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On the L^2-instability of fluid flows

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1. Introduction

1. Consider the flow of an ideal incompressible fluid in a bounded domain $M \subset \mathrm{R}^2$ with a smooth boundary Γ. It is described by the Euler equations

$$\frac{\partial u}{\partial t} + (u, \nabla)u + \nabla p = 0; \tag{1}$$

$$\nabla \cdot u = 0. \tag{2}$$

Here $u = u(x,t)$ is the velocity field, $p = p(x,t)$ is the pressure, $x \in M, t \in \mathbf{R}$. The boundary condition:

$$(u, n)\Big|_{\Gamma} = 0, \tag{3}$$

i.e. the fluid slips along the boundary. The initial condition:

$$u(x,t)\Big|_{t=t_0} = u_0(x). \tag{4}$$

This system of equations, though looking innocent, is extremely hard to analyse. It is well known that the solution of the problem (1)-(4) exists for all $t \in \mathbf{R}$, is unique and regular, provided the initial velocity field $u_0(x)$ is sufficiently regular (say, $C^{1+\varepsilon}$; recall that the space dimension is 2; see [MP]). The next problem is, how does solution behave on long time intervals, or what is its asymptotic behavior as $t \to \infty$. This is a question of indefinite difficulty. Its particular case is the stability problem.

Suppose $u_0(x)$ is a steady solution of the problem (1)-(4), i.e.

$$(u_0, \nabla)u_0 + \nabla p = 0; \tag{5}$$

suppose $v_0(x)$ is a velocity field, satisfying (2) and (3) and close in some sense to $u_0(x)$. Let $v(x,t)$ be a solution of (1). (2) with the initial condition

$v(x,0) = v_0(x)$. The question is, whether the flow $v(x,t)$ remains close to $u_0(x)$ for all $t \in \mathbf{R}$? For which flows u_0 this is true?

To be more accurate, we have to define in what sense we understand the closedness of velocity fields. Let X be a Banach space, whose elements are incompressible vector fields $u(x)$ in M, tangent to the boundary. We call the steady flow $u_0(x)$ stable in the space X, or simply X-stable, if for every $\varepsilon > 0$ there exists $\delta > 0$, such that if $v_0(x) \in X$ is a regular vector field, such that $||v_0 - u_0||_X < \delta$, then $||v(t) - u_0||_X < \varepsilon$ for all $t \in \mathbf{R}$, where $v(t) = v(x,t)$ is the solution of (1), (2) with the initial condition $v(x,0) = v_0(x)$. (Note that we *do not* require that the Euler equations (1), (2) are uniquelly solvable for all $v_0 \in X$.)

The linearized problem is already very nontrivial. The spectrum of the linearized system is symmetric w.r.t. the real and the imaginary axes (the last is due to the Hamiltonian nature of the fluid equations; see [A1]). Thus, it is impossible to establish the true stability in any space by the linear methods; at best we can prove the absence of a "fast" exponential instability.

The first positive result about the true nonlinear instability was obtained by V. Arnold [A2] and concerns the H^1-instability. Suppose that X is the space of incompressible vector fields $u(x)$ in M, tangent to the boundary, with the finite norm $||u||_X^2 = ||u||_{L^2}^2 + ||\nabla \times u||_{L^2}^2$; this norm is equivalent to the H^1-norm, due to the conditions (2), (3). (Note that X is not the correctness class for the Euler equations.) Let us call two fields $u, v \in X$ *equivortical*, if there exists a volume preserving diffeomorphism $\xi : M \to M$, such that $\nabla \times u(x) = (\nabla \times v)(\xi(x))$. Note that for every solution $u(x,t)$ of the Euler equations, $u(x,t_1)$ is equivortical to $u(x,t_2)$ for every t_1, t_2, due to the Kelvin-Helmholtz vorticity theorem. Arnold proved that if $V \subset X$ is a "leaf" of equivortical fields in X, i.e. an orbit of the action of the group D of volume preserving diffeomorphisma, and $E(u) = \frac{1}{2}||u||_{L^2}^2$ is the functional of the kinetic energy, then critical points of E on V correspond to steady flows; if a critical point u is a point of a strict local maximum or minimum of E on V, then the flow u is stable in X. Correspondingly, we have three sorts of stable flows: points of local maximum of E on V, points of local minimum, and one single flow (up to proportionality) with constant vorticity (in this case the orbit V reduces to a single point).

Thus, there are steady flows which are stable w.r.t. perturbations with small (in L^2) vorticity. But there is not less natural class of perturbations, namely the ones with small *energy* (i.e. perturbations small in L^2). Such perturbations may be created, for example, by small obstacles incerted in

the flow, or by small, but concentrated forces. Thus, we have arrived at the problem of stability in L^2. The methods used by Arnold in the proof of his results do not work here, and there appears no reason for stability of any nontrivial (i.e.different from zero) flow at all. And indeed, we prove that any flow of some restricted class is unstable in L^2. Suppose that the flow domain is the strip $|x_2| < 1$ in the (x_1, x_2)-plane, and we consider the flows and their perturbations having the same period L in the x_1-direction. Consider a basic steady flow having the form $u_0(x) = (U(x_2), 0)$, i.e. a parallel flow having velocity profile $U(x_2)$. Note that among such flows there are representatives of all three classes of the Arnold stable in H^1 flows.

But situation with their stability in L^2 is different. Our main result is the following

Theorem 1. *For every smooth profile $U(x_2) \neq$ const, the flow $u_0(x)$ is unstable in L^2. This means that there exists $C > 0$, s.t. for every $\varepsilon > 0$ there exist $T > 0$ and a smooth solution $v(x,t)$ of the Euler equations, such that $||v(x,0) - u_0(x)||_{L^2} < \varepsilon$, but $||v(x,T) - u_0(x)||_{L^2} > C$.*

Note that if X is the space of vector fields of class H^s, $s > 1$, then all nontrivial flows are unstable in X as well,as it was established by H. Koch. Thus, we see a curious, nonmonotonous change of situation with increasing of regularity, the case $s = 1$ being exceptional.

2. We may define a weaker notion of instability, namely instability w.r.t. the external forces. Consider the nonhomogeneous Euler equations

$$\frac{\partial u}{\partial t} + (u, \nabla)u + \nabla p = f, \tag{6}$$

$$\nabla \cdot u = 0. \tag{7}$$

Here $f = f(x,t)$ is the external force.

Let X be a Banach space of vector fields, as above. We say that the steady solution $u_0(x)$ is unstable w.r.t. external forces, if there exists $C > 0$, such that for every $\varepsilon > 0$ there exist $T > 0$, and a smooth force $f(x,t)$, defined for $0 \leq t \leq T$, such that

$$\int_0^T ||f(\cdot,t)||_X dt < \varepsilon, \tag{8}$$

and if $u(x,t)$ is a solution of (6), (7), satisfying $u(x,0) = u_0(x)$, then $||u(x,T) - u_0(x)||_X > C$.

Our next result is the following

Theorem 2. *Every nontrivial parallel flow with profile $U(x_2) \neq \text{const}$ is L^2-unstable w.r.t. external forces. Moreover, suppose that $u_0(x), v_0(x)$ are two parallel flows, whose velocity profiles $U(x_2), V(x_2)$ satisfy*

$$\int_{-1}^{1} U(x_2)dx_2 = \int_{-1}^{1} V(x_2)dx_2, \tag{9}$$

$$\int_{-1}^{1} \frac{1}{2}U^2(x_2)dx_2 = \int_{-1}^{1} \frac{1}{2}V^2(x_2)dx_2, \tag{10}$$

i.e. these flows have equal energies and momenta. Then for every $\varepsilon > 0$ there exist $T > 0$ and a smooth force $f(x,t)$, defined for $0 \leq t \leq T$, such that if $u(x,t)$ is the solution of (6), (7), satisfying $u(x,0) = u_0(x)$, then $u(x,T) = v_0(x)$, and

$$\int_0^T ||f(\cdot,t)||_{L^2}dt < \varepsilon. \tag{11}$$

Thus, not only are parallel flows unstable in L^2, they are perfectly controllable by arbitrarily small in L^2 forces. Note that this is not the case in H^1; every Arnold stable flow is stable in H^1 w.r.t. external forces (this statement is proven in the same way as the Arnold stability).

2. Constructions

1. We prove theorem 1 by constructing a flow $v(x,t)$, which is initially close in L^2 to u_0, and after some (may be long) time T deviates considerably from it. This is done by the variational method.

Let $\mathcal{D}$ be the group of volume preserving diffeomorphisms of the flow domain M. This is an infinite-dimensional manifold, and its tangent vectors may be identified with vector fields in M, which are incompressible and tangent to the boundary. The manifold $\mathcal{D}$ may be endowed with a Riemannian L^2-metric: for every tangent vector V, i.e. vector field $v(x)$, we define $|V| = ||v||_{L^2}$. The fluid flow is a curve $\xi_t \in \mathcal{D},\ 0 \leq t \leq T$. For every flow $\xi_t, 0 \leq t \leq T$,let

$$J\{\xi_t\} = \int_0^T \frac{1}{2}||\dot{\xi}_t^2||_{L^2}^2 dt \tag{12}$$

denote the action. The fluid motion in the absence of external forces is defined by the Hamiltonian principle: $\delta J\{\xi_t\} = 0$, provided $\delta\xi_0 = 0, \delta\xi_T = 0$. In other words, ξ_t is a geodesic trajectory (see [A2, AK]).

The simplest way to construct a geodesic on a Riemannian manifold is the following: we fix two points on the manifold, and look for a curve of minimal length connecting them (then after appropriate parametrization this trajectory delivers minimum to the action). This approach meets considerable difficulties in 3 dimensions (see [S], [AK], [B]); so, it should be applied with care.

The L^2-metric on the group $\mathcal{D}$ is right-invariant; therefore, we can always assume that the initial position of fluid ξ_0 is the identity diffeomorphism Id. However, for our present purpose it is better to consider ξ_0, ξ_T of different form. Let us look for the fluid map $\xi_t : M \to M$ in the form

$$\xi(x_1, x_2, t) = (x_1 + N \cdot s(x_1, x_2, t),\ h(x_1, x_2, t)), \tag{13}$$

where N is a constant assumed to be big enough, and $s(x_1, x_2, t), h(x_1, x_2, t)$ are smooth functions, periodic in x_1 with period L. The incompressibility condition, together with the boundary conditions $h(x_1, -1, t) = -1,\ h(x_1, 1, t) = 1$, expressing the fact that ξ_t is a diffeomorphism, is enough to define $h(x, t)$, provided $s(x, t)$ is given. So, we can express $h(x, t)$ as

$$h = F\{s\}, \tag{14}$$

where F is some nonlinear integro-differential operator.

Thus, we have a variational problem for one scalar function $s(x, t)$:

$$I\{s\} = \int_0^T \int_M \frac{1}{2}[|\dot{s}|^2 + |F\dot{\{s\}}|^2]dxdt \to \min. \tag{15}$$

Suppose $s(x, 0), s(x, T)$ are given, and T is of order N (say, $T = Nt_0$, where t_0 is chosen later). Let $\tau = t/Nt_0, \sigma(x, \tau) = s(x, Nt_0\tau)$. Then we obtain for the function $\sigma(x, \tau)$ the following variational problem:

$$K\{\sigma\} = \int_0^1 \int_M \frac{1}{2}\Big(|\frac{\partial}{\partial\tau}\sigma|^2 + \nu^2|\frac{\partial}{\partial\tau}\Phi\{\sigma\}|^2\Big)dxd\tau \to \min,\ \nu = \frac{1}{Nt_0}. \tag{16}$$

where Φ is a properly renormalized operator F. Boundary conditions: $\sigma(x,0) = \sigma_0(x), \sigma(x,1) = \sigma_1(x)$, where σ_0, σ_1 are given smooth functions.

For $\nu = 0$ we have a simple problem with solution $\sigma_0(x,t)$ linear in t. For ν small enough, we may apply simple considerations of convexity to proof that the local minimum is achieved at some close σ.

Now we can describe our construction. Suppose the flow $u_0(x)$ has a smooth velocity profile $U(x_2) \neq$ const, and $U'(x_2) < -c < 0$ for $a \leq x_2 \leq b$ (the case $U' > c > 0$ is considered similarly). We are going to define the functions $\sigma_0(x_1,x_2), \sigma_1(x_1,x_2)$ so that the flow, corresponding to the minimum action, is close at $t = 0$ to $u_0(x)$, but for $t = T$ deviates considerably from it. First of all, fix a (big) number A; it is fixed, while N is growing. Let us choose an integer n, and sufficiently small positive constants $\delta_1 << n^{-1}, \delta_2$.

Now let us define $\sigma_0(x), \sigma_1(x)$ as follows

$$\sigma_0(x_1,x_2) = \begin{cases} \begin{array}{r} Ax_2, \\ \text{if } -1 \leq x_2 \leq a; \\ \\ Aa + \frac{x_2-a}{n}, \\ \text{if } a+\delta_2 < x_2 < b-\delta_2, \\ \text{and } 0 < x_1 < \frac{L}{n} - \delta_1; \\ \\ Aa + \frac{b-a}{n} + \frac{x_2-a}{n}, \\ \text{if } a+\delta_2 < x_2 < b-\delta_2, \\ \text{and } \frac{L}{n} < x_1 < \frac{2L}{n} - \delta_1; \\ \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ \\ Aa + (b-a)\frac{n-1}{n} + \frac{x_2-a}{n}, \\ \text{if } a+\delta_2 < x_2 < b-\delta_2, \\ \text{and } \frac{L(n-1)}{n} < x_1 < L - \delta_2; \\ \\ Aa + (b-a) + A(x_2-b), \\ \text{if } b < x_2 < 1. \end{array} \end{cases} \tag{17}$$

For other x we continue σ smoothly, so that $\frac{\partial\sigma}{\partial x_2} > 0$, and $|\nabla\sigma| < C(\delta_1^{-1} + \delta_2^{-1})$.

The function $\sigma_1(x_1,x_2)$ is defined as follows:

$$\sigma_1(x_1,x_2)=\begin{cases} Ax_2+U(x_2), \\ \text{if } -1\le x_2\le a; \\ \\ Aa+\frac{x_2-a}{n}+U(a+\frac{b-a}{2n}), \\ \text{if } a+\delta_2<x_2<b-\delta_2, \\ \text{and } 0<x_1<\frac{L}{n}-\delta_1; \\ \\ Aa+\frac{b-a}{n}+\frac{x_2-a}{n}+U(a+\frac{3(b-a)}{2n}), \\ \text{if } a+\delta_2<x_2<b-\delta_2, \\ \text{and } \frac{L}{n}<x_1<\frac{2L}{n}-\delta_1; \\ \\ \dots\dots\dots\dots\dots\dots \\ \\ Aa+(b-a)\frac{n-1}{n}+\frac{x_2-a}{n}+U(b-\frac{b-a}{2n}), \\ \text{if } a+\delta_2<x_2<b-\delta_2, \\ \text{and } \frac{L(n-1)}{n}<x_1<L-\delta_2; \\ \\ Aa+(b-a)+A(x_2-b)+U(x_2), \\ \text{if } b<x_2<1. \end{cases} \tag{18}$$

These boundary conditions have the following meaning. The flow domain M may be regarded as a cylinder $[-1,1]\times(\mathbf{R}/L\mathbf{Z})$. Let us draw on M the line Γ^0 having equation

$$x_1=\begin{cases} NAx_2 \pmod L, \\ \text{if } -1\le x_2\le a; \\ \\ NAa+N(x_2-a) \pmod L, \\ \text{if } a+\delta_2<x_2<b-\delta_2; \\ \\ NAa+N(b-a)+NA(x_2-b) \pmod L, \\ \text{if } b\le x_2\le 1. \end{cases} \tag{19}$$

In the intervals $a<x_2<a+\delta_2$ and $b-\delta_2<x_2<b$ this function is continued smoothly. If we cut M along this line, we obtain a strip Σ^0, doing $NA(2-b+a)+N(b-a)$ revolutions. The width of Σ) is equal to $\frac{1}{NA}$ for

$x_2 \le a$ and $x_2 \ge b$, and it is equal to $\frac{1}{N}$ for $a+\delta_2 \le x_2 \le b+\delta_2$. This strip is, in its turn, cut into n substrips $\Sigma_1^0, \cdots, \Sigma_n^0$. In the domains $x_2 < a$ and $x_2 > b$ all these substrips have the same width, and their boundaries are parallel to the line Γ^0. But in the domain $a+\delta_2 < x_2 < a + \frac{b-a}{n} - \delta_2$ the substrip Σ_1^0 occupies almost all the width of Σ^0, while other substrips Σ_i^0 have the width of order $\frac{\delta_2}{nN}$. For $a + \frac{b-a}{n} + \delta_2 < x_2 < a + 2\frac{b-a}{n} - \delta_2$ the substrip Σ_2^0 occupies almost all the width of Σ^0, etc.

The line Γ^0 is the image of the line $x_1 = 0$ under the mapping $(x_1, x_2) \mapsto (N\sigma_0(x_1, x_2), h_0(x_1, x_1))$, while every subtrip Σ_i^0 is the image of the domain $\frac{L(i-1)}{n} < x_1 < \frac{Li}{n}$ (here $h_0(x)$ is the second component of the mapping ξ_0, defined uniquelly from the incompressibility and boundary conditions).

Now let us denote by Σ_i^1 the image of the same domain $\frac{L(i-1)}{n} < x_1 < \frac{Li}{n}$ under the mapping $(x_1, x_2) \mapsto (N\sigma_1(x_1, x_2), h_1(x_1, x_1))$. This is also a strip of variable width (having order of $1/N$). This width, which we denote by $\omega_i^1(x_1)$, is obtained from $\omega_i^0(x_1)$, the width of Σ_i^0, by the shift in the x_1-direction through the distance $TU_i = Nt_0U_i$, where $U_i = U(a + \frac{2i-1}{2n}(b-a))$.

The variational problem (16) has, for big N, solution which is close to linear in τ. Let Σ_i^t be the image of the domain $\frac{L(i-1)}{n} < x_1 < \frac{Li}{n}$ at the moment t. Then the width $\omega_i(x_1, t)$ of Σ_i^t is asymptotically ($N \to \infty$) close to $\omega_i^0(x_1 - U_i t)$, and the velocity in Σ_i^t is all the time close to U_i. (Here we describe the flow as if it were discontinuous. In fact, between the strips Σ_i^t there are intermediate layers of thicknes of order δ_1/N, where the velocity changes smoothly.)

Now comes the main point of our construction. We have supposed that $U'(x_2) < -c < 0$ on the interval $a < x_2 < b$. This means that $U_1 > U_2 > \cdots > U_n$, and $U_i - U_{i-1} > c(b-a)/n$. We started from the system of strips Σ_i^0; each of them has its "thick" part, where the main part of their mass is concentrated, in the domain $a + (i-1)(b-a)/n < x_2 < a + i(b-a)/n$. If the time $T = 2N/c$, then by this time the "thick" part of every strip Σ_i^t will be ahead of the "thick" part of the next strip Σ_{i+1}^t. Using the incompressibility condition, we find that actually the thick part of Σ_i^t is *above* the thick part of Σ_{i+1}^t. The velocity in Σ_i^t is all the time close to U_i. Thus, at $t = 0$ the velocity $v(x, 0)$ is close in L^2 to $u_0(x)$, while at $t = T$ it is L^2-close to the parallel flow $w(x) = (W(x_2), 0)$, where the new profile $W(x_2)$ is obtained from the initial profile $U(x_2)$ by the inversion on the interval (a, b):

$$W(x_2) = \begin{cases} U(x_2), \text{ if } -1 < x < a; \\ U(a+b-x_2), \text{ if } a < x < b; \\ U(x_2), \text{ if } b < x < 1. \end{cases} \tag{20}$$

Thus, the initial flow $u_0(x)$ is unstable in L^2.

It should be noted that the perturbation just constructed is very unstable itself. It would be desirable to find more realistic examples.

2. Theorem 2 is proven by an explicit construction of the flow.

We say that the force $f(x,t)$ transfers steady flow $u_0(x)$ into another steady flow $v_0(x)$ during the time T, if the solution $u(x,t)$ of equations (6), (7), satisfying initial condition $u(x,0) = u_0(x)$, satisfies also $u(x,T) = v_0(x)$.

Note first, that if $U_1, U_2, \cdots, U_N$ are velocity profiles, and Theorem 2 is true for every pair (U_i, U_{i+1}) of velocity profiles, then we can pass from U_1 to U_N, simply concatenating the flows connecting U_i and U_{i+1}; thus Theorem 2 is true for the pair (U_1, U_N). Therefore it is enough to construct the sequence of steady flows with profiles $U_1, \cdots, U_N$, and the intermediate nonsteady flows connecting every two successive steady ones.

Note also, that it is enough to construct a sequence of piecewise-smooth flows, for it is not difficult to smoothen them, so that the necessary force will have arbitrarily small norm in $L^1(0,T;L^2(M))$.

As a first step, we change the flow with the profile $U = U_1$ by a piecewise-constant profile U_2 with sufficiently small steps; this may be done by a force with arbitrarily small norm.

Thus, $U_2(x_2)$ is a step function, $U_2(x_2) = U_2^{(k)}$ for $x_2^{(k-1)} < x_2 < x_2^{(k)}$, $k = 1, \cdots, K$. Every next profile U_i is also a step-wise function. We are free to subdivide the steps and change a little the values of velocity, if these changes are small enough.

Every flow u_k is obtained from the previous one u_{k-1} by one of two operations, described in the following theorems.

Theorem 3. *Let $U(x_2)$ be a step function, $U(x_2) = U^{(k)}$ for $x_2^{(k-1)} < x_2 < x_2^{(k)}$; let $V(x_2)$ be another step function, obtained by transposition of two adjacent segments $[x_2^{(k-1)}, x_2^{(k)}]$ and $[x_2^{(k)}, x_2^{(k+1)}]$. Let $u(x_1,x_2)$, $v(x_1,x_2)$ be parallel flows with velocity profile $U(x_2), V(x_2)$. Then for every $\varepsilon > 0$ there exist $T > 0$ and a piecewise-smooth force $f(x,t)$, such that $\int_0^T \| f(\cdot,t) \|_{L^2} < \varepsilon$, and the force f transfers the flow u into the flow v during the time interval $[0,T]$.*

To formulate the next theorem, remind the law of an elastic collision of two bodies. Suppose that two point masses m_1 and m_2, having velocities u_1 and u_2, collide elastically. Then their velocities after collision will be $v_1 = 2u_0 - u_1,\ v_2 = 2u_0 - u_2$, where $u_0 = (m_1u_1 + m_2u_2)/(m_1 + m_2)$ is the velocity of the center of masses. The transformation $(u_1, u_2) \to (v_1, v_2)$ is called a *transformation of elastic collision.*

Theorem 4. *Suppose that the profile $U(x_2)$ is like in Theorem 3, and the profile $V(x_2)$ is equal to $U(x_2)$ outside the segment $x_2^{(k-1)} < x_2 < x_2^{(k+1)}$; on the last segment, $V(x_2) = v^{(k)}$, if $x_2^{(k-1)} < x_2 < x_2^{(k)}$, and $V(x_2) = v^{(k+1)}$, if $x_2^{(k)}) < x_2 < x_2^{(k+1)}$, where $(v^{(k)}, v^{(k+1)})$ is obtained from $(u^{(k)}, u^{(k+1))}$ by the transformation of elastic collision, the lengths $x_2^{(k)} - x_2^{(k-1)},\ x_2^{(k+1)} - x_2^{(k)}$ playing the role of masses m_1, m_1. Let $u(x_1, x_2), v(x_1, x_2)$ be parallel flows with profiles $U(x_2), V(x_2)$. Then for every $\varepsilon > 0$ there exist $T > 0$ and a force $f(x, t)$, such that $\int_0^T \parallel f(\cdot, t) \parallel_{L^2} < \varepsilon$, and the force f transfers the flow u into flow v.*

Suppose now, that $U(x_2)$ and $V(x_2)$ are two velocity profiles, having equal momenta and energies. Then it is not difficult to construct a sequence of step functions $U_2(x_2), U_3(x_2), \cdots, U_N(x_2)$, so that U_2 is L^2-close to $U_1 = U$, U_N is L^2-close to V, and every profile U_k is obtained from U_{k-1} by one of two operations, described in Theorems 3 and 4. Using these theorems and the notes above, we construct a piecewise-smooth force $f(x, t)$, such that $\int_0^T \parallel f(\cdot, t) \parallel_{L^2} dt < \varepsilon$, and f transfers U into V during the time interval $[0, T]$.

LIST OF REFERENCES

[A1] V. Arnold, *Mathematical methods of classical mechanics*, Springer-Verlag, New York, 1989.

[A2] V. Arnold, *Sur la Géométrie différentielle des groupes de Lie de dimension infinie et ses applications à l'hydrodynamique des fluides parfaits*, Ann. Inst. Fourier 16 (1966), 316-361.

[A-K] V. Arnold, B. Khesin, *Topological methods in hydrodynamics*, Applied Mathematical Sciences, v. 125, Springer-verlag, 1998.

[B] Y. Brenier, *The least action principle and the related concept of generalized flows for incompressible perfect fluids*, J. Amer. Math. Soc. **2** (1989), no. 2, 225-255.

[M-P] C. Marchioro, M. Pulvirenti, *Mathematical theory of incompressible nonviscous fluids*, Applied Mathematical Sciences, v. 96, Springer-Verlag, 1994.

[S] A. Shnirelman, *The geometry of the group of diffeomorphisms and the dynamics of an ideal incompressible fluid*, Math. USSR Sbornik **56** (1987), no. 1, 79-105.