presentation here illustrates the technique in the simplest case

(i): $\mid \xi \mid \leq r$ and $|\zeta| \geq \eta$,

where $r = \frac{V}{10}$. For symmetry reasons it is enough to consider $\xi \geq 0$ and $\zeta \geq \eta$. Take v_* with $\frac{V}{3} \leq \mid v_* \mid \leq \frac{2V}{3}$, with $\frac{v_*}{|v_*|}$ in that quarter of $-I_x$ corresponding to $\xi\xi_* < 0$ and $\zeta\zeta_* < 0$, and with the f^α-values also 'reasonably bounded with respect to n' at the boundary points of the characteristic through x in the v_*-direction. (For details see the complete proof in [5].) Take $\sigma \in S^2$ such that for $V'(v, v_*, \sigma)$ as defined by (1.2), $\frac{V'}{|V'|}$ belongs to I_x, $|V'| > \eta$, and $\zeta\zeta' > 0$. Such σ's form a set of surface area of magnitude $\geq i^{-2}$. For each v_* already chosen, also restrict the set of σ's so that the f^α-values also are 'reasonably bounded with respect to n' at the boundary points of the characteristic through x in the V'-direction. (Again, for details see the complete proof.) For these v the measure of the corresponding (v_*, σ) under the present construction is greater than or equal to $c'i^{-4}$, and $f^\alpha(V'(v, v_*, \sigma)) \leq n$.

Denote by

$$\mathbf{W_{x1}} := \{v \in V_x;\ \ |\xi| \leq r,\ \ \zeta \geq \eta,\ \ \text{meas} T_{xv} \geq \frac{c'i^{-4}}{2}\},$$

where

$$\mathbf{T_{xv}} := \{(v_*, \sigma)\ \text{ as defined above};\ f^\alpha(x, V_*'(v, v_*, \sigma)) \leq \frac{\lambda}{a_2(\lambda)}\},$$

and by

$$\mathbf{W_{x2}} := \{v \in V_x;\ \ |\xi| \leq r,\ \ \zeta \geq \eta\} \setminus W_{x1}.$$

(i)(a) For $v \in W_{x1}$, $(v_*, \sigma) \in T_{xv}$, and writing $f^\alpha = f$,

$$\frac{ff_*}{f'f_*'} \geq \frac{a_2(\lambda)}{n^2} \geq a_3(\lambda).$$

Here given n, the second inequality holds for λ large enough, and implies $\ln \frac{ff_*}{f'f_*'} \geq a_4(\lambda)$. Moreover,

$$f'f_*' \leq f'\frac{\lambda}{a_2(\lambda)} \leq f'\frac{f}{a_2(\lambda)} \leq n\frac{f}{a_2(\lambda)} \leq \frac{f}{2n} \leq \frac{ff_*}{2}.$$

Hence

$$f \leq \frac{2n}{a_4(\lambda)c_b\tilde{c}}B(ff_* - f'f_*')\ln\frac{ff_*}{f'f_*'},$$

where c_b is a positive lower bound of b, and $\tilde{c} = 1$ if $0 \leq \beta < 2$, $\tilde{c} = (2V)^\beta$ if $-3 < \beta < 0$. And so by a uniform entropy dissipation control, integration of this last inequality for $x \in O_{\alpha,i,n,\lambda}$, $v \in W_{x1}$ and $(v_*, \sigma) \in T_{xv}$ gives that

$$\int_{O_{\alpha,i,n,\lambda}} \int_{W_{x1}} f_\lambda^\alpha(x, v)dxdv \leq c\frac{ni^4}{a_4(\lambda)}.$$

(i)(b) For $v \in W_{x2}$, consider as a new set of v_*, the set $\{V_*'(v, v_*, \sigma); (v_*, \sigma) \notin T_{xv}\}$ with elements now denoted by v_*^1. Its volume is of order of magnitude at least i^{-2}. From this set of v_*^1, define $v^{1'}$ and $v_*^{1'}$ either as in (i)(a), or take $v^{1'}$ correspondingly but with $\zeta\zeta^{1'} < 0$, so that, again with $f^\alpha = f$,

$$|V'(v, v_*^1, \sigma)|, |V_*'(v, v_*^1, \sigma)| > \eta, \;\; f(x, v_*^1) \geq \frac{\lambda}{a_2(\lambda)} \geq a_1(\lambda),$$

for λ large enough, and $f(x, v^{1'}) \leq n$. Since the volume of $v_*^{1'}$ is of magnitude $\geq i^{-3}$, there is no loss of generality to restrict the domain of (v_*^1, σ) so that $f(x, v_*^{1'}) \leq f(x, v)$. Hence,

$$f(x,v)f(x,v_*^1) - f(x,v^{1'})f(x,v_*^{1'}) \geq f(x,v)(a_1(\lambda) - n) \geq f(x,v)a_2(\lambda),$$

for λ so large that $a_1(\lambda) - a_2(\lambda) \geq n$. Moreover,

$$\frac{f(x,v)f(x,v_*^1)}{f(x,v^{1'})f(x,v_*^{1'})} \geq \frac{a_1(\lambda)}{n} \geq a_2(\lambda),$$

for λ large enough, so that

$$\ln \frac{f(x,v)f(x,v^1)}{f(x,v^{1'})f(x,v_*^{1'})} \geq a_3(\lambda).$$

Thus again by the uniform entropy dissipation control,

$$\int_{O_{\alpha,i,n,\lambda}} \int_{W_{x2}} f_\lambda^\alpha(x,v)dxdv \leq \frac{ci^4}{a_2(\lambda)a_3(\lambda)}.$$

Together (a) and (b) give for case (i) that

$$\int_{O_{\alpha,i,n,\lambda}} \int_{\eta \leq |v| \leq V, |\xi| \leq r, \zeta \geq \eta} f_\lambda^\alpha(x,v)dxdv \leq ci^9\left(\frac{n}{a_4(\lambda)} + \frac{1}{a_2(\lambda)a_3(\lambda)}\right).$$

This estimate is of the type in the lemma. Adding the other cases, the statement of the lemma follows in a controlled number of steps.

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