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1 Introduction

In 1956 Mark Kac [5] invented a microscopic, linear model, from which the nonlinear Boltzmann equation describing the evolution of the velocity distribution for a system of colliding particles could be rigorously derived. Consider N particles in one dimension that interact through random collisions. We consider the spatially homogeneous case only, i.e., the case where the particles are uniformly distributed in configuration space, since our focus, as was Kac's, is on the collision mechanism.

The collisions in the model are binary. When the i-th and j-th particles collide, their pre–collisional velocities v_i and v_j are transformed into the post collisional velocities

$$v_i^*(\theta) = \cos(\theta)v_i + \sin(\theta)v_j \ , v_j^*(\theta) = -\sin(\theta)v_i + \cos(\theta)v_j \tag{1}$$

for some value of θ, the "scattering angle". Clearly, the total kinetic energy of these N particles, i.e.,

$$\vec{v}^2 = \sum_{i=1}^{N} v_i^2 = E \tag{2}$$

is conserved by these collisions. Denote the sphere defined by (2) by $S^{N-1}(\sqrt{E})$. Notice that the total momentum is not conserved. In one dimension, both momentum and energy conservation would require that either the particles keep their momenta or exchange them, a process that will certainly not be ergodic on $S^{N-1}(\sqrt{E})$.

Kac's model can now be described as follows. Let f be a function in $L^2(S^{N-1}(\sqrt{E}))$ and define the operator

$$Qf(\vec{v}) = \frac{1}{2\pi \binom{N}{2}} \sum_{i<j} \int_0^{2\pi} f(v_1, \cdots, v_i^*(\theta), \cdots, v_j^*(\theta), \cdots, v_N) \mathrm{d}\theta \ . \tag{3}$$

It is easily checked that this operator is selfadjoint on $L^2(S^{N-1}(\sqrt{E}))$, i.e., $(f, Qg) = (Qf, g)$ where $(\cdot, \cdot)$ denotes the inner product in $L^2(S^{N-1}(\sqrt{E}))$.

The following expression describes the random collision of these particles as a time evolution of the initial probability density $f_0 \in L^2(S^{N-1}(\sqrt{E}))$

$$f(\vec{v}, t) = \left[e^{-Nt(I-Q)} f_0 \right] (\vec{v}, t) . \tag{4}$$

The assumptions underlying this model will be discussed in Section 2.

The connection beteen this N–particle model and kinetic theory; i.e., with the Boltzmann equation, stems from what Kac called *propagation of chaos*: It is fairly obvious that if the initial condition f_0 in (4) is the product of its marginals, this property is not shared by $f(\cdot, t)$. However, this is *almost* true in the limit of infinitely many particles. More precisely, set $E = NT$ and for any probability distribution f on$S^{N-1}(\sqrt{E})$, define the *single particle marginal*

$$f_1^N(v_1) = \int_{v_2^2+\cdots+v_N^2=NT-v_1^2} f(\vec{v}) \mathrm{d}^{(N-2)} S \tag{5}$$

and more generally, define the *k-particle marginal*

$$f_1^N(v_1, \cdots, v_k) = \int_{v_{k+1}^2+\cdots+v_N^2=NT-v_1^2-\cdots-v_k^2} f(\vec{v}) \mathrm{d}^{(N-2)} S \tag{6}$$

for all fixed finite k. The following definition is due to Kac. A sequence of probability distributions $g_N(v_1, \cdots, v_N)$ has the **Boltzmann Property** if

$$\lim_{N\to\infty} g_k^N(v_1, \cdots, v_k) = \Pi_{j=1}^k \lim_{N\to\infty} g_1^N(v_j) . \tag{7}$$

Actually, as Maxwell and Boltzmann pointed out, $f_1^N(v_1, t)$ (for many purposes) contains all of the physically relevant information of $f(\vec{v}, t)$, where $f(\vec{v}, t)$ is given by (4). Now it follows from (4) that

$$\frac{\partial}{\partial t} f_1^N(v_1, t) = \left[\mathcal{L} f_2^N \right] (v_1, t) \tag{8}$$

for some linear operator $\mathcal{L}$. That is, one doesn't have an autonomous evolution equation for the marginal $f_1^N(v_1, t)$. But propagation of the Boltzmann property would imply that

$$f_2^N(v, w, t) = f_1^N(v, t) f_1^N(w, t) \tag{9}$$

which does render the evolution equation for $f_1^N(v_1, t)$ autonomous, but non–linear.

The key fact, discovered by Kac is that the Boltzmann property is in fact preserved under the timeevolution (4). Thus one has the theorem:

THEOREM 1.1 (Propagation of Chaos). *Suppose a sequence of distributions $g^N(v_1 \cdots, v_N)$ has the Boltzmann Property, and consider, for some fixed t, its timeevolved sequence $g^N(v_1 \cdots, v_N, t)$. Then $g^N(v_1 \cdots, v_N, t)$ has also the Boltzmann property and moreover the limit of the singe particle marginal $g_1^\infty(v_1, t)$ satisfies the following caricature of the Boltzmann equation*

$$\frac{\partial}{\partial t} f(v, t) =$$
$$\frac{1}{\pi} \int_{-\infty}^\infty \mathrm{d}w \int_0^{2\pi} \mathrm{d}\theta \, f(v\cos\theta + w\sin\theta, t) f(-v\sin\theta + w\cos\theta, t) \tag{10}$$
$$-2f(v, t) \int_{-\infty}^\infty \mathrm{d}w f(w, t) .$$

A very simple, but illuminating example is to consider the sequence of probability distributions c_N consisting of the constant function. It has been known at least since Maxwell that

$$\lim_{N\to\infty} c_N(v_1) = \frac{1}{\sqrt{2\pi T}} e^{-\frac{v_1^2}{2T}} = M(v_1) \tag{11}$$

recalling that we set $E = NT$. Similarly

$$\lim_{N\to\infty} c_N(v_1,\ldots,v_k) = \left(\frac{1}{\sqrt{2\pi T}}\right)^k e^{-\sum_{i=1}^k v_i^2/(2T)} = \Pi_{i=1}^k M(v_i) \tag{12}$$

Thus, the constant function is related to the Maxwell equilibrium distribution which is clearly a solution of (10).

For a proof of Theorem 1 we refer the reader to [5] and also to [8] and [6]. From now on, unless otherwise mentioned, we refer to equation (10) as the Boltzmann equation.

The advantages and disadvantages of the Boltzmann equation description and the N particle description are clear. In the latter case, one has to deal with a large number of particles while the time evolution is linear. The Boltzmann equation evolves functions of a single variable only, but with a complicated nonlinearity. While Kac's model is not derived from physical first principles, it is so far the simplest multiparticle dynamics rigorously connected with the Boltzmann equation, and one can regard it, as did Kac, as more fundamental since it describes the evolution of multiparticle correlations.

There have been a number of papers concerning the approach to equlibrium of solutions of (10). After the work of McKean [8] and Gruenbaum [7], it was shown in [2] that for a large class of initial conditions

$$\|f(\cdot,t) - M(v_1)\|_{L^1} \le C_\varepsilon e^{-(1-\varepsilon)\lambda_1 t} \tag{13}$$

for any $\varepsilon > 0$. The constant $\lambda_1 = 1/2$ is the gap of the linearized collision operator. This operator is quite easily obtained by perturbing the solution of the Bolzmann equation about the Maxwellian, i.e., set

$$f(v) = M(v)(1 + \varepsilon h(v)) \tag{14}$$

and expand.

It is reasonable to expect that the resulting linear operator should describe the solution for the Boltzmann equation reasonably well for initial conditions that are sufficiently close to the equilibrium. Thus the approach to equilibrium should be governed by the first nonzero eigenvalue λ_1 of this operator which is 1/2. Thus, the result of [2] is in some sense optimal. Not only is the linearized Boltzmann equation a good approximation close to the equilibrium but reasonable initial condition get driven close to the equilibrium with about the same rate.

It is therefore natural to investigate this question for the Kac model. In what sense should we study the approach to equilibrium? From a functional analysis point of view it is natural to consider the L^2 norm. It is easy to see that the semigroup (4) is ergodic, i.e., the largest eigenvalue of Q is one, it is nondegenerate and its eigenfuntion is the constant function. Thus, any initial condition will tend to this equilibrium state. Using the spectral theorem the rate of this approach is given by

$$\|\phi(t) - \overline{\phi}\|_2 \le e^{-\lambda_2 t} \|\phi(0) - \overline{\phi}\|_2 \tag{15}$$

where $\overline{\phi}$ is the average of ϕ (which is independent of t) and λ_2 is the smallest nonzero eigenvalues of the operator $N(I - Q)$, also called the *gap* of NQ.

The problem is now that the gap λ_2 is a function of N. It was conjectured by Kac that this function has a limit as $N \to \infty$ which is strictly positive. In section 3 we shall compute this number explicitly and show that it coincides with λ_1. Thus we have in the limit as $N \to \infty$

THEOREM 1.2 (Approach to equilibrium). *For any initial condition ϕ in $L^2(S^{N-1}(\sqrt{E}))$ we have that*

$$\|\phi(t) - \overline{\phi}\|_2 \leq e^{-t/2}\|\phi(0) - \overline{\phi}\|_2 . \tag{16}$$

In particular the rate does not depend on the energy and the particle number.

There were a number of results in this direction. In particular, Elise Janvresse [4] was the first to prove Kac's conjecture. She did so using using H.T. Yau's Martingale method. This method involves an induction on N, as does ours, but the nature of the induction is rather different and requires rather complicated estimates which preclude an quantitative estimate on the gap. An earlier result is due to Diaconis and Saloff–Coste [3], who proved that the second lowest eigenvalue of $I - Q$ is bounded below by c/N^3. In [3] it was also announced that Maslin has computed the gap precisely (as we do) using representation theory.

In the next section we give a brief description of Kac's model. In section 3 we give a short proof of Theorem 2 and in section 4 we mention some further results that can be proved by our methods such as random collision of Maxwellian molecules.

2 The Kac model

Kac's model cannot be considered as a fundamental model in that it has not been derived from first principles. It's justification is *a–posteriori* through the connection with the Boltzmann equation, as described in Theorem 1.1. However, it is enlightening to give an explanation of some of the heuristics that motivate the introduction of this model.

We consider N particles moving in one dimension undergoing random collisions. Suppose that at a given moment these particles have velocity $\vec{v} = (v_1, \cdots, v_N)$. We pick any pair of particles at random, say (i, j) with equal probability. Then their velocities after the collision are given by (1) with equal probability for all angles θ. More formally, consider a test function $\Psi(\vec{v})$ and its expectation value

$$E[\Psi](\vec{v}) = \frac{1}{\binom{N}{2}} \sum_{i<j} \frac{1}{2\pi} \int_0^{2\pi} \Psi(v_1, \cdots, v_i^*(\theta), \cdots, v_j^*(\theta), \cdots, v_N) \mathrm{d}\theta = Q\Psi(\vec{v}) \tag{17}$$

This is the expected value of Ψ after one collision, for given velocities $\vec{v}$.

Note that three particle collisions are not considered in this model although their contribution to equilibration may be significant in reality.

It is now fairly straightforward to calculate the probabilities for $\vec{v}(k)$, the velocities after k collisions. Certainly

$$E[\Psi(\vec{v}(k))] = E\left[E\{\Psi(\vec{v})|\vec{v}(k-1) = \vec{v}\}\right] = E\left[Q\Psi(\vec{v}(k-1))\right] , \tag{18}$$

i.e., we take first the expection given that the random variable $\vec{v}(k-1) = \vec{v}$ and then the expectation value with respect to $\vec{v}(k-1)$. Thus

$$E[\Psi(\vec{v}(k))] = E\left[Q^k \Psi(\vec{v})\right] . \tag{19}$$

If we denote the probability distribution of $\vec{v}(k)$ by $f_k(\vec{v})$ we obtain that

$$\int_{S^{(N-1)}} f_k(\vec{v})\Psi(\vec{v})\mathrm{d}^{(N-1)}S = \int_{S^{(N-1)}} f_0(\vec{v})Q^k\Psi(\vec{v})\mathrm{d}^{(N-1)}S , \tag{20}$$

and hence, since Ψ is arbitrary,

$$f_k(\vec{v}) = Q^k f_0(\vec{v}) . \tag{21}$$

A further assumption involves the distribution of the collisions in time. We assume first of all that each particle interacts with an environment that consists of other particles whose motions are *independent* of the given particle. This is of course not the case, but does become more and more reasonable as N increases.

Now assume that the collision times for a single particle interacting with its environment are exponentially distributed, i.e., the waiting time T_k for the kth particle between any two consecutive collisions is a random variable whose distribution is given by

$$P[T_k > t] = e^{-t/\tau} . \tag{22}$$

The time τ is the first average collision time for a particle interacting with a random environment. Under the assumptions made above, the T_k are independently and identically distributed, and hence if T denotes the waiting time for *any* collision to occur,

$$P[T > t] = P[\min\{T_1, \ldots, T_N\} > t] = e^{-Nt/\tau} . \tag{23}$$

That is, the rate at which collisions occur will be proportional to N.

Thus, the probability of having velocities $\vec{v}$ after exactly k collisions in the time interval $[0, t]$ is given by

$$\frac{e^{-kNt/\tau} t^k}{k! \tau^k} Q^k f_0(\vec{v}) , \tag{24}$$

and finally the probability of having velocities $\vec{v}$ after any number of collisions is

$$\sum_{k=0}^{\infty} \frac{e^{-kNt/\tau} t^k}{k! \tau^k} Q^k f_0(\vec{v}) = e^{-Nt(I-Q)/\tau} f_0(\vec{v}) . \tag{25}$$

From now on we set $\tau = 1$.

3 Approach to equilibrium

In this section we sketch a proof of Theorem 2. In order to avoid technicalities, we restrict ourselves to the physically interesting case where the probability distributions are symmetric functions of the particle labels. For the general case we refer the reader to [1].

THEOREM 3.1 (The gap for Q). *The gap Δ_N of Q is given by*

$$\Delta_N = \frac{N+2}{2N(N-1)} \, , \tag{26}$$

and the corresponding eigenfunction, which is unique up to a constant multiple, is given by

$$u(\vec{v}) = \sum_{j=1}^{N} |v_j|^4 - \frac{3}{N(N+2)} \, . \tag{27}$$

Certainly, this result gives as $N \to \infty$ the second eigenvalue for the linearized Boltzmann equation. Moreover, if one sets $E = NT$ as before and if one normalizes u, its marginal tends to the corresponding eigenfunction, i.e., the fourth order Hermite polynomial times a Gaussian as $N \to \infty$.

It is easy to see that the spectrum of Q is independent of E and hence we choose $E = 1$. It is a fairly simple calculation to verfy that the function u is indeed an eigenfunction of Q with the eigenvalue $1 - \Delta_N$. This eigenfuction is special in the sense that it is a sum of functions of a single variable. Most of the other eigenfunctions are not of this type.

Consider the operator

$$P_1 : L^2(S^{N-1}) \to L^2(S^{N-1}) \tag{28}$$

given by

$$P_1 f(\vec{v}) = \frac{1}{|S^{N-2}(\sqrt{1-v_1^2})|^{1/2}} \int_{v_2^2+\cdots+v_N^2=1-v_1^2} f(\vec{v}) \mathrm{d}^{(N-2)} S \tag{29}$$

and in a similar fashion we define P_k. Clearly $P_k^2 = P_k$ and $P_k^* = P_k$.

LEMMA 3.2. *Denote the gap of the operator*

$$P = \frac{1}{N} \sum_{j=1}^{N} P_j \tag{30}$$

by Λ_N. Then

$$\Delta_N \geq \Delta_{N-1} \Lambda_N \tag{31}$$

Proof. Define the operators

$$P_{i,j} f(\vec{v}) = \frac{1}{2\pi} \int_0^{2\pi} f(v_1, \cdots, v_i^*(\theta), \cdots, v_j^*(\theta), \cdots, v_N) \mathrm{d}\theta \tag{32}$$

and note that

$$Q = \frac{1}{\binom{N}{2}} \sum_{i<j} P_{i,j} \, . \tag{33}$$

Further, consider

$$Q_k = \frac{1}{\binom{N-1}{2}} \sum_{i<j, i,j \neq k} P_{i,j} \tag{34}$$

and note that

$$Q = \frac{1}{N} Q_k \, . \tag{35}$$

For any function $f \in L^2(S^{N-1})$, orthogonal to the constant function we compute

$$(f, Qf) = \frac{1}{N} \sum_{j=1}^{N} (f - P_k f, Q_k(f - P_k f)) + (f, Pf) . \tag{36}$$

We have used the fact that $P_k Q_k = P_k$. The function $f - P_k f \in L^2(S^{N-2}(\sqrt{1 - v_k^2}))$ is orthogonal to the constant function for every value of v_k. Since the spectrum of Q_k does not depend on the radius of the sphere $S^{N-2}(\sqrt{1 - v_k^2})$ it follows from (36) that

$$\frac{1}{N} \sum_{j=1}^{N} (f - P_k f, Q_k(f - P_k f)) \leq (1 - \Delta_{N-1}) \frac{1}{N} \sum_{j=1}^{N} \|f - P_k f\|^2 , \tag{37}$$

and hence

$$(f, Qf) \leq (1 - \Delta_{N-1})\|f\|^2 + \Delta_{N-1} (f, Pf) \tag{38}$$

from which the lemma follows. $\qquad\square$

Lemma 3.2 reduces the problem to the computation of the gap of the operator P, and this problem can be reduced to a one dimensional one as lemma 3.4 shows. This one dimensional, purely geometric problem can be described as follows. Denote by π_k the projection $\pi_k(\vec{v}) = v_k$. We consider the Hilbert space $\mathcal{H}$ of functions defined on the interval $[-1, 1]$ given by the inner product

$$< f, g >= (f \circ \pi_1, g \circ \pi_1) , \tag{39}$$

and the operator K defined by the bilinear form

$$< f, Kg >= (f \circ \pi_1, g \circ \pi_2) . \tag{40}$$

Clearly, the largest eigenvalue of K is one. It is again easy to see that the eigenvalues of K do not depend on the radius of the sphere, in particular we can choose it to be NT and let $N \to \infty$. In this limit, the expression (40) tends to the product of the expectation values of the functions g and f on the real line with respect to the Gaussian function (11). In other words, the correlation between the random variables π_1 and π_2 disappears as $N \to \infty$. The second eigenvalue of K the measures how fast this correlation tends to zero as $N \to \infty$. It is this number that ensures that the gap of Q does not decay faster than $1/N$.

There are various ways to analyze the operator K, but the following expression is most convenient

$$Kg(v) = \frac{|S^{N-3}|}{|S^{N-2}|} \int_0^\pi g\left(\sqrt{1 - v^2} \cos \theta\right) \sin^{N-3} \theta d\theta . \tag{41}$$

The lemma below describes all the eigenvalues.

LEMMA 3.3. *The operator K maps polynomials of degree n to polynomials of degree not larger than n and hence the eigenfunctions are polynomials of degree n with corresponding eigenvalues α_n. For n odd, $\alpha_n = 0$, and for $n = 2k$ the eigenvalues are given by the formula*

$$\alpha_{2k} = (-1)^k \frac{|S^{N-3}|}{|S^{N-2}|} \int_0^\pi (1 - \sin^2 \theta)^k \sin^{N-3} \theta d\theta . \tag{42}$$

In particular $|\alpha_{2(k+1)}| < |\alpha_{2k}|$ and

$$\alpha_2 = -\frac{1}{N-1} \quad \text{and} \quad \alpha_4 = \frac{3}{N^2 - 1} . \tag{43}$$

Proof. We only sketch the proof of this easy lemma. The first two assertions are obvious. Moreover the operator K maps even onto even functions and odd functions to zero.

Let g_{2k} be an even eigenfunction. Then, after suitable normalization, we can write this polynomial as $g_{2k} = v^{2k} + h(v)$ where $h(v)$ is an even polynomial of degree at most $2(k-1)$. Next, $\alpha_{2k}(v^{2k}+h(v)) = Kg_{2k} = Kv^{2k}+Kh(v)$, where $Kh(v)$ has order not more than $2(k-1)$. The formula for the nonzero eigenvalues follows, immediately from the formula of K since $Kv^{2k} = \alpha_{2k}v^{2k}+$ lower order where α_{2k} is given by the formula (43) . The monotonicity follows immediately from (42). $\qquad\square$

The next lemma computes the eigenvalues of P in terms of the eigenvalues of K.

LEMMA 3.4. *If λ is an eigenvalue of P then*

$$1 - \lambda = \frac{N-1}{N}(1-\mu) \ , \tag{44}$$

where μ is an eigenvalue of K. In particular the gap of P equals the gap of K multiplied by $(N-1)/N$.

Proof. Let f be an eigenfunction of P with eigenvalue λ. Then

$$\frac{1}{N}\sum_{j=1}^{N} P_j f = \lambda f \tag{45}$$

and hence f is of the form

$$f(\vec{v}) = \sum_{j=1}^{N} g(v_j) \ . \tag{46}$$

Note that it is here that we assume that f is symmetric in the particle labels. Applying the operator P to both sides of (45) and rearranging terms we obtain using (46) that

$$\sum_{j=1}^{N} \left[\frac{N-1}{N}(Kg)(v_j) + (\frac{1}{N} - \lambda)g(v_j) \right] = 0 \ . \tag{47}$$

One is tempted to deduce (44) from (47) directly. However this is not valid, since single variable functions are not independent on the sphere.

Applying the operator P_1 once more to both sides of (47) leeds to

$$\left[K + \frac{N}{N-1}\left(\frac{1}{N} - \lambda\right) \right] \left[K + \frac{1}{N-1} \right] g(v_1) = 0 \tag{48}$$

It follows from the Lemma 3.3 that the only solution of the equation

$$\left[K + \frac{1}{N-1} \right] g = 0$$

is given by the function $g(v) = v^2 - const.$ with the constant such that the function g is orthogonal to the constant function in $\mathcal{H}$. This implies that $f(\vec{v}) = \sum_{j=1}^{N}(v_j^2 - const.) \equiv 0$. Thus, we can invert $K + \frac{1}{N-1}$ and obtain for g the equation

$$\left[K + \frac{N}{N-1}\left(\frac{1}{N} - \lambda\right) \right] g = 0 \tag{49}$$

A simple calculation confirms the lemma.

$\qquad\square$

Proof of Theorem 3.1. Combining Lemma 3.2, Lemma 3.3 and Lemma 3.4 we learn that

$$\Delta_N \geq \frac{N-1}{N}\Delta_{N-1}(1-\alpha_4) \ . \tag{50}$$

Each of the operators $P_{i,j}$ is a projection and hence $\Delta_2 = 1$. This together with (50) yields a simple recursion relation that can easily be solved to yield (26) . To see that the eigenfunction is unique in the class of functions that are symmetric in the particle labels one notes that any eigenfunction for the second largest eigenvalue of Q must also be an eigenfunction for the second largest eigenvalue of P. But those are precisely determined by α_4 and the corresponding eigenfunction

$$v^4 - \frac{3}{N^2(N+2)} \ ,$$

which yields the theorem. □

4   Extension to Maxwellian molecules

A closer scrutiny of the above proof shows that it is flexible enough to cover more intricate, models.

Consider again the Kac model, but this time assume that the distribution of the scattering angle is not uniform but given by the probability measure $\rho(\theta)$. Thus the operator in (3) is replaced by

$$Qf(\vec{v}) = \frac{1}{\binom{N}{2}}\sum_{i<j}\int_0^{2\pi}\rho(\theta)f(v_1,\cdots,v_i^*(\theta),\cdots,v_j^*(\theta),\cdots,v_N)\mathrm{d}\theta \ . \tag{51}$$

In order that this operator be selfadjoint we require that $\rho(\theta) = \rho(-\theta)$.

The interesting fact is that equation (50) still holds, as can easily be seen. Thus, after solving the recursion, we obtaing immediately that

$$N\Delta_N \geq \frac{N+2}{4(N-1)}2\Delta_2 \ . \tag{52}$$

The only place where any detail about the collisions enters is in the computation of Δ_2 which is given by the second largest Fourier coefficient of ρ. Thus, Kac's conjecture also holds in this case.

Another, more interesting model is the following three dimensional one which is fairly close to a realistic model known as Maxwellian molecules. Consider again randomly colliding particles in three dimensions. The scattering map is now given by

$$v_i^*(\omega) = v_i - \omega \cdot (v_i - v_j)\omega \qquad v_j^*(\omega) = v_j + \omega \cdot (v_i - v_j)\omega \ , \tag{53}$$

for $\omega \in S^2$. Clearly, the total energy and the total momentum are conserved and we set the total energy equals to E and the total momentum equals to 0. Denote by P the set of all velocities satisfying the momentum condition. If we set

$$M^{3N-4} = S^{3N-1} \cap P \tag{54}$$

then we can define the Hilbert space $L^2(M^{3N-4}; d^{3N-4}S)$ where the measure $d^{3N-4}S$ is the measure induced from the Lebesgue measure on R^{3N}.

In the same fashion as before we define

$$P_{i,j}f(\vec{v}) = \frac{1}{4\pi} \int_{S^2} f(v_1, \ldots, v_i^*(\omega), \ldots, v_j^*(\omega), \ldots, v_N) d\omega \ , \tag{55}$$

and

$$Q = \frac{1}{\binom{N}{2}} \sum_{i<j} P_{i,j} \ . \tag{56}$$

Again this operator has 1 as its largest eigenvalue. Its unique eigenfunction is the constant function. We can prove [1]

THEOREM 4.1. *The gap Δ_N of Q satisfies the estimate*

$$\Delta_N \geq \frac{c}{N} \tag{57}$$

for some universal constant c.

The proof of this theorem follows along the same lines as the one for the Kac model, i.e., the problem is split into a purely geometric part and a purely dynamic part. The estimates, however, are more complicated.

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