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RESONANCE EXPANSIONS IN WAVE PROPAGATION

MACIEJ ZWORSKI

1. Introduction

The purpose of this lecture is to survey some very recent results obtained by Burq, Christiansen, Tang, and the author [2],[3],[29] and by Stefanov [24], on expanding propagators in terms of resonances. The results are technically quite simple, at least by the standards of the subject, and the appeal of this study lies in its connection with applied problems.

Resonances constitute a mathematical model of meta-stable states: a resonance is a complex number whose real part is the rest energy or the rate of oscillation of the state, and its imaginary part, the rate of decay of the state. They appear in many branches of mathematics, physics and chemistry, from particle physics to the theory of automorphic forms – see [32] for a general introduction. Our motivation comes mostly from molecular dynamics (semi-classical expansions), acoustic or electromagnetic scattering (wave expansions), and from hyperbolic scattering (expansions for the modular surface).

The intuition above is in its very nature dynamical. At the same time the clearest mathematical definitions of resonances are stationary: if the system is decribed by a Hamiltonian P, with propagation given by either the wave equation, $\sin t\sqrt{P}/\sqrt{P}$, or by the Schrödinger equation $\exp(itP/h)$, the resonances are defined as the poles of the meromorphic continuation of $(P-\lambda^2)^{-1}$, and of $(P-z)^{-1}$, respectively. A pure state, corresponding to a resonance, should behave as $\exp(it\lambda)$ and $\exp(-itz/h)$, as $t \to \infty$, with the convention that $\operatorname{Im}\lambda > 0$ and $\operatorname{Im} z < 0$. Of course, the two meromorphic continuations are clearly equivalent, and the different conventions come from the traditions of wave and Schrödinger equations.

The dynamical nature of resonances was emphasized early by Lax and Phillips [7] and their celebrated semi-group still provides the most elegant connection between the stationary and dynamical definitions. They showed, that for obstacle problems in odd dimensions, the restriction of the wave group $U(t)$ to a naturally defined interaction space (roughly speaking, obtained by taking the orthocomplement of the spaces of incoming and outgoing initial data), gives a semi-group, $Z(t)$, and

$$Z(t) = e^{itB}\,,$$

with the spectrum of the non-self-adjoint operator B given by the poles of the meromorphic continuation of $(P-\lambda^2)^{-1}$ from $\operatorname{Im}\lambda < 0$ to $\mathbb{C}$ – see [22] for a careful analysis in greater generality.

One of the main applications of this point of view was the trace formula for resonances in odd dimension proved by Bardos-Guillot-Ralston [1] (for $|t| > R$) and Melrose [11] (for $|t| > 0$),

$$\operatorname{tr}\,(U(t) - U_0(t)) = \sum_{\text{resonances}} e^{i\lambda|t|}\,, \quad t \in \mathbb{R} \setminus \{0\}\,, \tag{1.1}$$

where $U_0(t)$ is the free wave group.

As far as expanding $U(t)$ in terms of resonances,

$$U(t) \sim \sum_{\text{resonances}} e^{i\lambda t} w_\lambda \otimes w_\lambda \,, \quad w_\lambda \ \text{ a resonant state} \,, \quad t \to \infty \,, \tag{1.2}$$

the semi-group method provides expansions in non-trapping situations only – see Theorem 3.1 for the precise statement and Sect.3 of [29] for a review of Vainberg's direct proof [27] .

The recent advances in the understanding of the trace formula (1.1) in all dimensions and in more general settings, by Guillopé and the author [6], Sjöstrand [19],[20], and the author [30],[31], permit now some extensions of (1.2) to more general situations. This development owes a lot of the work of Sjöstrand [19], Stefanov and Vodev [26], and to some previous work of Tang and the author (see [28] and references given there). In works surveyed here, [2],[3],[24],[29],[27], the contour in the spectral decomposition of the propagator is deformed. The resonance expansion comes from a residue calculation, and the main issue is estimating the resolvent in the non-physical half-plane (that is, the half-plane where the poles are).

We remark that the time dependent theory of resonances has also been investigated recently in [12] and [23]. The study there is motivated by the *Fermi Golden Rule* and is concerned with the time behaviour of a *single* state obtained by perturbing a bound state embedded in the continuous spectrum. That method is also applied to non-linear problems and that constitutes an exciting new development.

We are concerned with situations which are classically *trapping* and where, at high energies, we expect a lot of resonances near the real axis. Eventually, it is the influence of large clouds of resonances on propagation that should of interest. At the energies at which molecular reactions take place there are normally many resonant states and it is their contributions to the propagation of a state that seems to be of interest – see [15] and references given there. The general expansions in Sect.3 and 4 are all in the "non-overlapping régime":

$$\frac{\langle \Gamma \rangle}{\Delta E} \ll 1 \,,$$

where $\langle \Gamma \rangle$ is the average decay rate and ΔE is the average energy level spacing: in semi-classical Theorem 4.1, $\Gamma < h^M$ and we expect $\Delta E \sim h^n$.

On the other hand the expansions in Theorem 6.2 are concerned with resonances in the "overlapping régime": when we rescale to the semi-classical situation ($h \sim |\lambda|^{-1}$) then

$$\frac{\langle \Gamma \rangle}{\Delta E} \sim 1 \,, \quad \Gamma \sim h \,, \quad \Delta E \sim h \,.$$

In Theorem 6.1, we still expect $\Gamma \sim h$, but $\Delta E \sim h(\log(1/h))^{-1}$. We remark that the random matrix methods used to study such phenomena in transition state theory [15] remain inaccessible in rigorous work on quantum mechanics.

A better understanding of resonance expansions should lead to new ways of computing resonances or to inverse results. The classical method of Prony [18] has been used in non-trapping situations [8] to compute resonances, and it is frequently used in applied work – see for instance [9] and references given there. Roughly speaking, by measuring a signal at different times, the complex frequencies in the expansion of the wave can be detected.

In connection with computing and detecting resonances we make the final comment of this introduction. The standard way of seeing resonances in experiments is through the *Breit-Wigner approximation*: a resonance at $E_0 - i\Gamma_0$, will show up at real energies E through terms

$$\frac{\Gamma_0}{(E - E_0)^2 + \frac{1}{4}\Gamma_0^2}$$

appearing in measured quantities (see [13, (2.13)] for an example in the context of modern chemistry), such as the derivative of the scattering phase, or more realistically (from the experimental point of view), the scattering cross-section. Both have been rigorously studied in the case of a *single* semi-classical resonance by Gérard-Martinez-Robert [5], and the former, for many resonances at high energies by Petkov and the author [16],[17]. The methods used to prove the expansions presented here, should be also useful in extending the results on the scattering phase to scattering amplitudes and cross-sections. Roughly speaking, the relation of the Breit-Wigner formula of [16] to the Breit-Wigner formulæ for scattering matrices, should be the same as the relation of the trace formula for resonances (1.1) to the resonance expansions (1.2).

2. Assumptions on the operator

To make the statements precise we recall here very general assumptions on the operators we consider. They are made in order to avoid the analysis of specific aspects of obstacle, potential, or metric scattering. Thus we follow the “black box” formalism introduced in [21] and generalized further in [19]. The two most interesting and easy to state cases are

$$P = -\Delta \quad \text{on} \quad \mathbb{R}^n \setminus \mathcal{O} \quad \text{with the Dirichlet or Neumann boundary condition},$$

in the case of wave expansions and

$$P = -h^2\Delta + V(x)\,, \quad |V(x)| \leq C\langle x \rangle^{-\epsilon}\,, \quad V \text{ analytic in a conic neighbourhood of infinity},$$

in the case of semi-classical expansions.

Our general operator, P, $\mathcal{H}$, a complex Hilbert space with an orthogonal decomposition

$$\mathcal{H} = \mathcal{H}_{R_0} \oplus L^2(\mathbb{R}^n \setminus B(0,R_0))\,,$$

where $R_0 > 0$ is fixed and $B(x,R) = \{y \in \mathbb{R}^n : |x-y| < R\}$.

The corresponding orthogonal projections are denoted by $u|_{B(0,R_0)}$ and $u|_{\mathbb{R}^n\setminus B(0,R_0)}$ or by $1\!\!1_{B(0,R_0)}u$ and $1\!\!1_{\mathbb{R}^n\setminus B(0,R_0)}u$ respectively, where $u \in \mathcal{H}$.

We work in the semi-classical setting and for each $h \in (0,h_0]$, we have

$$P(h) : \mathcal{H} \longrightarrow \mathcal{H}$$

with the domain $\mathcal{D}$, independent of h, and satisfying

$$1\!\!1_{\mathbb{R}^n\setminus B(0,R_0)}\mathcal{D} = H^2(\mathbb{R}^n \setminus B(0,R_0))$$

uniformly with respect to h (see [19] or [28] for a precise meaning of this statement).

We also assume that

$$1\!\!1_{B(0,R_0)}(P(h)+i)^{-1} : \mathcal{H} \longrightarrow \mathcal{H}_{R_0} \quad \text{is compact}\,, \tag{2.1}$$

$$1\!\!1_{\mathbb{R}^n\setminus B(0,R_0)}P(h)u = Q(h)(u|_{\mathbb{R}^n\setminus B(0,R_0)}), \quad \text{for } u \in \mathcal{D}\,, \tag{2.2}$$

where $Q(h)$ is a formally self-adjoint operator on $L^2(\mathbb{R}^n)$ given by

$$Q(h)v = \sum_{|\alpha|\leq 2} a_\alpha(x;h)(hD_x)^\alpha v \quad \text{for } v \in C_0^\infty(\mathbb{R}^n)$$

such that $a_\alpha(x;h) = a_\alpha(x)$ is independent of h for $|\alpha| = 2$, $a_\alpha(x;h) \in C_b^\infty(\mathbb{R}^n)$ are uniformly bounded with respect to h, here $C_b^\infty(\mathbb{R}^n)$ denotes the space of C^∞ functions on $\mathbb{R}^n$ with bounded derivatives of all orders,

$$\sum_{|\alpha|=2} a_\alpha(x;h)\xi^\alpha \geq (1/c)|\xi|^2, \quad \forall \xi \in \mathbb{R}^n\,,$$

for some constant $c > 0$, $\sum_{|\alpha|\leq 2} a_\alpha(x;h)\xi^\alpha \longrightarrow \xi^2$ uniformly with respect to h as $|x| \to \infty$.

The meromorphic continuation is guaranteed by the following analyticity assumption: there exist $\theta \in [0, \pi)$, $\epsilon > 0$ and $R \geq R_0$ such that the coefficients $a_\alpha(x; h)$ of $Q(h)$ extend holomorphically in x to

$$\{r\omega : \omega \in \mathbb{C}^n, \operatorname{dist}(\omega, \mathbf{S}^n) < \epsilon, r \in \mathbb{C}, |r| > R, \arg r \in [-\epsilon, \theta_0 + \epsilon)\}$$

with $\sum_{|\alpha|\leq 2} a_\alpha(x; h)\xi^\alpha \longrightarrow \xi^2$ uniformly with respect to h as $|x| \to \infty$ remains valid in this larger set of x's.

We use $P(h)$ to construct a self-adjoint operator $P^\sharp(h)$ on

$$\mathcal{H}^\sharp = \mathcal{H}_{R_0} \oplus L^2(M \setminus B(0, R_0))$$

as in [21] where $M = (\mathbb{R}/R\mathbb{Z})^n$ for some $R \gg R_0$. Let $N(P^\sharp(h), I)$ denote the number of eigenvalues of $P^\sharp(h)$ in the interval I, we assume

$$N(P^\sharp(h), [-\lambda, \lambda]) = \mathcal{O}((\lambda/h^2)^{n^\sharp/2}), \quad \text{for } \lambda \geq 1\,, \tag{2.3}$$

for some number $n^\sharp \geq n$.

Under the above assumptions on $P(h)$, the resonances close to the real axis can be defined by the method of complex scaling (see [19] and references given there). They coincide with the poles of the meromorphic continuation of the resolvent $(P(h) - z)^{-1}$ from $\operatorname{Im} z > 0$ to a conic neighbourhood of the positive half axis in the lower half plane. The set of resonances of $P(h)$ will be denoted by $\operatorname{Res}P(h)$ and we include them with their multiplicity.

Suppose that the operator P satisfies our assumptions with $h = 1$. The wave group of P can be defined abstractly by

$$\mathcal{U}(t) = \exp\begin{pmatrix} 0 & I \\ P & 0 \end{pmatrix} = \begin{pmatrix} D_t U(t) & U(t) \\ D_t^2 U(t) & D_t U(t) \end{pmatrix} \quad \text{with} \quad U(t) = i\left(\frac{\sin t\sqrt{P}}{\sqrt{P}}\right) \tag{2.4}$$

As usual we define the iterated domain by

$$\mathcal{D}^L = (P + i)^{-L}\mathcal{H}. \tag{2.5}$$

3. Expansions of scattered waves

We first recall the result on expansions of scattered waves in odd dimensions. It is due to Lax-Phillips [7] who proved it using their semi-group, and Vainberg [27] who provided a direct argument. In the obstacle case, the hard problem of showing that the classical non-trapping implies the "quantum" non-trapping (3.1) was resolved by Andersson, Melrose, Morawetz, Ralston, Sjöstrand, Strauss and Taylor – see [10], [7, Appendix to 2nd Edition], and references given there.

In the context of black-box scattering we consider the following "quantum" non-trapping condition: Let $\chi \in C_0^\infty(\mathbb{R}^n)$ be identically one near $B(0, R_0)$.

$$\begin{gathered} \forall\, a > R_0\,,\ |y| \leq a \ \ \exists\, C_a > 0 \\ U(t, x, y) \in C^\infty \text{ if } |x| < t - C_a, |y| > R_0 \\ \chi U(t)\chi : \mathcal{H} \to \mathcal{D}^\infty \end{gathered} \tag{3.1}$$

Theorem 3.1. *(Lax-Phillips* [7]*, Vainberg* [27]*) Assume the nontrapping condition* (3.1) *and let* $\chi \in C_0^\infty(\mathbb{R}^n)$ *be identically* 1 *in a neighbourhood of* $B(0, R_0)$*. Then, for any* $A > 0$ *and small* $\epsilon > 0$*, we have for* $t > 0$ *sufficiently large, and* $g \in \mathcal{H}$

$$\chi U(t)\chi g = \sum_{\substack{\lambda_j \in \operatorname{Res}(P) \\ \operatorname{Im}\lambda < A}} \sum_{m=0}^{M_j} e^{it\lambda_j} t^m \chi w_{j,m} + E_A(t)g$$

where $w_{j,m} \in \mathcal{D}^\infty$ are the resonant states corresponding to λ_j, and $E_A(t)g$ is the error term satisfying the following estimate

$$\|E_A(t)\|_{\mathcal{H}\to\mathcal{H}} \leq C(\epsilon)e^{-(A-\epsilon)t}$$

By resonant states we mean the elements of the range of the residue of the continuation of $(P-\lambda^2)^{-1}$ at λ_j. In particular,

$$(P-\lambda_j)^{m+1}w_{j,m} = 0\,.$$

To formulate the result in trapping situations let us first give a "quantum" trapping condition:

$$\exists\, \lambda_j \in \mathrm{Res}(P) \quad \mathrm{Im}\,\lambda_j = \mathcal{O}(|\lambda_j|^{-N}) \quad \text{for any } N>0. \tag{3.2}$$

This assumption is made in order to make the expansion non-trivial. In a great variety of classically trapping situations (roughly speaking, whenever an elliptic closed orbit is present) (3.2) holds, as was shown in successive generality and detail by Stefanov-Vodev [26], Tang and the author [28], and Stefanov [25].

We also need to assume that

$$\exists\, l \quad \lambda,\mu \in \mathrm{Res}(P)\cap\{\zeta \;:\; \mathrm{Im}\,\zeta < \langle\zeta\rangle^{-K}\}\,,\ \lambda\neq\mu\,,\ \Longrightarrow\ |\lambda-\mu|\geq C(\max\{|\lambda|,|\mu|\})^{-l}\,. \tag{3.3}$$

and, in the same region,

$$\exists\, \alpha \quad \text{algebraic multiplicity of } \lambda \leq \alpha \text{ for all } \lambda\in\mathrm{Res}(P)\,. \tag{3.4}$$

Weaker assumptions are also possible but none can be verified – all that is needed is a verification of the same assumption for the eigenvalues of the reference operator $P^\sharp$, and that is believed to be true generically. See also [29, Sect.5] for an application of the method in for spherically symmetric metric perturbations.

Theorem 3.2. *(Tang-Zworski,[29]) Assume that P satisfies the trapping condition* (3.2)*, the separation condition* (3.3)*, and the condition on the uniform bound on the multiplicity of resonances* (3.4)*. Let $\chi \in C_0^\infty(\mathbb{R}^n)$ be identically equal to 1 near the ball $B(0,R_0)$. Then, for any sufficiently large M, we have*

$$\chi U(t)\chi g = \sum_{\substack{\lambda_j\in\mathrm{Res}P\\ \mathrm{Im}\,\lambda_j\leq\langle\lambda_j\rangle^{-K}}} \sum_{m=1}^{m_j} e^{it\lambda_j}t^m w_{j,m} + E(t)g\,, \quad g\in\mathcal{D}^M\,, \tag{3.5}$$

for some K depending on k, α and $n^\sharp$. As in the nontrapping case, $w_{j,m}$'s are the cut-off resonance states associated with the resonance λ_j and $E(t)$ is the error term satisfying the following estimate

$$\|E(t)\|_{\mathcal{D}^M\to\mathcal{H}} = \begin{cases} C_N t^{-N} & \text{when } n \text{ is odd} \\ Ct^{-n+1} & \text{when } n \text{ is even}\,, \end{cases} \tag{3.6}$$

for any large constant N depending on M, α, l and $n^\sharp$.

The double sum in (3.5) should be understood as follows: the convergence of the outer sum is absolute in $\mathcal{L}(\mathcal{D}^M,\mathcal{H})$. However we *cannot* control the absolute convergence of the double sums. In principle, there could be cancellations in the inner sum.

By giving up convergence and considering instead sums of finitely many terms, with number of terms depending on time an unconditional result can be obtained. It is slightly different depending on finer assumptions on P which we state as three cases:

Case 1	$P\|_{\mathbb{R}^n\setminus B(0,R_0)} = -\Delta\|_{\mathbb{R}^n\setminus B(0,R_0)}$	n odd
Case 2	$P\|_{\mathbb{R}^n\setminus B(0,R_0)} = -\Delta\|_{\mathbb{R}^n\setminus B(0,R_0)}$	n even
Case 3	$P\|_{\mathbb{R}^n\setminus B(0,R_0)} = Q\|_{\mathbb{R}^n\setminus B(0,R_0)}$	any n

where Q is an elliptic operator close to the Laplacian at infinity – see (2.2) with $h = 1$.

Theorem 3.3. *(Burq-Zworski, [2]) Let P be an operator satisfying the assumptions of Sect.2 with $h = 1$. Let $\chi \in \mathcal{C}_c^\infty(\mathbb{R}^n)$ be equal to one on a neighbourhood of $B(0, R_0)$ and $\Psi \in C^\infty(\mathbb{R})$ be an even function such that*

$$\Psi(x) = 1 \begin{cases} \text{for } x \in \mathbb{R} \text{ in cases 1 and 2} \\ \text{for } x \geq 1 \text{ in case 3} \end{cases}, \qquad \Psi(x) = 0 \quad \text{near } 0 \text{ in case 3} \tag{3.7}$$

For every $M > M_0$, there exist $\epsilon = \epsilon(M) > 0$, a function $c(t)$ satisfying $|c(t) - t^\epsilon| \leq C$, and

$$\begin{gathered} \chi U(t)\Psi(\sqrt{P})\chi = \sum_{\substack{\operatorname{Im}\lambda_j > -\langle\lambda_j\rangle \\ 1<|\operatorname{Re}\lambda_j|<c(t)}} \chi \operatorname{Res}(e^{-it\bullet}R(\bullet), \lambda_j)\chi + E(t) + W(t) \\ \lambda_j^2 \in \operatorname{Res}(P)\,,\ \operatorname{Im}\lambda_j < 0\,, \\ \|E(t)\|_{\mathcal{D}^L \to \mathcal{H}} \leq \begin{cases} C_M t^{K_0 - \epsilon L} & \text{in cases 1 and 3} \\ C t^{-n+1} & \text{in case 2}\,, \end{cases} \end{gathered} \tag{3.8}$$

where K_0 is a fixed constant and L is large enough in case 2, and $W(t)$ corresponds to the contribution from the pure point spectrum of P.

We note that when the algebraic multiplicity is equal to the geometric multiplicity we have

$$\operatorname{Res}(e^{-it\bullet}R(\bullet), \lambda_j) = e^{-it\lambda_j}\operatorname{Res}(R(\bullet), \lambda_j)\,,$$

while in general, powers of t will appear in the expansion.

The term $W(t)$ has the usual expression:

$$W(t) = i \sum_{\mu_k \in \sigma_{pp}(P)} \chi \frac{\sin(t\sqrt{\mu_k})}{\sqrt{\mu_k}} \Psi(\sqrt{\mu_k})\Pi_{\mu_k}\chi\,,$$

where Π_μ is the orthogonal projection on the eigenspace of μ.

4. Semi-classical expansions

In semi-classical situations, we would like to obtain results for the Schrödinger equation, and with errors which are small as $h \to \infty$, rather than only as $t \to \infty$. Unlike in the wave-equation case, that is probably the best one can hope for. Since inverse of the distance to the real axis gives the life-span of a resonance, the times at which the expansions are valid have to be large enough to eliminate the contribution of other resonances (see the remark at the end of Sect.3). Our method also gives an expansion of scattered classical waves in terms of scattering poles close to the real axis. This expansion is weaker than the expansion presented in [29] but the advantage is that it does not depend on any hard to verify conditions (which was the case in [29]).

Theorem 4.1. *(Burq-Zworski, [2]) Let $P(h)$ be an operator satisfying the assumptions of Sect.2 and let $\chi \in \mathcal{C}_c^\infty(\mathbb{R}^n)$ be equal to one on a neighbourhood of $B(0, R_0)$. Let $\psi \in \mathcal{C}_c^\infty((0, \infty))$ and let* chsupp $\psi = [a, b]$. *We put $\mu(z) = z$ or $\sqrt{z}$, with the convention that $\sqrt{z} > 0$ for $z > 0$. There exists $0 < \delta < c(h) < 2\delta$ such that for every $M > M_0$ there exists $L = L(M)$, and we have*

$$\begin{gathered} \chi e^{-it\mu(P)/h}\chi\psi(P) = \sum_{z \in \Omega(h)\cap \operatorname{Res}(P)} \chi \operatorname{Res}(e^{-it\mu(\bullet)/h}R(\bullet, h), z)\chi\psi(P) \\ + \mathcal{O}_{\mathcal{H}\to\mathcal{H}}(h^\infty)\,, \quad \text{for } t > h^{-L}\,, \\ \Omega(h) = (a - c(h), b + c(h)) - i[0, h^M)\,, \end{gathered} \tag{4.1}$$

and where $\operatorname{Res}(f(\bullet), z)$ *denotes the residue of a meromorphic family of operators, f, at z.*

The function $c(h)$ depends on the distribution of resonances: roughly speaking we cannot "cut" through a dense cloud of resonances. Even in the very well understood case of the modular surface (Theorem 6.1 below) there is, currently at least, the need for some non-explicit grouping of terms. This is eliminated by the separation condition (3.3) which however is hard to verify.

5. Expansions of elasticity waves

By an ingenious exploitation of a pole free region, Stefanov recently obtained another unconditional result, with stronger convergence properties than in Theorem 3.3:

Theorem 5.1. *(Stefanov, [24]) Suppose that for K, p, $K \gg p$, sufficiently large there are* no *resonances in the region*

$$\langle\lambda\rangle^{-K} \leq \operatorname{Im}\lambda \leq \langle\lambda\rangle^{-K+p}\,.$$

Let $\chi \in C_0^\infty(\mathbb{R}^n)$ be identically equal to 1 near the ball $B(0, R_0)$. Then,

$$\chi U(t)\chi = \sum_{l=0}^{\infty} \sum_{\substack{\lambda_j \in \operatorname{Res} P\,,\, Re\lambda_j \in I_l \\ \operatorname{Im}\lambda_j \leq \langle\lambda_j\rangle^{-K}}} \chi \operatorname{Res}(e^{-it\bullet}R(\bullet), \lambda_j)\chi + E(t)\,, \tag{5.1}$$

where I_l are any sequence of intervals, $I_l = [a_l, b_l]$, $a_l < b_l < a_{l+1}$, satisfying

$$b_{l+1} - a_l > a_l^{-k}\,,\ k > n^\sharp\,,\ \operatorname{Re}(\operatorname{Res}(P) \cap \{\operatorname{Im}\lambda < \langle\lambda\rangle^{-K}\} \subset \bigcup_{l=0}^{\infty} I_l\,.$$

The error term satisfies the following estimate

$$\|E(t)\|_{\mathcal{D}^M \to \mathcal{H}} = \begin{cases} C_N t^{-N} & \text{when } n \text{ is odd} \\ C t^{-n+1} & \text{when } n \text{ is even}\,, \end{cases} \tag{5.2}$$

where M is sufficiently large and N can be made arbitrarily large by increasing M.

The existence of the intervals I_l is guaranteed by the upper bound $\mathcal{O}(r^{n^\sharp})$ for the number of resonances in a disc of radius r. The convergence of the outer sum in (5.1) is absolute but the absolute convergence of the total sum is not known.

A very natural and physical example to which this theorem can be applied comes from the work of Stefanov and Vodev [26]:

Theorem 5.2. *(Stefanov, [24]) Let $\mathcal{O} \subset \mathbb{R}^n$ have a smooth and strictly convex boundary. Suppose that $U(t)$ is the wave group associated with the Neumann problem in linear elasticity. Then, in the notation of Theorem 5.1, $\chi U(t)\chi$, has an expansion* (5.1) *in terms of resonances.*

6. Expansions in hyperbolic scattering

Hyperbolic scattering provides some interesting examples in which we can study resonance expansions. It would be very interesting to find out to what extend such expansions are valid for general Riemann surfaces.

Let $X_0 = \mathbb{H}^2/PSL(2;\mathbb{Z})$ be the quotient of the hyperbolic upper half plane by $PSL(2;\mathbb{Z})$, and let Δ be the Laplacian on X_0. To have an operator with the continuous spectrum starting at 0 we put

$$P = -\Delta + \frac{1}{4}\,.$$

This is a natural choice for any Riemann surface and we define the wave group for P as at the end of Sect.2.

The scattering matrix for P is well know to be given by

$$S(s) = \sqrt{\pi}\frac{\Gamma(s-\frac{1}{2})}{\Gamma(s)}\frac{\zeta(2s-1)}{\zeta(2s)} \tag{6.1}$$

where Γ is the Euler Γ-function and ζ is the Riemann ζ-function. One consequence of this is that the poles of the scattering matrix other than $s = 1$ correspond to the non-trivial zeros of $\zeta(2s)$.

Theorem 6.1. *(Christiansen-Zworski, [3]) Let $f, \chi \in C_c^\infty(X_0)$. Then there exist $v_{jk} \in C_c^\infty(X_0)$ such that as $t \to \infty$,*

$$\chi U(t) f = \frac{1}{2i}\sum_{\lambda_j \in \sigma_p(\Delta)} \left(\frac{e^{i\sqrt{\lambda_j - 1/4}t} - e^{-i\sqrt{\lambda_j - 1/4}t}}{\sqrt{\lambda_j - 1/4}} \right) \chi(z)\phi_j(z)(f, \phi_j) + \sum_{s_j \text{ poles of } S(s)} e^{(s_j - 1/2)t(\operatorname{sgn}(1/2 - \operatorname{Re} s_j))} \sum_{k \leq m(s_j)-1} v_{jk} t^k + \mathcal{O}(e^{-Nt})$$

for any N. Here ϕ_j are the eigenfuctions of $-\Delta$ on X_0.

The second example we consider is the hyperbolic half-cylinder $Y_l^0 \simeq (\mathbb{R}_+)_r \times (\mathbb{R}/l\mathbb{Z})_\theta$ with metric $dr^2 + \cosh^2 r d\theta^2$. The analysis is equally applicable to the case of the full cylinder $Y_l \simeq (\mathbb{R})_r \times (\mathbb{R}/l\mathbb{Z})_\theta$ with the same metric. In both cases the trapped set consists of one closed hyperbolic orbit which is well known to generate resonances on a lattice (see [6] and references given there).

In order to be consistent with our first example, we shall use as the variable $s = 1/2 - ik$. The scattering matrix $S_{0l}(s)$ for the hyperbolic half-cylinder with Dirichlet boundary conditions is

$$S_{0l}(s) = \bigoplus_{m \in \mathbb{Z}} s_{lm}(s) \tag{6.2}$$

with

$$s_{lm}(s) = \frac{2^{2s-1}\Gamma(1/2 - s)\Gamma((1 + s - i2\pi m/l)/2)\Gamma((1 + s + i2\pi m/l)/2)}{\Gamma(s - 1/2)\Gamma((2 - s - i2\pi m/l)/2)\Gamma((2 - s + i2\pi m/l)/2)} \tag{6.3}$$

and the resonances of the Dirichlet Laplacian associated to $s_{lm}(s)$ are $\pm i2\pi m/l - n$, $n \in 2\mathbb{N} - 1$.

Theorem 6.2. *(Christiansen-Zworski, [3]) Let $f \in \dot{C}_c^\infty(Y_l^0)$ and let $\chi \in C_c^\infty(Y_l^0)$. Then there exist $v_{m,n}, w_n \in C_c^\infty(Y_l^0)$ such that as $t \to \infty$,*

$$\chi U(t) f = \sum_{\substack{0 < n < \beta \\ m \in \mathbb{Z},\, n \in 2\mathbb{N}-1}} e^{(i2m\pi/l - n - 1/2)t} v_{m,n} + \sum_{\substack{0 < n < \beta \\ n \in 2\mathbb{N}-1}} e^{(-n-1/2)t} w_n t + \mathcal{O}(e^{(-\beta - 1/2)t})$$

if $\beta \notin 2\mathbb{N} - 1$. The same conclusion holds for $f \in C_c^\infty(Y_l)$ and Δ_l with the resonance set replaced by the resonance set of the full hyperbolic cylinder $\pm 2\pi i m/l - n$, $m \in \mathbb{N}_0$, $n \in \mathbb{N}_0$.

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