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On the blow up phenomenon for the critical nonlinear Schrödinger equation in 1D

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0. Introduction

Consider the nonlinear Schrödinger equation

$$i\psi_t = -\psi_{xx} - |\psi|^{2p}\psi, \quad x \in \mathbb{R} \tag{1}$$

with initial data

$$\psi|_{t=0} = \psi_0 \in H^1.$$

It is well known that for $p \geq 2$ the problem has solutions that blow up in finite time. The case $p = 2$ marks the transition between the global existence and the blow up phenomenon. In this paper we study the participation of nonlinear bound states in singularity formation in the critical case $p = 2$.

The NLS (1) has an important solution of special form- soliton: $e^{it}\varphi_0(x)$, where φ_0 is the "ground state solitary wave". We consider the Cauchy problem for (1) with initial data close to a soliton:

$$\psi|_{t=0} = \varphi_0 + \chi_0,$$

where χ_0 is small in suitable sense. We show that for a certain class of initial perturbations the solution ψ blows up in finite time T^*, admitting the following asymptotic representation

$$\psi(t,x) \sim e^{i\mu(t)}\lambda^{1/2}(t)\varphi_0(\lambda(t)x), \quad t \to T^*, \tag{2}$$

$$\lambda(t) \sim (T^* - t)^{-1/2} \left(\ln|\ln(T^* - t)|\right)^{1/2}, \ \mu(t) \sim \ln(T^* - t)\ln|\ln(T^* - t)|.$$

Thus, up to a phase factor the formation of the singularity is self-similar with a profile given by the ground state. The behavior (2) was predicted in [FS,KSZ,LPSS1,LPSS2,Ma,SS].

1. Preliminary facts and formulation of the result

1.1. The nonlinear equation.

We formulate here the necessary facts about Cauchy problem for the equation

$$i\psi_t = -\psi_{xx} - |\psi|^4\psi \tag{1.1}$$

with initial data in H^1.

Proposition 1.1. *The Cauchy problem for equation* (1.1) *with initial data* $\psi(0,x)=\psi_0(x)$, $\psi_0 \in H^1$ *has a unique solution* ψ *in the space* $C([0,T^*)\to H^1)$ *with some* $T^*>0$ *and*

(i) ψ *satisfies the conservation laws*

$$\int dx|\psi|^2 = const, \quad H(\psi)=\int dx[|\psi_x|^2-\frac{1}{3}|\psi|^6] = const;$$

(ii) if $T^*<\infty$, *then* $\|\psi_x\|_2\to\infty$ *as* $t\to T^*$ *and*

$$\|\psi_x\|_2 \geq c(T^*-t)^{-1/2};$$

(iii) if $H(\psi_0)<0$ *then* $T^*<\infty$.

Suppose in addition that $x\psi_0\in L_2$. *Then* $x\psi\in C([0,T^*)\to L_2)$ *and* ψ *satisfies the pseudo-conformal conservation law*

$$\int dx|(x+2it\partial_x)\psi|^2-\frac{4}{3}t^2\int dx|\psi|^6 = const.$$

Equation (1.1) is invariant with respect to transformations:

$$\psi(x,t)\to(a+bt)^{-1/2}e^{i\omega+i\frac{bx^2}{4(a+bt)}}\psi(\frac{x}{a+bt},\frac{c+dt}{a+bt}), \tag{1.2}$$

where $\omega\in\mathbb{R}$, $\begin{pmatrix} a & b\\ c & d\end{pmatrix}\in SL(2,\mathbb{R})$.

1.2. Exact blow up solutions. The equation (1.1) has a family of soliton solutions

$$e^{i\frac{\alpha^2}{4}t}\varphi_0(x,\alpha), \quad \alpha>0,$$

where φ_0 is a positive even smooth decreasing function satisfying the equation

$$-\varphi_{0xx}+\frac{\alpha^2}{4}\varphi_0-\varphi_0^5=0.$$

As $|x|\to\infty$, $\varphi_0\sim\varphi_\infty e^{-\frac{\alpha}{2}|x|}$.

One has a relation

$$\varphi_0(x,\alpha)=\left(\frac{\alpha}{2}\right)^{1/2}\varphi_0(\frac{\alpha}{2}x),$$

where $\varphi_0(x)$ stands for $\varphi_0(x,2)$.

Applying transformations (1.2) to the soliton solution $e^{it}\varphi_0(x)$ one gets a 3-parameter family of solutions

$$e^{i\mu(t)-i\beta(t)z^2/4}\lambda^{1/2}(t)\varphi_0(z), \; z=\lambda(t)x, \tag{1.3}$$

where μ,β,λ are given by

$$\lambda(t)=(a+bt)^{-1}, \; \beta(t)=-b(a+bt), \; \mu(t)=\frac{c+dt}{a+bt}.$$

Remark that $\lambda(t)$, $\beta(t)$, $\mu(t)$ satisfy the system

$$\lambda^{-3}\lambda_t=\beta,\ \lambda^{-2}\beta_t+\beta^2=0,\ \lambda^{-2}\mu_t=1.$$

If $b\neq 0$ solution (1.3) blows up in finite time. It is known that equation (1.1) has no blow-up solutions in the class

$$\{\psi\in H^1(\mathbb{R}),\|\psi\|_2<\|\varphi_0\|_2\},$$

solutions (1.3) being the only blow-up solutions (up to Galilei invariance) with minimal mass, see [W1,Me].

1.3. Extended manifold of blow-up solutions. 3-parameter family (1.3) can be considered as the boundary $a=0$ of the 4-parameter family of formal solutions $w(x,\sigma(t))$,

$$w(x,\sigma)=e^{i\mu-i\beta z^2/4}\lambda^{1/2}\varphi(z,a),\ z=\lambda x,$$

$\sigma=(\frac{\mu}{2},\lambda,\beta,a),\ \ \lambda\in\mathbb{R}_+,\ \beta,\ \mu,\ a\in\mathbb{R}$. Here

$$\varphi(z,a)=\sum_{k=0}^{\infty}a^k\varphi_k(z) \tag{1.4}$$

is a formal solution of the equation

$$-\varphi_{zz}+\varphi-\frac{az^2}{4}\varphi-\varphi^5=0,$$

all φ_k being even smooth exponentially decreasing (as $|z|\to\infty$) functions.

$w(x,\sigma(t))$ is a formal solution of (1.1) if $\sigma(t)$ satisfies the system

$$\lambda^{-3}\lambda_t=\beta,\ \lambda^{-2}\beta_t+\beta^2=a,\ \lambda^{-2}\mu_t=1,\ a_t=0, \tag{1.5}$$

which gives, in particular, $\lambda=(d_2t^2+d_1t+d_0)^{-1/2}$, $a=d_1^2/4-d_2d_1$. Here d_j are constant.

We shall use the notation $\varphi^N(z,a)=\sum_{k=0}^{N}a^k\varphi_k(z)$,

$$\varphi^N(z,\alpha,a)=\left(\frac{\alpha}{2}\right)^{1/2}\varphi^N(\frac{\alpha}{2}x,\frac{16a}{\alpha^4}).$$

1.3. Linearization of (1.1) on a soliton. Consider the linearization of (1.1) on the soliton $e^{it}\varphi_0(x)$:

$$i\chi_t=-\chi_{xx}-\varphi_0^4\chi-2\varphi_0^4(\chi+e^{2it}\bar{\chi}).$$

Introduce function f: $\chi=e^{it}f$. Then f satisfies the equation

$$i\vec{f}_t=H_0\vec{f},\quad \vec{f}=\begin{pmatrix}f\\ \bar{f}\end{pmatrix},$$

$$H_0=(-\partial_x^2+1)\sigma_3+V(\varphi_0),\quad V(\xi)=-3\xi^4\sigma_3-2i\xi^4\sigma_2,$$

σ_2, σ_3 being the standard Pauli matrices.

H_0 is considered as a linear operator in $L_2(\mathbb{R} \to \mathbb{C}^2)$ defined on the natural domain. Here and later L_2 stands for the subspace of the standard L_2 consisting of even functions. The operator H_0 satisfies the relations

$$\sigma_3 H_0 \sigma_3 = H_0^*, \quad \sigma_1 H_0 \sigma_1 = -H_0. \tag{1.6}$$

The continuous spectrum of H_0 consists of two semi-axes $(-\infty, -1]$, $[1, \infty)$ and is simple.

The point $E = 0$ is an eigenvalue of the multiplicity 4. By differentiating the solution w with respect to the parameters it is easy to distinguish an eigenfunction $\vec{\xi}_0$

$$\vec{\xi}_0 = i\varphi_0 \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \quad H_0 \vec{\xi}_0 = 0,$$

and three associated functions $\vec{\xi}_j$, $j = 1, 2, 3$,

$$\vec{\xi}_1 = \frac{1}{4}(1 + 2x\partial_x)\varphi_0 \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad \vec{\xi}_2 = -i\frac{1}{8}x^2\varphi_0 \begin{pmatrix} 1 \\ -1 \end{pmatrix},$$

$$\vec{\xi}_3 = \frac{1}{2}\varphi_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad H_0 \vec{\xi}_j = i\vec{\xi}_{j-1},$$

φ_1 being the second coefficient in the expansion (1.4). φ_1 can be characterized by the equation

$$L_+ \varphi_1 = \frac{x^2}{4}\varphi_0, \quad L_+ = -\partial_x^2 + 1 - 5\varphi_0^4,$$

the operator L_+ is invertible being restricted on the subspace of even function .

One can show that $E = 0$ is the only eigenvalue of H_0 (see [W2, G, P], for example).

1.4. Main theorem.

Consider the Cauchy problem for equation (1.1) with initial data

$$\psi|_{t=0} = \psi_0, \; \psi_0(x) = e^{-i\beta_0 x^2/4}(\varphi^N(x, \beta_0^2) + f_0(x)), \; \beta_0 > 0, \tag{1.7}$$

where $f_0(x) = f_0(-x)$ and f_0 satisfies the estimate

$$\|f_0\|_X \leq \beta_0^{2N}. \tag{1.8}$$

Here $\|f\|_X = \|f\|_{H^1} + \|xf\|_{L_2}$.

Assume that

(i) β_0 is sufficiently small;

(ii) N is sufficiently large.

These conditions give, in particular,

$$H(\varphi^N(\beta_0^2) + f_0) = -\frac{\beta_0^2}{4}e + O(\beta_0^4) < 0, \quad e = \int dx x^2 \varphi_0^2,$$

which together with conformal invariance implies that the solution ψ of Cauchy problem (1.1,1.7) blows up in finite time $T^* < \infty$.

Our main result is the following.

Theorem 1.1.
The solution ψ of the Cauchy problem (1.1,7) *blows up in finite time* $T^* = \frac{1}{2\beta_0}(1+o(1))$, *as* $\beta_0 \to 0$, *and there exist* $\lambda(t),\ \mu(t) \in C^1([0,T^*))$,

$$\lambda(t) = const(T^*-t)^{-1/2}(\ln|\ln(T^*-t)|)^{1/2}(1+o(1)), \tag{1.9}$$

$$\mu(t) = const\ln(T^*-t)\ln|\ln(T^*-t)|(1+o(1)), \quad t \to T^*,$$

such that ψ admits the representation

$$\psi(x,t) = e^{i\mu(t)}\lambda^{1/2}(t)\left(\varphi_0(z) + f(z,t)\right), \;\; z = \lambda(t)x,$$

where f is small in $L_2 \cap L_\infty$ uniformly with respect to $t \in [0,T^*)$. Moreover, $\|f\|_\infty = o(1)$, as $t \to T^*$. The constants in (1.9) *are independent of initial data.*

It is worth mentioning that due to the conformal invariance the same result remains valid for initial data of the form

$$\tilde{\psi}_0(x) = e^{i\omega - ibz^2/4}\lambda^{1/2}\psi_0(z), \;\; z = \lambda x,$$

where $\omega \in \mathbb{R},\ \ \lambda \in \mathbb{R}_+,\ \ b > -\frac{1}{T^*}$.

2. Some words about the proof

This section contains the outline of the proof. The details can be found in [P].

2.1. Splitting of motions.
The main idea repeats the main idea of the works [SW1,2], [BP] where the asymptotic stability of solitary waves were considered. We start by introducing some new coordinates for the description of the solution with initial data (1.7). The new coordinates posses an important property: they allow us to split the motion into two parts, the first part being a finite- dimensional dynamics on the manifold of formal solutions $\{w(\cdot,\sigma)\}$ and the second part remains small in some sense for all $t \in [0,T^*)$.

To describe these coordinates we need to introduce a modified ground state $\tilde{\varphi}(z,\alpha,a)$ which is characterized by the equation

$$-\tilde{\varphi}_{zz} + \frac{\alpha^2}{4}\tilde{\varphi} - \frac{az^2}{4}\theta(hz)\tilde{\varphi} - \tilde{\varphi}^5 = 0, \;\; h = \sqrt{|a|} > 0, \tag{2.1}$$

$\alpha,\ a \in \mathbb{R}$. Here $\theta \in C_0^\infty(\mathbb{R}),\ \ \theta(\xi) = \theta(-\xi)$,

$$\theta(\xi) = \begin{cases} 1, & |\xi| \le 2-\delta \\ 0, & |\xi| > 2-\delta/2 \end{cases},$$

δ being a sufficiently small fixed number. One has the following proposition.

Proposition 2.1. *For α in some finite vicinity of* 2 *and for a sufficiently small, equation* (2.1) *has a unique positive even smooth decreasing solution $\tilde{\varphi}(z,\alpha,a)$ which is close to $\varphi_0(z,\alpha)$. Moreover,*

(i) *as $a \to 0$, $\tilde{\varphi}(z,\alpha,a)$ admits the asymptotic expansion* (1.4) *in the sense*

$$|\tilde{\varphi} - \varphi^N| \le c|a|^{N+1} < x >^{3(N+1)} e^{-\frac{1}{h}\tilde{S}_{\alpha,a}(h|x|)}.$$

Here $\tilde{S}_{\alpha,a}(\xi)=\frac{1}{2}\int_0^\xi ds\sqrt{\alpha^2-(a)_+s^2\theta(s)}$;

(ii) $\|e^{\frac{1}{h}\tilde{S}_{\alpha,a}(h|x|)}\tilde{\varphi}(\alpha,a)\|_\infty\le c$,

where $S_{\alpha,a}(\xi)=\frac{1}{2}\int_0^\xi ds\sqrt{\alpha^2-\operatorname{sgn} a s^2\theta(s)}$;

The similar formulas are valid for the derivatives of $\tilde{\varphi}$ *with respect to* z, α, a.

Introduce a linearized operator $\tilde{H}(a)$ associated to the modified ground state $\tilde{\varphi}(z,a)$ (as before, $\tilde{\varphi}(z,a)=\tilde{\varphi}(z,2,a)$)

$$\tilde{H}(a)=(-\partial_x^2+1-\frac{az^2}{4}\theta)\sigma_3+V(\tilde{\varphi}(a)).$$

The continuous spectrum of $\tilde{H}(a)$ is the same as in the case of the operator H_0. The point $E=0$ is an eigenvalue of $\tilde{H}(a)$ of the multiplicity 2. There are an eigenfunction $\vec{\zeta}_0(a)$

$$\vec{\zeta}_0(a)=i\tilde{\varphi}(a)\begin{pmatrix}1\\-1\end{pmatrix},\quad \tilde{H}\vec{\zeta}_0=0,$$

and an associated function $\vec{\zeta}_1(a)$

$$\vec{\zeta}_1(a)=\partial_\alpha\tilde{\varphi}(\alpha,a)|_{\alpha=2}\begin{pmatrix}1\\1\end{pmatrix},\quad \tilde{H}\vec{\zeta}_1=\vec{\zeta}_0.$$

A more detailed description of the discrete spectrum can be obtained by means of the standard perturbation methods. In particular, the following proposition can be proved.

Proposition 2.2. *For* a *sufficiently small, the discrete spectrum of the operator* $\tilde{H}(a)$ *in some finite vicinity of the point* $E=0$ *consists of* 0 *and two simple eigenvalues* $\pm\lambda(a)$, $\lambda(a)=i\sqrt{a}\lambda'(a)$, *where* λ' *is a smooth real function of* a. *As* $a\to 0$, $\lambda'(a)=2+O(a)$. *Let* $\vec{\zeta}_2(a)$ *be an eigenfunction corresponding to* $\lambda(a)$ *normalized by the condition*

$$\left\langle\vec{\zeta}_2(a),\vec{\xi}_0\right\rangle=-i\left\langle\vec{\zeta}_0(a),\vec{\xi}_0\right\rangle+i\lambda^2(a)\left\langle\vec{\xi}_2,\vec{\xi}_0\right\rangle.$$

Then $\vec{\zeta}_2(a)$ *is a smooth function of* $a^{1/2}$ *admitting the following asymptotic expansion as* $a\to 0$

$$\vec{\zeta}_2=-i\vec{\zeta}_0-\lambda\vec{\zeta}_1+i\lambda^2\vec{\xi}_2+\lambda^3\vec{\xi}_3+a\lambda^2\begin{pmatrix}1\\-1\end{pmatrix}(h_0+O(a))+a\lambda^3\begin{pmatrix}1\\1\end{pmatrix}(h_1+O(a)),$$

where h_i, $i=1,2$ *are some real even smooth exponentially decreasing functions.* $O(a)$ *corresponds to the* L_∞*-norm with the weight* $e^{\frac{1-\gamma}{h}\tilde{S}_{\alpha,a}(h|x|)}$, $\gamma=O(h)$.

In the subspace generated by $\vec{\zeta}_j(a)$, $j=0,\dots 3$, $\vec{\zeta}_3=-\sigma_1\vec{\zeta}_2$ being an eigenfunction corresponding to the eigenvalue $-\lambda$, we introduce a new basis $\{\vec{e}_j(a)\}_{j=0}^3$:

$$\vec{e}_0=\vec{\zeta}_0,\quad \vec{e}_1=\vec{\zeta}_1,$$

$$\vec{e}_2=-\frac{i}{2\lambda^2}\left(\vec{\zeta}_2+\vec{\zeta}_3+2i\vec{\zeta}_0\right),\quad \vec{e}_3=\frac{1}{2\lambda^3}\left(\vec{\zeta}_2-\vec{\zeta}_3+2\lambda\vec{\zeta}_1\right),$$

$\vec{e}_2 = e_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix}$, $\vec{e}_3 = e_3 \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $\bar{e}_j = (-1)^{j-1} e_j$. It follows from proposition 2.2 that as $a \to 0$,

$$\vec{e}_2 = \vec{\xi}_2 - iag_0 \begin{pmatrix} 1 \\ -1 \end{pmatrix} + O(a^2),$$

$$\vec{e}_3 = \vec{\xi}_3 + ag_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + O(a^2).$$

Return to the Cauchy problem (1.1,7). Using the profile $\tilde{\varphi}$ one can rewrite the initial data ψ_0 in the form: $\psi_0 = e^{-i\frac{\beta_0 x^2}{4}} (\tilde{\varphi}(\beta_0^2) + f_0')$, $\|f_0'\|_X = O(\beta_0^{2N})$. Below we shall omit " $'$ " in the notation of f_0'. Write the solution ψ as the sum

$$\psi(x,t) = e^{i\Phi} \lambda^{1/2}(t) \left(\tilde{\varphi}(z, a(t)) + f(z,t) \right), \quad \Phi = \mu(t) - \frac{\beta}{4} z^2, \;\; z = \lambda(t) x, \tag{2.2}$$

where $\sigma(t) = (\frac{\mu(t)}{2}, \lambda(t), \beta(t), a(t))$ is an arbitrary curve in $\mathbb{R}_+ \times \mathbb{R}^3$, it is not a solution of (1.5) in general.

The decomposition can be fixed by the orthogonality conditions

$$\left\langle \vec{f}(t), \sigma_3 \vec{e}_j(a(t)) \right\rangle = 0, \quad j = 0, \dots, 3. \tag{2.3}$$

This means that σ has to satisfy the system

$$F_j(\psi, \sigma) = 0, \quad j = 0, \dots 3, \tag{2.4}$$

$$F_j(\psi, \sigma) = \lambda^{1/2} \left\langle \vec{\psi}, \sigma_3 e^{i\Phi\sigma_3} \vec{e}_j(\lambda \cdot, a) \right\rangle - \langle \vec{e}_0(a), \vec{e}_j(a) \rangle = 0, \quad \vec{\psi} = \begin{pmatrix} \psi \\ \bar{\psi} \end{pmatrix}.$$

The solvability of (2.4) for ψ in some small L_2- vicinity of φ_0 is guaranteed by the smoothness of the basis $\vec{e}_j(a)$, $j = 0, \dots, 3$ and the non-degeneration of the corresponding Jacobi matrix

$$B_0 = \left\{ \frac{\partial F_j}{\partial \sigma_k} \right\} \Big|_{\substack{\psi = \varphi \\ \sigma = (1,0,0,0)}}.$$

It is not difficult to check that

$$B_0 = -2 \left\{ \left\langle \vec{\xi}_k, \sigma_3 \vec{\xi}_j \right\rangle \right\}_{k,j=0}^{3}, \quad \det B_0 = \left| 2 \left\langle \vec{\xi}_1, \sigma_3 \vec{\xi}_2 \right\rangle \right|^4 = e^4 \neq 0.$$

So, one can assume that the initial decomposition (1.7) obeys conditions (2.3).

To prove the existence of a trajectory $\sigma(t)$ we need the following orbital stability result:

Proposition 2.3. *For any $\epsilon > 0$ there exist $\delta > 0$ such that for any ψ_0, $\|\psi_0 - \varphi_0\|_{H^1} \le \delta$, $E(\psi_0) < 0$, there exists $\mu(t) \in C([0, T^*))$ such that the solution ψ corresponding to the initial data ψ_0 satisfies the inequality*

$$\|\psi(t) - \lambda^{1/2}(t) e^{i\mu(t)} \varphi_0(\lambda(t) \cdot)\|_2 \le \epsilon, \quad 0 \le t < T^*,$$

where $\lambda(t)$ *is given by*

$$\lambda(t)=\frac{\|\psi_x(t)\|_2}{\|\varphi_{0x}\|_2}.$$

See [W2,W3,LBSK] for the proof.

By (1.8), $\tilde{\psi}_0$, $\tilde{\psi}_0=\tilde{\varphi}(\beta_0^2)+f_0$ satisfies the conditions of the above proposition. Thus, the corresponding solution $\tilde{\psi}(t)$ admits the representation

$$\tilde{\psi}(x,t)=e^{i\tilde{\Phi}}\tilde{\lambda}^{1/2}(t)\left(\tilde{\varphi}(z,\tilde{a}(t))+\tilde{f}(z,t)\right),\quad \tilde{\Phi}=\tilde{\mu}(t)-\frac{\tilde{\beta}}{4}z^2,\ \ z=\tilde{\lambda}(t)x,$$

where $\tilde{\sigma}(t)=(\frac{\tilde{\mu}(t)}{2},\tilde{\lambda}(t),\tilde{\beta},\tilde{a}(t))$, $\tilde{\sigma}(0)=(0,1,0,\beta_0^2)$ is a continuous trajectory satisfying (2.3), $\|\tilde{f}\|_2$, $\tilde{\lambda}\frac{\|\varphi_{0x}\|_2}{\|\psi_x(t)\|_2}-1$, $\tilde{\beta}$, $\tilde{a}$ being small uniformly with respect to t.

By conformal invariance we can write now the solution $\psi(t)$ of the Cauchy problem (1.1,7) in the form (2.2) where

$$\mu(t)=\tilde{\mu}(\rho),\ \ \lambda(t)=(1-\beta_0 t)^{-1}\tilde{\lambda}(\rho),$$

$$\beta(t)=\beta_0(1-\beta_0 t)\tilde{\lambda}^{-2}+\tilde{\beta}(\rho),\quad a(t)=\tilde{a}(\rho),\ \ \rho=\frac{t}{1-\beta_0 t},$$

$f(z,t)=\tilde{f}(z,\rho)$ satisfying the orthogonality conditions (2.3).

By (i) of proposition 1.1, λ admits the estimate

$$\lambda(t)\geq c(T^*-t)^{-1/2}. \tag{2.5}$$

Remark that since $\psi(t)\in C^1([0,T^*)\to H^{-1})$ the trajectory $\sigma(t)$ belongs in fact, to C^1.

2.3. Differential equations. We write a system of equations for σ and f in explicit form. Introduce a new time variable τ:

$$\tau=\int_0^t ds\lambda^2(s).$$

By (2.5), $\tau\to\infty$ as $t\to T^*$.

In terms of f (1.1) takes the form

$$i\vec{f}_\tau=\tilde{H}(a)\vec{f}+N, \tag{2.6}$$

where

$$N=N_0(a,f)+N_1(\tilde{\varphi},f)+l(\sigma)\left(\tilde{\varphi}\begin{pmatrix}1\\1\end{pmatrix}+\vec{f}\right)-ia_\tau\tilde{\varphi}_a\begin{pmatrix}1\\1\end{pmatrix},$$

$$N_0(a,f)=\frac{az^2}{4}(\theta(hz)-1)\sigma_3(\tilde{\varphi}\begin{pmatrix}1\\1\end{pmatrix}+\vec{f}), \tag{2.7}$$

$$N_1(\tilde{\varphi},f)=-|\tilde{\varphi}+f|^4\sigma_3(\tilde{\varphi}\begin{pmatrix}1\\1\end{pmatrix}+\vec{f})+\tilde{\varphi}^5\begin{pmatrix}1\\-1\end{pmatrix}-V(\tilde{\varphi})\vec{f},$$

$$l(\sigma) = (\mu_\tau - 1)\sigma_3 + i(\beta - \frac{\lambda_\tau}{\lambda})(z\partial_z + \frac{1}{2}) + (a - \beta_\tau + \beta^2 - 2\beta\frac{\lambda_\tau}{\lambda})\frac{z^2}{4}\sigma_3.$$

Substitute the expression for $\vec{f}_\tau$ from (2.6,7) into the derivative of the orthogonal conditions. The result can be written down as follows:

$$(A_0(a) + A_1(a, f))\vec{\eta} = \vec{g}(a, f). \tag{2.8}$$

Here

$$\vec{\eta} = (\frac{\mu_\tau - 1}{2}, \frac{\lambda_\tau}{\lambda} - \beta, \beta_\tau - \beta^2 + 2\beta\frac{\lambda_\tau}{\lambda} - a, a_\tau),$$

$$A_0 = 2\begin{pmatrix} 0 & 0 & 0 & -(\tilde{\varphi}_a, \tilde{\varphi}) \\ 2(\tilde{\varphi}, \tilde{\varphi}_E) & 0 & -(\frac{z^2}{4}\tilde{\varphi}, \tilde{\varphi}_E) & 0 \\ 0 & -i((z\partial_z + \frac{1}{2})\tilde{\varphi}, e_2) & 0 & -i(\tilde{\varphi}_a, e_2) \\ 2(\tilde{\varphi}, e_3) & 0 & -(\frac{z^2}{4}\tilde{\varphi}, e_3) & 0 \end{pmatrix},$$

$$(A_1\vec{\eta})_j = \left\langle l(\sigma)\vec{f}, \sigma_3\vec{e}_j \right\rangle + ia_\tau \left\langle \vec{f}, \sigma_3\vec{e}_{ja} \right\rangle,$$

$$g_j(a, f) = -\left\langle N_0 + N_1, \sigma_3\vec{e}_j \right\rangle.$$

By propositions 2.1,2, as $a \to 0$,

$$A_0(a) = iB_0 + O(a). \tag{2.9}$$

In principle the system (2.8) can be solved with respect to the derivatives η and together with equation (2.6) constitutes a complete system for $\sigma, \vec{f}$:

$$i\vec{f}_t = H(a)\vec{f} + N'(a, f), \tag{2.10}$$

$$\vec{\eta} = G(a, f), \tag{2.11}$$

$$f|_{t=0} = f_0, \quad \sigma|_{t=0} = (0, 1, \beta_0, \beta_0^2).$$

Here $H(a) = (-\partial_z^2 + 1 - \frac{az^2}{4})\sigma_3 + V(\tilde{\varphi}(a))$, $N' = N - a\frac{z^2}{4}(\theta - 1)\sigma_3\vec{f}$.

2.4. Effective equations.

In order to derive a system of effective equations consider the main nonlinear terms of (2.10), (2.11). Below it will become clear that the function a depends slowly on τ . More precisely,

$$a \sim \ln^{-2}(\tau + \tau^*), \tag{2.12}$$

with some $\tau^* = O(e^{\frac{2S_0}{\beta_0}}\beta_0^3)$. We shall also see that the contribution f of the continuous spectrum asymptotically is of the order $\Gamma^{1/2}$ (in the uniform norm), $\Gamma = e^{-\frac{2S_0}{h}}$, $h = \sqrt{a}$, $S_0 = \int_0^2 ds\sqrt{1 - s^2/4}$, and of the order Γ for z not too large. In its turn the vector η also has order Γ. We shall use these facts while deriving the equations. At this stage we are not worrying about formal justification.

The main terms of N are generated by the expression

$$N \sim F_0(a)\begin{pmatrix}1\\-1\end{pmatrix}, \quad F_0(a) = a\frac{z^2}{4}(\theta-1)\tilde{\varphi}. \tag{2.13}$$

Thus, it is clear that in the region $|z| \geq const\, h^{-1}$ the main order term of f is given by the expression

$$f \sim -(L(a) - i0)^{-1}F_0(a), \tag{2.14}$$

where $L(a) = -\partial_z^2 + 1 - a\frac{z^2}{4}$. The sign "-" (in $-i0$) is essential: it means that $e^{-ih\frac{z^2}{4}}(L(a)-i0)^{-1}F_0(a)$ has finite energy.

For the following it is convenient to write $f = f^0 + f^1$, $f^0 = -(L(a)-i0)^{-1}F_0(a)$. It will become clear later that in the region $|z| \geq const\, h^{-1}$ f^0 and f^1 are of the order $\Gamma^{1/2}$ and Γ respectively while for $|z| \sim 1$ both f^0 and f^1 have order Γ.

Consider (2.11). The main term of G is given by the expression

$$G \sim A_0^{-1}(a)\vec{g}^0(a),$$

where $g_j^0 = -\langle N_0(a, f_0), \sigma_3 \vec{e}_j\rangle$. So we rewrite (2.11) in the form

$$\vec{\eta} = G_0(a) + G_R(a, f). \tag{2.15}$$

Here $G_0(a) = A_0^{-1}(a)\vec{g}^0(a)$, G_R being the remainder.

The behavior of $f^0(a)$, $G_0(a)$ in the limit $a \to 0$ is described by the following proposition.

Proposition 2.4. *For $a > 0$ sufficiently small, $f^0(a)$, $G_0(a)$ satisfy the estimates*

$$\|f^0(a)\|_\infty \leq c\Gamma^{1/2-\epsilon}, \quad \|G_0(a)\| \leq c\Gamma^{1-\epsilon}.$$

Moreover, G_0^3 admits the following representation

$$G_0^3(a) = -2\nu_0\Gamma(1 + O(a^{1/2})), \quad \nu_0 = \frac{8\varphi_\infty^2}{e}.$$

This asymptotic estimate can be differentiated any number of times with respect to a.

Here and in what follows the letter ϵ is used as a general notation for small positive constants that depends on the choice of the cut off function θ and tend to zero as $\delta \to 0$.

In order to estimate qualitatively the behavior of a, consider the last equation of (2.15) neglecting the remainder G_R:

$$a_\tau = G_0^3(a).$$

We denote by $a_0(\tau)$ solution of this equation with initial data $a_0(0) = \beta_0^2$. It is easy to check that $h_0 = \sqrt{a_0}$ admits the representation

$$h_0^{-1}(\tau) = \frac{1}{2S_0}(\ln \nu_1(\tau+\tau^*) + 3\ln\ln \nu_1(\tau+\tau^*)) + O(\frac{\ln\ln(\tau+\tau^*)}{\ln(\tau+\tau^*)}), \tag{2.16}$$

as $\tau + \tau^* \to +\infty$, $\nu_1 = \frac{\nu_0}{4S_0^2}$, $\tau^* = \frac{\beta_0^3}{2S_0\nu_0}e^{\frac{2S_0}{\beta_0}}(1 + O(\beta_0))$.

2.5. Estimates of soliton parameters.

Following [BP] we consider system (2.10,11) on some finite interval $[0,\tau_1]$ and later investigate the limit $\tau_1 \to \infty$.

On the interval $[0,t_1]$, $t_1 = t(\tau_1)$ we approximate the trajectory $\sigma(t)$ by $\sigma_1(t)$ where $\sigma_1(t) = (\frac{\mu(t)}{2}, \lambda_1(t), \beta_1(t), a_1(t))$ is the solution of the following Cauchy problem

$$\lambda_1^{-3}\lambda_1' = \beta_1,\ \lambda_1^{-2}\beta_1' + \beta_1^2 = a_1,\ a_1' = 0,$$

$$\lambda_1(t_1) = \lambda(t_1),\ \ \beta_1(t_1) = a^{1/2}(t_1),\ \ a_1(t_1) = a(t_1).$$

Introduce a natural system of norms for the components of the solution ψ on the interval $[0,\tau_1]$:

$$s_0(\tau) = \sup_{s\le\tau} |h(s) - h_0(s)| h_0^{-2}(s),$$

$$s_1(\tau) = \sup_{s\le\tau} |\beta(s) - h(s)| h_0^{-2}(s) p^{-1}(s;\kappa_1,r_1),$$

$$s_2(\tau) = \sup_{\tau\le s\le\tau_1} \left|\beta(s) - \beta_1(s)\frac{\lambda_1^2(s)}{\lambda^2(s)}\right| h_0^{-1}(s) p^{-1}(s;\kappa_2,r_2),$$

$$M_0(\tau) = \sup_{s\le\tau} \|f(s)\|_\infty p^{-1}(s;\kappa_0,r_0),$$

$$M_1(\tau) = \sup_{s\le\tau} \|(1+|z|)^{-\nu} f^1(s)\|_\infty p^{-1}(s;\kappa_3,r_3),\ \ \nu\ge 2,$$

$$M_2(\tau) = \sup_{s\le\tau} \|\rho f(s)\|_2 p^{-1}(s;\kappa_4,r_4),$$

where

$$p(\tau;\kappa,r) = e^{-\kappa\int_0^\tau ds h_0(s)} + e^{-r\frac{S_0}{h_0(\tau)}},\ \ \rho = e^{\frac{(1-\delta_1)}{h_0}\int_0^{h_0|z|} ds\sqrt{1-\frac{s^2}{4}\theta(s)}},$$

$\kappa_4 = \frac{d_0}{4}$, $\kappa_0 = \kappa_3 = \frac{7}{8}\kappa_4$, $\kappa_1 = \frac{3}{2}\kappa_4$, $\kappa_2 = \frac{5}{4}\kappa_4$, $r_0 = \frac{3}{4}$, $r_1 = \frac{15}{8}$, $r_2 = \frac{7}{4}$, $r_3 = \frac{4}{3}$, $r_4 = \frac{3}{2}$, $\delta_1 > 0$ is supposed to be a sufficiently small fixed number.

At last, set

$$\hat s_j = s_j(\tau_1),\ j = 0,1,\quad \hat s_2 = s_2(0),\quad \hat M_j = M_j(\tau_1).$$

Consider equation (2.11). It follows immediately from (2.7,8,9) and from proposition 2.4 that

$$|\eta| \le W(M,s)\left[\Psi_0(M) e^{-2\kappa_3\int_0^\tau ds h_0(s)} + e^{-(2-\epsilon)\frac{S_0}{h_0(\tau)}}\right], \tag{2.17}$$

$$|G_R| \le W(M,s)\Psi_1(M)\left[e^{-\frac{3\kappa_3}{2}\int_0^\tau ds h_0(s)} + e^{-\frac{3r_4}{2}\frac{S_0}{h_0(\tau)}}\right], \tag{2.18}$$

$$\Psi_0(M) = M_2 M_0^4 + \beta_0^2 M_1^2 + M_2^2,$$

$$\Psi_1(M) = e^{-\gamma/\beta_0} + M_2 M_0^4 + M_2^2,$$

with some $\gamma > 0$.

We use $W(M,s)$ as a general notation for functions of M_j, $j=0,1,2$, s_k, $k=0,1,2$, defined on $\mathbb{R}^6$, which are bounded in some finite neighbourhood of 0 and may acquire the infinite value $+\infty$ outside some larger neighborhood. It will be assumed that W does not depend on β_0. In all the formulas where W appear it would be possible to replace them by some explicit expressions but such expressions are useless for our aims.

Using (2.17,18) and proposition 2.4 it is not difficult to prove the following inequalities

$$s_0 \le W(M,s)\beta_0^{-4}\Psi_1(M),$$

$$s_1 \le W(M,s)\left(e^{-\frac{\gamma}{\beta_0}} + \beta_0^{-3}\Psi_0(M)\right), \tag{2.19}$$

$$s_2 \le W(\hat{M},\hat{s})\left(e^{-\frac{\gamma}{\beta_0}} + \beta_0^{-3}\Psi_0(\hat{M})\right), \ \ \gamma > 0.$$

2.6. Estimates of f. For f one can get the following set of estimates

$$M_0, M_1 \le W(\beta_0^{-1}M,s)\left[\beta_0^{2N} + \beta_0^{N-1}M_0 + \beta_0^{-1}(M_2 + (M_0+M_1)^2)\right], \tag{2.20}$$

$$M_2 \le W(\hat{M},\hat{s})\beta_0^{-K_0}[\beta_0^{2N} + (\hat{M}_0 + \hat{M}_1 + \hat{M}_2)^2)], \tag{2.21}$$

with some $K_0 \ge 0$.

By the way of explanation we remark that the deriving of these inequalities is based on the fact that H depends slowly on τ and on some suitable estimates of the group $e^{-i\tau H(a)}$, a being fixed.

2.7. Estimates of majorants. Consider the system of inequalities (2.19,20,21). Introduce new scales:

$$\hat{M}_j = \beta_0\hat{\mathbb{M}}_j, \ j=0,1, \quad \hat{M}_2 = \beta_0^{K_0+2}\hat{\mathbb{M}}_2.$$

Remark that one can choose the function W to be spherically symmetric and monotone. Then in terms of $\hat{\mathbb{M}}_j$ the inequalities (2.19-21) can be written in the form

$$\beta_0\hat{s}_0,\ \hat{s}_1,\ \hat{s}_2 \le W(\hat{\mathbb{M}},\hat{s})\left[e^{-\frac{\gamma}{\beta_0}} + \beta_0(\hat{\mathbb{M}}_0+\hat{\mathbb{M}}_1)^2 + \beta_0^{2K_0+1}\hat{\mathbb{M}}_2^2)\right],$$

$$\hat{\mathbb{M}}_2 \le W(\hat{\mathbb{M}},\hat{s})\left[\beta_0^{2N-2K_0-2} + \beta_0^2\hat{\mathbb{M}}_2^2 + \beta_0^{-2K_0}(\hat{\mathbb{M}}_0+\hat{\mathbb{M}}_1)^2\right],$$

$$\hat{\mathbb{M}}_0 + \hat{\mathbb{M}}_1 \le W(\hat{\mathbb{M}},\hat{s})\left[\beta_0^{2N-1} + \beta_0^{N-1}(\hat{\mathbb{M}}_0+\hat{\mathbb{M}}_1) + \beta_0^{K_0}\hat{\mathbb{M}}_2\right].$$

Fix the ball $\|\hat{\mathbb{M}}\|^2 + \|\hat{s}\|^2 \le R$ where $W(\hat{\mathbb{M}},\hat{s})$ is a bounded by a constant. Then the above inequalities can be simplified

$$\begin{aligned} \beta_0\hat{s}_0,\ \hat{s}_1,\ \hat{s}_2 &\le W_1\left[\beta_0^{4N-1} + \beta_0^{2K_0+1}\hat{\mathbb{M}}_2^2\right], \\ \hat{\mathbb{M}}_2 &\le W_2\left[\beta_0^{2N-2K_0-2} + \hat{\mathbb{M}}_2^2\right], \\ \hat{\mathbb{M}}_0 + \hat{\mathbb{M}}_1 &\le W_3\left[\beta_0^{2N-1} + \beta_0^{K_0}\hat{\mathbb{M}}_2\right], \end{aligned} \tag{2.22}$$

where W_j, $j = 1, 2, 3$, some constants that do not depend on β_0 provided $N > 1$, β_0 is sufficiently small.

Choosing $N > 1 + K_0$ one gets that for β_0 sufficiently small the solution of (2.22) can belong either to a small neighborhood of 0 or to some domain whose distance from 0 is bounded uniformly with respect to β_0. Since all $\hat{\mathbb{M}}_j$, s_j are continuous functions of τ_1 and for $\tau_1 = 0$ are small only the first possibility can be realized.

As a consequence, one finally obtains

$$M_0, M_1 \leq c\beta_0^{2N-K_0-1}, \quad M_2 \leq c\beta_0^{2N-K_0},$$

$$\beta_0 s_0,\ s_1 \leq c\beta_0^{4N-2K_0-3},\ \ \tau \leq \tau_1.$$

The constant c here does not depend either on β_0 or on τ_1. Since τ_1 is arbitrary these estimates are valid, in fact, for $\tau \in \mathbb{R}$.

The statement of the theorem 1.1 is a simple consequence of the above inequalities and (2.17).

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