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SCATTERING MATRIX IN CONFORMAL GEOMETRY

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1. Statement of the results

This talk describes recent work [9] on the connection between scattering matrices on conformally compact asymptotically Einstein manifolds and conformally invariant objects on their boundaries at infinity. This connection is a manifestation of the general principle that the far field phenomena on a conformally compact Einstein manifold are related to conformal theories on its boundary at infinity. This relationship was proposed in [4] as a means of studying conformal geometry, and the principle forms the basis of the AdS/CFT correspondence in quantum gravity – see [14],[23],[11],[8] and references given there.

We first define the basic objects discussed here. By a conformal structure on a compact manifold M we mean an equivalence class $[h]$ determined by a metric representative h:

$$\hat{h} \in [h] \iff \hat{h} = e^{2\Upsilon} h \,, \quad \Upsilon \in \mathcal{C}^\infty(M) \,.$$

Let X be a compact $n+1$-manifold with $\partial X = M$.[1] Let x be a defining function of ∂X in X:

$$x|_{\overset{\circ}{X}} > 0 \,, \quad x|_{\partial X} = 0 \,, \quad dx|_{\partial X} \neq 0 \,.$$

We say that g is a conformally compact metric on X with conformal infinity $[h]$ if

$$g = \frac{\overline{g}}{x^2} \,, \quad \overline{g}|_{T\partial X} \in [h] \,,$$

where $\overline{g}$ is a smooth metric on X. Since we can choose different defining functions, the metric g determines only the conformal class $[h]$. A conformally compact metric is said to be asymptotically hyperbolic if its sectional curvatures approach -1 at ∂X; this is equivalent to $|dx|_{\overline{g}} = 1$ on ∂X. The basic example is hyperbolic space $\mathbb{H}^{n+1}$, with boundary given by $\mathbb{R}^n$ in the half-space model and by $\mathbb{S}^n$ in the ball model: the two being conformally equivalent.

One of the results of [4] is that given a conformal structure $[h]$ on M, one can construct a conformally compact metric g with conformal infinity $[h]$ which satisfies

$$\text{Ric}(g) + ng = \begin{cases} O(x^\infty) & \text{for } n \text{ odd} \\ O(x^{n-2}) & \text{for } n \text{ even.} \end{cases} \tag{1.1}$$

[1] In general such X might not exist, but we can always consider instead $M \times [0,1]$ with a suitable boundary condition on the second boundary component – see §2.

When n is even, the condition (1.1) is augmented by a vanishing trace condition to the next order. We call a metric g satisfying these conditions asymptotically Einstein. When n is odd, the condition (1.1) together with an asymptotic evenness condition uniquely determine g mod $O(x^\infty)$ up to diffeomorphism. We shall call a metric g which is asymptotically Einstein and which, if n is odd, also satisfies the asymptotic evenness condition, a Poincaré metric associated to $[h]$.

Our first theorem relates the scattering matrix of a Poincaré metric g to the "conformally invariant powers of the Laplacian" on $(M, [h])$. The scattering matrix of (X, g) is a meromorphic family $S(s)$ of pseudodifferential operators on M defined in terms of the behaviour at infinity of solutions of $[\Delta_g - s(n-s)]u = 0$, which we discuss in §2. The conformally invariant powers of the Laplacian are a family P_k, $k \in \mathbb{N}$ and $k \leq n/2$ if n is even, of scalar differential operators on M constructed in [6], which we discuss in §3. These operators are *natural* in the sense that they can be written in terms of covariant derivatives and curvature of a representative metric h, and they are *invariant* in the sense that if $\hat{h} = e^{2\Upsilon}h$, then

$$\widehat{P}_k = e^{(-n/2-k)\Upsilon} P_k \, e^{(n/2-k)\Upsilon} \, . \tag{1.2}$$

The operator P_k has the same principal part as Δ^k (in our convention the Laplacian is a positive operator) and equals Δ^k if h is flat.

Theorem 1. *Let $(M^n, [h])$ be a compact manifold with a conformal structure, and let (X, g) be a Poincaré metric associated to $[h]$. Suppose that $k \in \mathbb{N}$ and $k \leq n/2$ if n is even, and that $(n/2)^2 - k^2$ is* not *an L^2-eigenvalue of Δ_g. If $S(s)$ is the scattering matrix of (X, g), and P_k the conformally invariant operator on M, then $S(s)$ has a simple pole at $s = n/2 + k$ and*

$$c_k P_k = -\mathrm{Res}_{\, s=n/2+k} \, S(s) \, , \quad c_k = (-1)^k [2^{2k} k!(k-1)!]^{-1} \, , \tag{1.3}$$

where $\mathrm{Res}_{\, s=s_0} \, S(s)$ denotes the residue at s_0 of the meromorphic family of operators $S(s)$.

We remark that the condition on the spectrum is automatically satisfied if $k \geq n/2$ and in general can be guaranteed by perturbing the metric g in the interior. Since $S(s)$ is self-adjoint for $s \in \mathbb{R}$, as an immediate consequence of Theorem 1 and of this remark we obtain

Corollary. *The conformally invariant operators, P_k, are self-adjoint.*

This was previously known only for small values of k for which the operators can be explicitly calculated. The first two of the invariant operators are the conformal Laplacian

$$P_1 = \Delta + \frac{n-2}{4(n-1)} R$$

and the Paneitz operator

$$P_2 = \Delta^2 + \delta T d + (n-4)(\Delta J + \frac{n}{2} J^2 - 2|P|^2)/2.$$

Here R denotes the scalar curvature, $J = R/(2(n-1))$, $P_{ij} = \frac{1}{n-2}(R_{ij} - Jh_{ij})$ where R_{ij} is the Ricci curvature, $T = (n-2)Jh - 4P$ acting as an endomorphism on 1-forms, $|P|^2 = P_{ij}P^{ij}$, and δ is the adjoint of d (the divergence operator).

Another important notion of conformal geometry is Branson's Q-curvature in even dimensions. It is a scalar function on M constructed from the curvature tensor and its covariant derivatives, with an invariance property generalizing that of scalar curvature in dimension two: if once again $\hat{h} = e^{2\Upsilon}h$, then

$$e^{n\Upsilon}\widehat{Q} = Q + P_{n/2}\Upsilon\,. \tag{1.4}$$

There has been great progress recently in understanding the Q-curvature and its geometric meaning in low dimensions and on conformally flat manifolds – see [3] for an example of recent work. However, in general it remains a rather mysterious quantity – its definition (given in [2] and reviewed in §3 below) in the general case is via analytic continuation in the dimension. In dimension two it is given by $Q = R/2$, and in dimension four by $6Q = \Delta R + R^2 - 3|\mathrm{Ric}|^2$.

If n is even, then the operator $P_{n/2}$ has no constant term, i.e. $P_{n/2}1 = 0$. It therefore follows from Theorem 1 that $S(s)1$ extends holomorphically across $s = n$, so $S(n)1 = \lim_{s\to n} S(s)1$ is a well-defined function on M.

Theorem 2. *With the notation of Theorem 1, for n even, we have*

$$c_{n/2}Q = S(n)1\,. \tag{1.5}$$

Theorem 2 can be used as an alternative definition of the Q-curvature. The conformal transformation law (1.4) is an easy consequence of Theorems 1 and 2. It follows from (1.4), the self-adjointness of $P_{n/2}$, and the fact that $P_{n/2}1 = 0$, that $\int_M Q$ is a conformal invariant. For $(M,[h])$ conformally flat, $\int_M Q$ is a multiple of the Euler characteristic $\chi(M)$.

A specific mathematical object which appeared in the study of the AdS/CFT correspondence is the renormalized volume of an asymptotically hyperbolic manifold (X,g) – see [5] for a discussion and references. It has also appeared in geometric scattering theory – see §2. A metric h in the conformal class on M uniquely determines a defining function x and a product identification near ∂X so that g takes the form $g = x^{-2}(dx^2 + h_x)$, where h_x is a one-parameter family of metrics on M with $h_0 = h$. The renormalized volume is defined as the finite part in the expansion of $\mathrm{vol}_g(\{x > \epsilon\})$ as $\epsilon \to 0$. For asymptotically Einstein metrics the expansions take a special form

$$\mathrm{vol}_g(\{x > \epsilon\}) = c_0\epsilon^{-n} + c_2\epsilon^{-n+2} + \cdots + c_{n-1}\epsilon^{-1} + V + o(1)$$

for n odd,

(1.6)

$$\mathrm{vol}_g(\{x > \epsilon\}) = c_0\epsilon^{-n} + c_2\epsilon^{-n+2} + \cdots + c_{n-2}\epsilon^{-2} + L\log(1/\epsilon) + V + o(1)$$

for n even.

It turns out that for asymptotically Einstein metrics g, V is independent of the conformal representative h on the boundary at infinity when n is odd, and L is independent of the conformal representative when n is even. The dependence of V on the choice of h for n even is the so-called holographic anomaly – see [11],[5]. Anderson [1] has recently identified V when $n = 3$. In an appendix to [21], Epstein shows that for conformally compact hyperbolic manifolds, the invariants L for n even and V for n odd are each multiples of $\chi(X)$.

Using the connection with the scattering matrix, we are able to identify L in terms of the Q-curvature:

Theorem 3. *Let n be even and let L be defined by* (1.6)*. Then*

$$L = 2c_{n/2}\int_M Q\,, \tag{1.7}$$

where $c_{n/2}$ is defined in (1.3)*.*

We should stress that despite the fact that the scattering matrix is a global object, in some sense our results are all formal Taylor series statements at the boundary of X. In fact, it is possible to give direct proofs of the self-adjointness of the P_k's and of Theorem 3 based on the same ideas but which avoid the analytic continuation via the scattering matrix. It is nevertheless worthwhile to proceed with the full scattering theory: as a byproduct, this allows us to clarify certain confusing issues about the infinite rank poles of the scattering matrix at $s = n/2 + k/2$, $k \in \mathbb{N}$.

In the remaining two sections we will describe scattering theory on conformally compact manifolds and the relevant aspects of conformal geometry.

2. Scattering Theory

To introduce scattering theory in the context of our work, it is best to start with the traditional presentation from any basic text in quantum mechanics. Thus we will consider scattering on a half line, $X = [0,\infty)$ with a compactly supported real valued potential $V \in L^\infty_{\mathrm{comp}}(X;\mathbb{R})$. The quantum Hamiltonian is given by

$H = -\partial_y^2 + V(y)$ and we impose, say, the Dirichlet boundary condition at $y = 0$. We are interested in the properties of generalized eigenfunctions at energies $-s^2$:

$$(H + s^2)u(y) = 0\,, \quad u(0) = 0\,,$$

which for large values of y satisfy

$$u(y) = A(s)e^{sy} + B(s)e^{-sy}. \tag{2.1}$$

For $s \in i\mathbb{R}$, consideration of the Wronskian of u and $\bar{u}$ shows that $|A(s)|^2 = |B(s)|^2$. Normalizing, we define the scattering matrix, $S(s)$ (which in this case is a one-by-one matrix!) by

$$S(s) = \frac{B(s)}{A(s)}\,.$$

This defines a meromorphic function of $s \in \mathbb{C}$. The definition shows that $S(s) = S(-s)^{-1}$ and that for $s \in i\mathbb{R}$, $S(s)^* = S(s)^{-1}$. Hence for all s we have

$$S(-\bar{s})^* = S(s)^{-1}\,, \quad S(\bar{s})^* = S(s)\,, \tag{2.2}$$

so that, in particular, $S(s)$ is self-adjoint for s real. When V is compactly supported then L^2-eigenvalues of H, $E \leq 0$, correspond to the poles of $S(s)$, $E = -s^2$, $\operatorname{Re} s > 0$. This follows easily from (2.1) and self-adjointness of H. The poles of S for $\operatorname{Re} s < 0$ correspond to *resonances* – see [24].

When V is not compactly supported, then under relatively mild assumptions we still have the scattering matrix for $s \in i\mathbb{R}$ but its meromorphic continuation becomes very sensitive to the behaviour of V at infinity. The first physical case in which these difficulties[2] occurred was that of Yukawa potential, $V(y) = e^{-y}$.

We can study the Yukawa potential scattering using simple regular singular point analysis, not unlike what one encounters in the study of the free hyperbolic space. We start by making a change of variables:

$$x = e^{-y}\,, \quad H + s^2 = -(x\partial_x)^2 + x + s^2\,, \quad X = (0, 1]\,.$$

Note that the definition of the scattering matrix in the new variables is

$$\begin{aligned} &(H + s^2)u(x) = 0\,, \quad u(1) = 0\,, \quad s \in i\mathbb{R}\,, \\ &u(x) = x^{-s} + S(s)x^s + O(x)\,, \quad x \to 0\,. \end{aligned} \tag{2.3}$$

We obtain $u(x)$ from the two solutions of $(H + s^2)G = 0$:

$$G_\pm(x, s) = x^{\pm s} \sum_{j=0}^{\infty} b_j^\pm(s)x^j\,, \quad b_j^\pm(s) = [j!\Gamma(\pm 2s + j + 1)]^{-1}\,. \tag{2.4}$$

[2]This is potentially rather confusing. The "false" poles of the scattering matrix discussed below almost led Heisenberg to abandoning his S-matrix formalism, until a clear explanation was provided by Jost – see [20].

These are independent provided $2s \notin \mathbb{N}$. The scattering matrix is obtained by finding a combination matching the boundary conditions:

$$S(s) = -\frac{G_-(1,s)b_0^+(s)}{G_+(1,s)b_0^-(s)}.$$

For $k \in \mathbb{N}$, $G_+(x,k/2) = G_-(x,k/2)$ and $b_0^+(k/2) \neq 0$, $b_0^-(k/2) = 0$. Hence $S(s)$ has a simple pole at $s = k/2$, $k \in \mathbb{N}$.

Another independent solution for $s = k/2$ is obtained by taking

$$\begin{aligned} &\partial_s(G_-(x,s) - G_+(x,s))|_{s=k/2} \\ &= x^{-\frac{k}{2}} \left(\sum_{j=0}^{k-1} \partial_s b_j^-(k/2)x^j - 2b_k^-(k/2)x^k \log x + O(x^k) \right), \end{aligned}$$

and dividing by $\partial_s b_0^-(k/2)$ gives us a solution with the prescribed leading part, as in (2.3):

$$u_{k/2}(x) = x^{-k/2} + \cdots + p_{k/2}x^{k/2}\log x + O(x^{k/2}). \tag{2.5}$$

Comparison of the definition of the scattering matrix with the expansions gives

$$p_{k/2} = 2\mathrm{Res}_{\,s=k/2}\, S(s), \tag{2.6}$$

where $p_{k/2}$ appears in (2.5).

Comparison of this elementary discussion with the way in which the invariant operators P_k are formulated in §3 provides motivation for Theorem 1. To make this clear we now move to the geometric setting.

The class of asymptotically hyperbolic manifolds, (X, g), comes with a well developed scattering theory, which originated in the study of infinite volume hyperbolic quotients $\Gamma\backslash\mathbb{H}^{n+1}$ by Patterson, Lax-Phillips, Agmon, Guillopé, Perry and others – see [22], and references given there.

In the general setting the crucial result was obtained by Mazzeo-Melrose [17]: the resolvent

$$R_g(s) = (\Delta_g - s(n-s))^{-1},$$

continues meromorphically from $\mathrm{Re}\, s > n$ to $\mathbb{C}$, with only real poles in $\mathrm{Re}\, s \geq n/2$, $s \neq n/2$, corresponding to eigenvalues $s(n-s)$ of Δ_g. This was done constructively so that the structure of $R_g(s)$ became well understood, as will be briefly reviewed below. The work of Mazzeo [15], [16] provided a description of the mapping properties of R_g, and of the asymptotic expansions of the solutions of the eigen-equation. The characteristic exponents are $s, n-s$; one concludes that when the difference $2s - n$ is not an integer, regular solutions of

$$[\Delta_g - s(n-s)]u = 0 \tag{2.7}$$

have the form

$$u = x^{n-s}A + x^s A' \tag{2.8}$$

with $A, A' \in \mathcal{C}^\infty(X)$. As we shall see below, for such s satisfying also $\operatorname{Re} s \geq n/2$ and $s(n-s) \notin \operatorname{spec}(\Delta_g)$, the scattering matrix $S(s)$ is the operator which maps $A|_{\partial X} \to A'|_{\partial X}$.

Scattering theoretical consequences of the resolvent construction were sketched in [18],[10], and carried out in [12] (both [10] and [12] had different specific goals). The starting point here is the pairing formula on the unitarity axis, the analogue in this setting of Green's identity: let $\dot{\mathcal{C}}^\infty(X)$ denote functions vanishing to infinite order at ∂X, then for $\operatorname{Re} s = n/2$, if u_1, u_2 satisfy

$$(\Delta_g - s(n-s))u_i = r_i \in \dot{\mathcal{C}}^\infty(X),$$
$$u_i = x^{n-s}a_i^- + x^s a_i^+ + O(x^{n/2+1}), \quad a_i^\pm \in \mathcal{C}^\infty(\partial X),$$

then

$$\int_X (u_1\bar{r}_2 - r_1\bar{u}_2)\, dv_g = (2s-n)\int_{\partial X} \left(a_1^-\bar{a}_2^- - a_1^+\bar{a}_2^+\right) dv_h, \tag{2.9}$$

where, since we chose the defining function of ∂X, x, we have a natural choice in the conformal class, h. To have the coefficients $a_i^\pm$ invariantly defined we can introduce density bundles of [19], $|N^*\partial X|^s$, which keep track of changes of the defining function, or equivalently of the choice of the metric in the conformal class[3]: somewhat informally,

$$f \in \mathcal{C}^\infty(\partial X, |N^*\partial X|^s) \iff f = a|dx|^s, \; a \in \mathcal{C}^\infty(\partial X).$$

Using the resolvent and its mapping properties we can construct a holomorphic family of *Poisson operators*, for s with $\operatorname{Re} s \geq n/2$, $s(n-s) \notin \operatorname{spec}(\Delta_g)$, having the properties:

$$\begin{gathered} P(s) : \mathcal{C}^\infty(\partial X, |N^*\partial X|^{n-s}) \longrightarrow \mathcal{C}^\infty(\overset{\circ}{X}), \\ (\Delta_g - s(n-s))P(s)f = 0, \quad f = a|dx|^{n-s}, \\ P(s)f = ax^{n-s} + o(x^{n-s}), \; \operatorname{Re} s > n/2, \\ P(s)f = ax^{n-s} + a'x^s + O(x^{n/2+1}), \quad a' \in \mathcal{C}^\infty(\partial X), \; \operatorname{Re} s = n/2. \end{gathered} \tag{2.10}$$

The pairing formula (2.9) gives an expression for the kernel of $P(s)$ in terms of that of the resolvent, much in the same spirit as in the classical derivation of the Poisson kernel from the Green function:

$$P(s) = (2s-n)x'^{-s}R_g(s)|_{x'=0}, \tag{2.11}$$

where x' denotes the defining function in the integration variables.

[3]These bundles are equivalent to the density bundles of conformal geometry – see §3.

The scattering matrix, $S(s)$, is defined as in the one dimensional case (2.3) (corresponding to $n = 0$), as the map taking a to a' in (2.10). It now turns out to have a meromorphic extension to $\mathbb{C}$ as a family of pseudo-differential operators:

$$S(s) \in \Psi^{2\operatorname{Re} s - n}(\partial X; |N^*\partial X|^{n-s}, |N^*\partial X|^{s}) \,,$$

relating the incoming and outgoing fields.

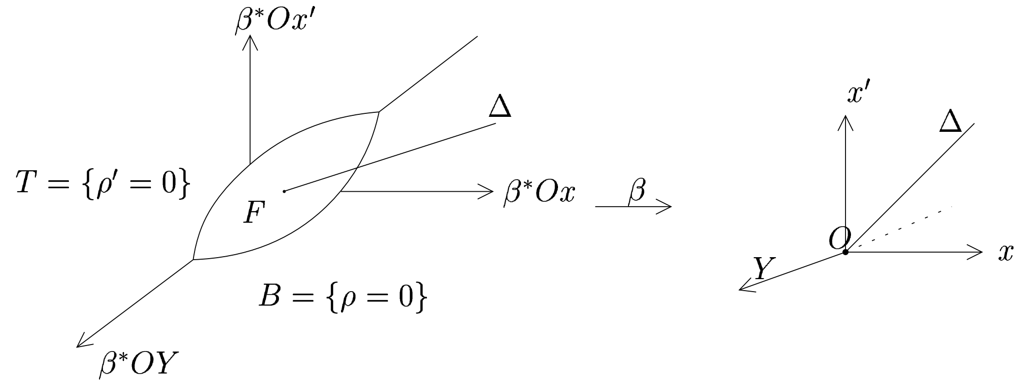

FIGURE 1. The boundary faces of the blown-up space $X \times_0 X$. The variable Y stands for the defining function of the diagonal of in $\partial X \times \partial X$, $Y = y - y'$, in local coordinates.

Following [10] we can give an elegant description of the structure of the scattering matrix using the Mazzeo-Melrose construction. That implies the pseudo-differential character of the scattering operator and describes the structure of the non-spectral poles in $\operatorname{Re} s > n/2$. To do this we recall the blow-down map of [17]:

$$\beta \,:\, X \times_0 X \longrightarrow X \times X \,,$$

where $X \times_0 X$ is the blow-up of $X \times X$ along the boundary diagonal, illustrated in Fig.1. The restriction to the boundary gives us a blow-up of the diagonal in $\partial X \times \partial X$ with the corresponding blow-down map:

$$\beta_\partial \,:\, T \cap B \longrightarrow \partial X \times \partial X \,,$$

where T and B are the top and bottom faces in the blow-up, as shown in Fig.1.

In these terms we obtain, for the kernels of the operators, essentially from (2.11),

$$S(s) = (2s - n)(\beta_\partial)_* \left(\beta^* \left(x^{-s} x'^{-s} R_g(s) \right) |_{T \cap B} \right) \,. \tag{2.12}$$

Roughly speaking, the results of [17] show that away from the lift of the diagonal to $X \times_0 X$, the lift of the kernel of the resolvent is a smooth function multiplied by $\rho^s \rho'^s$ where ρ, ρ' are the defining functions of the top and bottom faces respectively.

This analysis gives a precise description of scattering matrices:

$$\begin{gathered} S(s) \text{ is meromorphic in } \mathbb{C}, \\ S(\bar{s})^* = S(s)\,, \quad S(s)S(n-s) = Id\,, \\ \sigma(S(s)) = 2^{n-2s}\frac{\Gamma(n/2-s)}{\Gamma(s-n/2)}\sigma(\Delta_h^{s-n/2}). \end{gathered}$$

In addition to the spectral poles, $S(s)$ has poles of infinite rank at $s = n/2 + k$, and possibly at $s = n/2 + k - 1/2$, $k \in \mathbb{N}$. These are the values of s for which logarithmic terms may enter in (2.8). The poles at $s = n/2 + k - 1/2$ do not occur for Poincaré metrics.

When $n = 1$, the renormalized volume V in (1.6) appears in the asymptotics of the scattering phase, which is a natural object obtained from the scattering matrix – see [10]. It is expected that analogous expansions should be valid in higher dimensions. In that direction, Joshi and Sá Barreto [13] showed that the coefficient of the leading singularity of the "0-trace" of the wave group on an asymptotically hyperbolic manifold is essentially given by V.

We conclude with the comment that the scattering theory still goes through if X has additional boundary components near which g is smooth and on which one imposes elliptic boundary conditions. Therefore, for any conformal manifold $(M, [h])$, we can take $X = M \times [0, 1]$, with a metric which is asymptotically Einstein at $x = 0$ and smooth at $x = 1$. For Theorem 2 we choose the Neumann boundary condition so that $u \equiv 1$ is a solution for $s = n$. Alternately, we can take g to be asymptotically Einstein with conformal infinity $(M, [h])$ at each boundary component.

3. Conformal Geometry

Branson's Q-curvature is defined in terms of the invariant operators P_k, which were constructed in [6] via the ambient metric of [4]. We begin by reviewing the constructions of these objects, then discuss how the P_k may be reformulated in terms of Poincaré metrics, and conclude with a brief indication of the proofs of Theorems 1, 2, and 3.

The metric bundle of $(M, [h])$ is the ray subbundle $G \subset S^2T^*M$ of multiples of the metric: if h is a representative metric, then the fiber of G over $p \in M$ is $\{t^2h(p) : t > 0\}$. Denote by $\pi : G \to M$ the natural projection, and by h_0 the tautological symmetric 2-tensor on G defined for $(p, h) \in G$ and $X, Y \in T_{(p,h)}G$ by $h_0(X, Y) = h(\pi_*X, \pi_*Y)$. There are dilations $\delta_s : G \to G$ for $s > 0$ given by $\delta_s(p, h) = (p, s^2h)$, and we have $\delta_s^*h_0 = s^2h_0$. Denote by T the infinitesimal dilation vector field $T = \frac{d}{ds}\delta_s|_{s=1}$. Define the ambient space $\tilde{G} = G \times (-1, 1)$. Identify G with its image under the inclusion $\iota : G \to \tilde{G}$ given by $\iota(g) = (g, 0)$ for $g \in G$. The dilations δ_s and infinitesimal generator T extend naturally to $\tilde{G}$.

The ambient metric $\tilde{g}$ is a Lorentzian metric on $\tilde{G}$ which satisfies the initial condition $\iota^*\tilde{g} = h_0$, is homogeneous in the sense that $\delta_s^*\tilde{g} = s^2\tilde{g}$, and is an asymptotic solution of $\mathrm{Ric}(\tilde{g}) = 0$ along G. For n odd, these conditions uniquely determine a formal power series expansion for $\tilde{g}$ up to diffeomorphism, but for n even and $n > 2$, a formal power series solution exists in general only to order $n/2$.

The space of conformal densities of weight $w \in \mathbb{C}$ is

$$\mathcal{E}(w) = \mathcal{C}^\infty(M; G^{\frac{w}{2}}) \,,$$

where by abuse of notation we have denoted by G also the line bundle associated to the ray bundle defined above. There is a canonical isomorphism

$$\mathcal{E}(w) \simeq \mathcal{C}^\infty(M, |N^*\partial X|^{-w}) \,,$$

in the notation of the previous section. The invariance property (1.2) can be reformulated as the statement that P_k is an invariantly defined operator

$$P_k : \mathcal{E}(-n/2 + k) \to \mathcal{E}(-n/2 - k).$$

An element of $\mathcal{E}(w)$ can be regarded as a homogeneous function of degree w on G. One of the ways that P_k is derived in [6] is as the obstruction to extending $f \in \mathcal{E}(-n/2 + k)$ to a smooth function F on $\tilde{G}$, such that F is homogeneous of degree $-n/2 + k$ and satisfies $\tilde{\Delta}F = 0$, where $\tilde{\Delta}$ denotes the Laplacian in the metric $\tilde{g}$. The Taylor expansion of F is formally determined to order $k - 1$, but there is an obstruction at order k which defines the operator P_k. Equivalently, one may include a logarithmic term at this order in the expansion for F, and $P_k f$ is a multiple of the coefficient of this log term, normalized so that the principal part of P_k agrees with that of Δ^k. Note that if n is even, then $P_{n/2}$ is invariantly defined from $\mathcal{E}(0)$ to $\mathcal{E}(-n)$, that is, from $C^\infty(M)$ to the space of volume densities. Since the constant function $1 \in \mathcal{E}(0)$ has a smooth homogeneous extension annihilated by $\tilde{\Delta}$, we have $P_{n/2}1 = 0$.

We next define the Q-curvature as in [2]. For this discussion we will denote by $P_{n,k}$ the operator P_k in dimension n. Fix $k \in \mathbb{N}$. One consequence of the construction above is that the operator $P_{n,k}$ is natural in the strong sense that $P_{n,k}f$ may be written as a linear combination of complete contractions of products of covariant derivatives of the curvature tensor of a representative for the conformal structure with covariant derivatives of f, with coefficients which are rational in the dimension n. Also, the zeroth order term of $P_{n,k}$ may be written as $(n/2 - k)Q_{n,k}$ for a scalar Riemannian invariant $Q_{n,k}$ with coefficients which are rational in n and regular at $n = 2k$. (This is consistent with the fact mentioned above that $P_{n,n/2}1 = 0$.) The Q-curvature in even dimension n is then defined as $Q = Q_{n,n/2}$.

Branson also derived the transformation law (1.4) by analytic continuation in the dimension. Again fix k. Apply (1.2) to the function 1 to obtain

$$(n/2 - k)e^{(n/2+k)\Upsilon}\widehat{Q_{n,k}} = (n/2 - k)Q_{n,k} + P_{n,k}(e^{(n/2-k)\Upsilon} - 1). \tag{3.1}$$

This relation holds for any metric in dimension $n \geq 2k$. Given a fixed metric h^0 in dimension $2k$, we apply (3.1), taking $h = h^0 + h^E$, where h^E denotes the Euclidean metric in $\mathbb{R}^{n-2k}$, and taking Υ to be independent of the additional $\mathbb{R}^{n-2k}$ variables. Since the curvature of h can be identified with that of h^0, we deduce that (3.1) holds for a fixed metric in dimension $2k$, where now $n \in \mathbb{N}$, $n \geq 2k$ is a formal parameter. However, since the coefficients of $Q_{n,k}$ and $P_{n,k}$ are rational in n, we may divide by $n/2 - k$ and continue analytically to $n = 2k$ to conclude that for n even we have (1.4).

In order to make the connection with scattering theory, it is necessary to reformulate the operators P_k in terms of Poincaré metrics. As described in [4], the formal Poincaré metric associated to a conformal structure can be constructed from the ambient metric and vice versa; the two constructions are equivalent. In the ambient space $\tilde{G}$, the equation $\tilde{g}(T,T) = -1$ defines a hypersurface $\overset{\circ}{X}$ which lies on one side of G and which intersects exactly once each dilation orbit on this side of G. The Poincaré metric g is the pullback to $\overset{\circ}{X}$ of $\tilde{g}$. The equation $\mathrm{Ric}(\tilde{g}) = 0$ is equivalent to $\mathrm{Ric}(g) = -ng$. To see this, one uses the fact from [4] that in suitable coordinates on $\tilde{G}$, the ambient metric takes the form

$$\tilde{g} = 2tdtd\rho + 2\rho dt^2 + t^2 \sum_{i,j=1}^{n} \overline{h}_{ij}(y,\rho) dy^i dy^j. \tag{3.2}$$

Here ρ is a defining function for $G \subset \tilde{G}$, t is homogeneous of degree 1 with respect to the dilations on $\tilde{G}$, and the y^i arise from a coordinate system on M. In these coordinates we have $T = t\partial_t$, so $\overset{\circ}{X} = \{2\rho t^2 = -1\}$. Introduce a new variable $x = \sqrt{-2\rho}$ and set $s = xt$ so that $\overset{\circ}{X} = \{s = 1\}$. A straightforward calculation shows that (3.2) becomes

$$\tilde{g} = s^2 g - ds^2, \tag{3.3}$$

where

$$g = x^{-2}[h_{ij}(y,x) dy^i dy^j + dx^2] \tag{3.4}$$

and $h_{ij}(y,x) = \overline{h}_{ij}(y,\rho)$. The equivalence of $\mathrm{Ric}(\tilde{g}) = 0$ and $\mathrm{Ric}(g) = -ng$ is a straightforward calculation given the relationship (3.3) (see Proposition 5.1 of [7]). Therefore, g is asymptotically Einstein and (3.4) shows that g is conformally compact with conformal infinity $(M, [h])$. Note that $h_{ij}(y,x)$ is even in x; this is the asymptotic evenness condition referred to in §1. If n is odd, the Taylor expansion of $h_{ij}(y,x)$ is uniquely determined if we impose this evenness condition; otherwise not.

Rewrite (3.3) as $\tilde{g} = s^2(g - ds^2/s^2)$ and transform under a conformal change to obtain $\tilde{\Delta} = s^{-2}[\Delta_g + (s\partial_s)^2 + ns\partial_s]$. If F is homogeneous of degree w, we therefore have

$$\tilde{\Delta} F = s^{-2}[\Delta_g + w(w+n)]F.$$

Let $u = F|_{\overset{\circ}{X}}$. We may regard u as a function homogeneous of degree 0 and write $F = s^w u = t^w x^w u$. As described above, the operator P_k arises as the obstruction to finding a smooth F solving $\tilde{\Delta} F = 0$ and prescribed at $\rho = 0$. In order for F to be smooth up to $\rho = 0$, we require therefore that $x^w u$ be smooth up to $x = 0$ (and be even in x). Taking $w = -n/2 + k$, we see that P_k may be characterized as the normalized obstruction operator for the problem of finding a function u on $\overset{\circ}{X}$ solving $(\Delta_g - (n/2-k)(n/2+k))u = 0$ with $x^{-n/2+k}u$ smooth in X and prescribed at $x = 0$. As above, this characterization may be reformulated in terms the coefficient of the log term in a nonsmooth formal solution u.

As described in §2, the scattering matrix $S(s)$ is defined in terms of the behaviour at $\{x = 0\}$ of solutions of (2.7). Observe that if $s = n/2 + k$, this is precisely the equation which arose above in the characterization of P_k. If $\operatorname{Re} s > n/2$ and $s \notin n/2 + \mathbb{N}$, then given $f = a|dx|^{n-s} \in \mathcal{E}(s-n)$ with $a \in \mathcal{C}^\infty(M)$, one can construct a formal solution u to (2.7) with $x^{s-n}u$ smooth and equal to a at $x = 0$. The coefficients in the expansion of this solution are differential operators on M applied to a, depending on s. Certain of these coefficients have poles for $s = n/2 + k$ corresponding to the fact that the formal solution breaks down for such s. On the other hand, according to (2.10), $u = P(s)f$ is a solution which varies holomorphically in s right across $s = n/2 + k$. By comparing with (2.8), one can derive that the poles in the scattering matrix must cancel the poles in the formal solution, leading to Theorem 1. Theorem 2 is proved by a similar analysis using the fact that $P(n)1 = 1$.

Theorem 3 follows from a more complicated version of the boundary pairing (2.9), on the real axis and involving taking finite parts. More precisely, we have

Proposition. *Suppose $s \in \mathbb{R}$, $s > n/2$ and $s \notin \{n/2 + k : k \in \mathbb{N}\}$, $s(n-s) \notin \operatorname{spec}(\Delta_g)$. Let $f_1, f_2 \in \mathcal{E}(s-n)$ be real-valued and set $u_j = P(s)f_j$, $j = 1, 2$. Then*

$$\operatorname{pf} \int_{x>\epsilon} [\langle du_1, du_2\rangle - s(n-s)u_1u_2]dv_g = -n \int_M f_1 S(s) f_2 dv_h\,. \tag{3.5}$$

Here, pf denotes the finite part of the divergent integral.

A special case of (3.5) was discussed by Witten [23] in a physical context, for $X = \mathbb{H}^{n+1}$, and using explicit formulæ for the Poisson operator and the scattering matrix. Our project originated from an attempt to understand that discussion in the general setting. Theorem 3 follows upon taking $f_1 = f_2 = 1 \cdot |dx|^{n-s}$ in (3.5) and letting $s \to n$. By Theorem 2, the right hand side converges to $-nc_{n/2} \int Q$. A rather intricate analysis shows that the left hand side converges to $-nL/2$.

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