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The wave equation with oscillating density : observability at low frequency

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1 Introduction and results

Let Ω be a smooth bounded domain in $\mathbb{R}^d$, and $\rho(x, y)$ a smooth function on $\mathbb{R}^d \times \mathbb{R}^d$, such that

(1.1) $$0 < \rho_{\min} \leq \rho(x, y) \leq \rho_{\max} \quad \forall\, (x, y)$$

(1.2) ρ is 2π-periodic with respect to the second variable, i.e.

$$\rho(x, y) = \rho(x, y + 2\pi\ell) \quad \forall\, \ell \in \mathbb{Z}^d\,.$$

For $\varepsilon > 0$, let $(\omega_n^\varepsilon, e_n^\varepsilon(x))$ be the spectrum of the Dirichlet problem for the operator $-\rho^{-1}(x, x/\varepsilon)\Delta_g$ on $L^2\left(\Omega; \rho(x, x/\varepsilon) d_g x\right)$ normalized in the form

(1.3)
$$\begin{cases} \rho(x, x/\varepsilon)(\omega_n^\varepsilon)^2\, e_n^\varepsilon(x) = -\Delta_g e_n^\varepsilon(x) & \text{in } \Omega \\ e_n^\varepsilon(x) = 0 & \text{on } \partial\Omega \\ \int_\Omega e_n^\varepsilon(x)\overline{e_m^\varepsilon(x)}\rho(x, x/\varepsilon) d_g x = \delta_{n,m} \qquad ; \qquad 0 < \omega_1^\varepsilon \leq \omega_2^\varepsilon \leq \dots & \end{cases}$$

Here, Δ_g denotes the Laplace operator for some fixed smooth metric g on $\overline{\Omega}$, and $d_g x$ is the volume form associated to g.

For any given $\gamma_0 > 0$, we shall denote by $J^\varepsilon_{\gamma_0}$ the space of solutions $u^\varepsilon(t,x)$ of the wave equation with oscillating density ρ

$$(1.4) \qquad \begin{cases} (\rho(x,x/\varepsilon)\partial_t^2 - \Delta_g)\, u^\varepsilon(t,x) = 0 \quad \text{in } \mathbb{R}_+ \times \Omega \\ u^\varepsilon(t,x)_{|x\in\partial\Omega} = 0 \end{cases}$$

with maximum frequency less than γ_0/ε.
In other words, $J^\varepsilon_{\gamma_0}$ is the set

$$(1.5) \qquad J^\varepsilon_{\gamma_0} = \left\{ u^\varepsilon(t,x) = \sum_{\varepsilon\omega_n^\varepsilon \le \gamma_0} \left(u_{+,n}e^{it\omega_n^\varepsilon} + u_{-,n}e^{-it\omega_n^\varepsilon}\right) e_n^\varepsilon(x) \right\}.$$

Let $\{u_k^{\varepsilon_k}\}$ be a bounded sequence (in $L^2_{\mathrm{loc}}(\mathbb{R}_t, L^2(\Omega))$, of solutions of (1.4), with $\lim \varepsilon_k = 0$. It is well known that any weak limit of this sequence will satisfy the homogeneized wave equation in Ω

$$(1.6) \qquad \begin{cases} (\underline{\rho}(x)\partial_t^2 - \Delta_g)u(t,x) = 0 \quad \text{in } \mathbb{R} \times \Omega \\ u(t,x)_{|x\in\partial\Omega} = 0 \end{cases}$$

where $\underline{\rho}(x) = \oint \rho(x,y)dy$ is the mean value of ρ.
Let V be an open subset of Ω, and $T_0 > 0$.
One says that waves solution of (1.6) are observable from V in time T_0 if there exists a constant C_0 s.t for any L^2-solution of (1.6) one has

$$(1.7) \qquad \int_0^{T_0}\int_\Omega |u(t,x)|^2\underline{\rho}(x)dtd_gx \le C_0 \int_0^{T_0}\int_V |u(t,x)|^2\underline{\rho}(x)dtd_gx\,.$$

If $u = \sum_{\pm,n} u_{\pm,n}e^{\pm it\omega_n}e_n(x)$ is the Fourier series of u in the spectral decomposition of $(-\underline{\rho})^{-1}(x)\Delta_g$, this condition is equivalent to the following

$$(1.8) \qquad \begin{cases} \exists C_0 \text{ s.t. } \quad \forall (u_{+,n},u_{-,n})_n \in \ell^2\times\ell^2 \\ \sum_n |u_{+,n}|^2 + |u_{-,n}|^2 \le C_0 \int_0^{T_0}\int_V |u(t,x)|^2\underline{\rho}(x)dtd_gx \end{cases}.$$

It is proved in [B.L.R.] that (1.7) holds true under the geometric-control hypothesis

(1.9)
1) there is no infinite order of contact between the boundary $\partial\Omega$ and the bicharacteristics of $\underline{\rho}(x)\partial_t^2 - \Delta_g$
2) any generalized bicharacteristic of $\underline{\rho}(x)\partial_t^2 - \Delta_g$ parameterized by $t \in]0, T_0[$ meets $\overline{V}$

Here the generalized bicharacteristic flow is the one defined by Melrose and Sjöstrand in [M-S].

Our main result is the following theorem, which asserts that the estimate (1.7) remains true under the hypothesis (1.9) for $\underline{\rho}(x)$, for solutions of (1.4) in $J^{\varepsilon}_{\gamma_0}$, if γ_0 is small enough.

Theorem 1.1 *Let the hypothesis (1.9) be satisfied. There exist small positive constants* γ_0, ε_0 *and a constant* C_0*, such that for any* $\varepsilon \in]0, \varepsilon_0[$ *and any* $u^{\varepsilon} \in J^{\varepsilon}_{\gamma_0}$

(1.10)
$$\int_0^{T_0} \int_{\Omega} |u^{\varepsilon}(t,x)|^2 \rho(x, x/\varepsilon) dt d_g x \leq C_0 \int_0^{T_0} \int_V |u^{\varepsilon}(t,x)|^2 \rho(x, x/\varepsilon) dt d_g x \,.$$

This is clearly a stability result of the observability estimate (1.7) under the singular perturbation $\underline{\rho}(x) \to \rho(x, x/\varepsilon)$. Let us recall that Theorem 1.1 has been proved in the 1-d case by C. Castro and E. Zuazua [C-Z], and that in the 1-d case, the counter-example of Avallaneda-Bardos-Rauch shows that (1.10) failed for γ_0 large. Indeed, in the 1-d case, when $\rho = \rho(x/\varepsilon)$, C. Castro [C] has shown that the greatest value of γ_0 such that (1.10) holds true for some T_0 (when $V \Subset [a,b] = \Omega$) is related with the first instability interval of the Hille equation on the line $(\frac{d}{dy})^2 + \omega^2 \rho(y)$. In the multi-d case, the understanding of the best value of γ_0 such that (1.10) holds true will clearly involved the understanding of the localization and propagation of Bloch waves for the boundary value problem (1.4) : this highly difficult problem is out of the scoop of the present work.

The conserved energy for solutions of (1.4) is

$$E(u^{\varepsilon}) = \frac{1}{2} \int_{\Omega} \{ |\partial_t u^{\varepsilon}|^2 \rho(x, x/\varepsilon) + |\nabla_g u^{\varepsilon}|^2 \} d_g x \,. \tag{1.11}$$

Applying the estimate (1.10) to $\partial_t u^{\varepsilon}$, one easily gets the energy observability estimate

Corollary 1.1 *Under the hypothesis and with the notations of Theorem 1.1, there exists a constant* C_0 *s.t. for any* $\varepsilon \in]0, \varepsilon_0[$ *and any* $u^{\varepsilon} \in J^{\varepsilon}_{\gamma_0}$ *one has*

$$E(u^{\varepsilon}) \leq C_0 \int_0^{T_0} \int_V |\partial_t u^{\varepsilon}|^2 \rho(x, x/\varepsilon) dt d_g x \,. \tag{1.12}$$

2 Sketch of proof

1. **Reduction to a semi-classical estimate**
2. **The Bloch-wave**
3. **Lopatinski estimate**
4. **Propagation estimate**

1. In the first part, using a Littlewood-Paley decomposition, we reduce the proof of the inequality (1.10) to the assertion

$$(2.1)\quad \begin{cases} \text{There exist } \gamma_0, \varepsilon_0, h_0, C_0 \text{ such that for any } \varepsilon \in]0, \varepsilon_0[, \text{ and} \\ h \in [\varepsilon/\gamma_0, h_0] \text{ the inequality (1.10) holds true for any } u^\varepsilon \in I_h^\varepsilon, \\ \text{where} \quad I_h^\varepsilon = \left\{ u^\varepsilon = \sum_{0.9 \le \omega_n^\varepsilon h \le 2.1} (u_{+,n}\, e^{it\omega_n^\varepsilon} + u_{-,n}\, e^{-it\omega_n^\varepsilon}) e_n^\varepsilon(x) \right\} \end{cases} .$$

2. In the second part, we choose a coordinate system near the boundary

$$(2.2)\quad \begin{cases} \partial\Omega \times [0, r_0] \xrightarrow{\Theta} \mathbb{R}^d \\ (x', x_d) \mapsto \Theta(x', x_d) \end{cases}$$

which satisfies

$$(2.3)\quad \begin{cases} i)\ \Theta(\partial\Omega \times [0, r_0]) \subset \overline{\Omega} \\ ii)\ \text{for } x_d \text{ small }, x_d \mapsto \Theta(x', x_d) \text{ is the geodesic normal to the} \\ \text{boundary at } x' \in \partial\Omega \text{ , for the metric } g \text{ on } \overline{\Omega}. \end{cases}$$

In these coordinates, the Laplace operator takes the form

$$(2.4)\quad \begin{cases} \Delta_g = \frac{\partial}{\partial x_d}\left(A_0(x) \frac{\partial}{\partial x_d} + A_1(x, \partial_{x'}) \right) + A_2(x, \partial_{x'}) \ ; \\ x = (x', x_d) \, , \, x' \in \partial\Omega \end{cases}$$

where $A_j(x, \partial_{x'})$ are differential operators of order j on $\partial\Omega$, with x_d as parameter. Let $a_j(x, \xi')$ be the principal symbol of A_j. The dual metric $g^{-1}(x, \xi) \overset{\text{def}}{=} \|\xi\|_x^2$ on the cotangent bundle $T^*\Omega$ is

$$(2.5)\quad \|\xi\|_x^2 = a_0(x)\xi_d^2 + a_1(x, \xi')\xi_d + a_2(x, \xi') \, .$$

Let $\mathbb{T}^d = (\mathbb{R}/2\pi\mathbb{Z})^d$ be the d-dimensional torus and for $\varepsilon > 0$, $S_\varepsilon \subset \partial\Omega \times [0, r_0] \times \mathbb{T}^d_y$ the subvariety

$$S_\varepsilon = \left\{(x, y)\,;\, y = \Theta(x)/\varepsilon \bmod (2\pi\mathbb{Z})^d\right\} \tag{2.6}$$

Let $f(x)$ be a function on $\partial\Omega \times [0, r_0]$. We define a distribution $T(f)$ on $\partial\Omega \times [0, r_0] \times \mathbb{T}^d_y$ by the formula

$$T(f) = \sum_{\ell\in\mathbb{Z}^d} e^{i\ell(y-\Theta(x)/\varepsilon)} f(x) = (2\pi)^d \delta_{y=\Theta(x)/\varepsilon} \otimes f(x). \tag{2.7}$$

If X is a vector field on $\partial\Omega \times [0, r_0]$, we shall denote by X^*_ε the lift of X on S_ε. If $x' = (x_1, \dots, x_{d-1})$ is a local coordinate system on $\partial\Omega$, and $(\Theta_1(x), \dots, \Theta_d(x)) = \Theta(x)$ are the Cartesian coordinates of Θ, one has

$$(\frac{\partial}{\partial x_k})^*_\varepsilon = \frac{\partial}{\partial x_k} + \frac{1}{\varepsilon} \sum_{j=1}^{d} \frac{\partial\Theta_j}{\partial x_k}(x)\, \frac{\partial}{\partial y_j} \quad \text{for } 1 \le k \le d\,. \tag{2.8}$$

and

$$\left(\frac{\partial}{\partial x_k}\right)^*_\varepsilon T(f) = T\left(\frac{\partial}{\partial x_k} f\right) \quad \text{for } 1 \le k \le d\,. \tag{2.9}$$

The Bloch-operator on $\partial\Omega \times [0, r_0] \times \mathbb{T}^d$ is defined by

(2.10)

$$\left\{\begin{array}{l} \mathbb{B}_\varepsilon(x, \varepsilon\partial_x, \varepsilon\partial_t; y, \partial_y) = \hat\rho(x, y)(\varepsilon\partial_t)^2 - \varepsilon^2(\Delta_g)^*_\varepsilon\ ;\ \hat\rho(x, y) = \rho(\Theta(x), y) \\ (\Delta_g)^*_\varepsilon = (\frac{\partial}{\partial x_d})^*_\varepsilon \left(A_0(x)(\frac{\partial}{\partial x_d})^*_\varepsilon + A_1(x, (\partial_{x'}))^*_\varepsilon\right) + A_2\,(x, (\partial_{x'})^*_\varepsilon) \end{array}\right.$$

It satisfies the identity

$$\mathbb{B}_\varepsilon\left(T(u(x, t))\right) = \varepsilon^2 T\left(\rho(\Theta(x), \Theta(x)/\varepsilon)\partial_t^2 - \Delta_g\right)(u(x, t)) \tag{2.11}$$

Let $\widetilde{A}_j$ be the operators

$$\widetilde{A}_j = A_j(x, (\partial_{x'})^*_\varepsilon) \tag{2.12}$$

and let $e_k(x)$ $1 \le k \le d$ be the vectors of $\mathbb{R}^d$

$$e_k(x) = \frac{\partial\Theta}{\partial x_k}(x) \tag{2.13}$$

If $v(t,x,y)$ is a distribution on $\overset{\circ}{X} \times \mathbb{T}^d$, with $X = \mathbb{R}_t \times (\partial\Omega \times [0, r_0])$, we shall write the equation $\mathbb{B}_\varepsilon(v) = 0$ as a 2×2 system for the vector $w = \mathcal{A}(v)$.

$$\mathcal{A}(v) = \begin{bmatrix} w_0 \\ w_1 \end{bmatrix} \overset{\text{def}}{=} \begin{bmatrix} v \\ (A_0(x)(\varepsilon \frac{\partial}{\partial x_d})^*_\varepsilon + \varepsilon \widetilde{A}_1)v \end{bmatrix} \tag{2.14}$$

This system takes the form

$$\begin{cases} \varepsilon \frac{\partial}{\partial x_d} w + \mathbb{M}w = 0 \\ \mathbb{M} = \begin{bmatrix} e_d(x)\cdot\partial_y + \varepsilon A_0^{-1}(x)\widetilde{A}_1 & -A_0^{-1}(x) \\ \varepsilon^2 \widetilde{A}_2 - \hat{\rho}(x,y)(\varepsilon\partial_t)^2 & e_d(x)\cdot\partial_y \end{bmatrix} \end{cases} \tag{2.15}$$

The operator $\mathbb{M}$ will be seen as a semi-classical operator in $t, x, \frac{\varepsilon}{i}\partial_{x'} = \xi'$, $\frac{\varepsilon}{i}\partial_t = \tau$ with operator values in the fiber $\mathbb{T}^d$

$$\mathbb{M} = \sum_{j=0}^{2} (\frac{\varepsilon}{i})^j \mathbb{M}^j(x, \xi', \tau; y, \partial_y)\,. \tag{2.16}$$

The differential degree in y of $\mathbb{M}^j$ is at most $2-j$ and the principal symbol $\mathbb{M}^0$ is the matrix

(2.17)

$$\mathbb{M}^0(x,\xi',\tau;y,\partial_y) = \begin{bmatrix} e_d(x)\cdot\partial_y + \ a_0^{-1}(x)a_1(x, i\xi' + e'(x)\cdot\partial_y) & -a_0^{-1}(x) \\ a_2(x, i\xi' + e'(x)\cdot\partial y) + \hat{\rho}(x,y)\tau^2 & e_d(x)\cdot\partial_y \end{bmatrix}$$

Let $E^\bullet = \{E^s,\ s\in\mathbb{R}\}$ be the scale of Hilbert spaces on the torus

$$E^s = H^s(\mathbb{T}^d) \oplus H^{s-1}(\mathbb{T}^d). \tag{2.18}$$

For any $\rho = (x, \xi', \tau)$, $\mathbb{M}^j(\rho, y, \partial_y)$ maps E^s into E^{s-1+j} and $\mathbb{M}^0$ is an elliptic operator. Let $\mathbb{M}^0_0$ be the restriction of $\mathbb{M}^0$ to the zero section $\xi' = \tau = 0$.

(2.19)

$$\mathbb{M}^0_0(x,\partial_y) = \mathbb{M}^0(x,0,0,y,\partial_y) = \begin{bmatrix} e_d(x)\cdot\partial_y + \ a_0^{-1}a_1(x, e'(x)\cdot\partial_y) & -a_0^{-1}(x) \\ a_2(x, e'(x)\cdot\partial_y) & e_d(x)\cdot\partial_y \end{bmatrix}$$

The eigenvalues $\lambda^0_{\pm,\ell}(x)$ of $\frac{1}{i}\mathbb{M}^0_0(x,\partial_y)$ on the space $e^{i\ell y}\mathbb{C}^2$, for $\ell \in \mathbb{Z}^d$ are the complex roots of the equation

$$a_0(x)(-\lambda + e_d.\ell)^2 + (-\lambda + e_d.\ell)a_1(x, e'.\ell) + a_2(x, e'.\ell) = 0 \tag{2.20}$$

which is equivalent to

$$\|{}^t d\Theta(x)(\ell) - \lambda(0, \cdots, 0, 1)\|_x^2 = 0\,. \tag{2.21}$$

In particular we have

$$\inf_x \min_{\ell \neq 0} |\lambda^0_{\pm,\ell}(x)| > 0 \tag{2.22}$$

so the double eigenvalue $\lambda^0_{\pm,0}(x) = 0$ is isolated in the spectrum of $\mathbb{M}^0_0(x, \partial_y)$.

In the sequel, we shall restrict the values of the Sobolev index of regularity s on the torus to some fixed large interval, $s \in [-\sigma_0, \sigma_0]$, $\sigma_0 \gg \frac{d}{2}$.

Let $X = \partial\Omega \times \mathbb{R}_t \times [0, r_0]$. We denote by ${}^tT^*X$ the tangential cotangent bundle

$${}^tT^*X = T^*(\partial\Omega \times \mathbb{R}_t) \times [0, r_0] \tag{2.23}$$

Let $W_1 \Subset W_0$ be two small neighborhoods of the set $\{\xi' = \tau = 0\} \times \{t \in [-T_0, 2T_0]\}$ in ${}^tT^*X$.

We choose a non-negative function $\chi_0 \in C_0^\infty(W_0)$, such that $\chi_0 \equiv 1$ on W_1.

If W_0 is small enough, we define the map $p_0(x, t, \xi', \tau) : E^{\bullet} \to \mathbb{C}^2$ by the formula

$$p_0[w] = \chi_0 . \oint_{\mathbb{T}^d} \left\{ \frac{1}{2i\pi} \int_{\partial D} \frac{dz}{z - \mathbb{M}^0} \right\} [w] \quad w \in E^s\ ,\ s \in [-\sigma_0, \sigma_0] \tag{2.24}$$

(where $D \subset \mathbb{C}$ is a small disk with center $z = 0$).

It satisfies the estimates

$$\exists\, C\, \forall\, s \in [-\sigma_0, \sigma_0]\ \forall\, w \in E^s \quad \|p_0(w) - \chi_0 \oint_{\mathbb{T}^d} w\|_{\mathbb{C}^2} \leq C\tau^2 \|w\|_{E^s} \tag{2.25}$$

and there exists $L^0(x, t, \xi', \tau) \in C^\infty({}^tT^*X; M_2(\mathbb{C}))$, defined near $\xi' = \tau = 0$ such that

$$p_0 \circ \mathbb{M}^0 = L^0 \circ p_0\,. \tag{2.26}$$

By a Taylor expansion near $\xi' = \tau = 0$, one gets

$$L^0 = \begin{bmatrix} a_0^{-1}(x) a_1(x, i\xi') & -a_0^{-1}(x) \\ a_2(x, i\xi') + \underline{\hat{\rho}}(x)\tau^2 & 0 \end{bmatrix} + O(\tau^4) \tag{2.27}$$

We then suitably quantize the above construction and we obtain tangential pseudo differential operators

$$(2.28)\quad \begin{cases} \Pi_0(\varepsilon, t, x, \varepsilon\partial_t, \varepsilon\partial_{x'}) & : L^2(X; E^s) \to L^2(X, \mathbb{C}^2), \ s \in [-\sigma_0, \sigma_0] \\ L(\varepsilon, t, x, \varepsilon\partial_t, \varepsilon\partial_{x'}) & : L^2(X; \mathbb{C}^2) \to L^2(X, \mathbb{C}^2) \end{cases}$$

with principal symbol $\sigma(\Pi_0) = p_0$, $\sigma(L) = L^0$, which satisfy the relation

$$(2.29)\quad \Pi_0(\varepsilon\partial_{x_d} + \mathbb{M}) = (\varepsilon\partial_{x_d} + L)\Pi_0 + R(\varepsilon, t, x, \varepsilon\partial_t, \varepsilon\partial_{x'})$$

In (2.29), the error term $R : L^2(X; E^s) \to L^2(X, \mathbb{C}^2)$ will be a tangential pseudo differential operator such that for any tangential o.p.d. Q with essential support in W_1 and any $s \in [-\sigma_0, \sigma_0]$, one has

$$(2.30)\quad \|Q \circ R;\, L^2(X; E^s) \to L^2(X, \mathbb{C}^2)\| \in O(\varepsilon^\infty)$$

Definition 2.1 *For $u^\varepsilon \in I_h^\varepsilon$, we define the Bloch-wave $\Gamma(u^\varepsilon) \in L^2(X; \mathbb{C}^2)$ by the formula*

$$(2.31)\quad \Gamma(u^\varepsilon) = \begin{bmatrix} \Gamma_0(u^\varepsilon) \\ \Gamma_1(u^\varepsilon) \end{bmatrix} = \Pi_0 \mathcal{T}(u^\varepsilon) \quad (\mathcal{T} = \mathcal{A} \circ T)$$

■

Let $\gamma_0, \varepsilon_0, h_0$ be given small enough, $\varepsilon \in]0, \varepsilon_0]$, $h \in [\varepsilon/\gamma_0, h_0]$. For $u_\varepsilon \in I_h^\varepsilon$, $u^\varepsilon = \sum_{0.9 \le \omega_n^\varepsilon h \le 2.1} (u_{+,n} e^{it\omega_n^\varepsilon} + u_{-,n} e^{-it\omega_n^\varepsilon}) e_n^\varepsilon(x)$, we define $\|u^\varepsilon\|^2 \left(\simeq \int_0^{T_0} \int_\Omega |u^\varepsilon|^2\right)$ by

$$(2.32)\quad \|u^\varepsilon\|^2 = \sum_{0.9 \le \omega_n^\varepsilon h \le 2.1} |u_{+,n}|^2 + |u_{-,n}|^2$$

Let $X_{T_0} = \partial\Omega \times [-T_0, 2T_0] \times [0, r_0]$, and let K be the compact subset of ${}^tT^*X$, $K = \partial\Omega \times [0, T_0] \times [0, r_0/2] \times \{\xi' = 0, \tau = 0\}$. By a localization argument near the zero section (γ_0 small), and a propagation argument in the interior, we first verify

Proposition 2.1 *Let $Q(\varepsilon, t, x, \varepsilon\partial_{x'}, \varepsilon\partial_t)$ be a zero order tangential opd on X, equal to Id near K. If $\gamma_0, \varepsilon_0, h_0$ are small enough, there exists a constant $C > 0$, such that for any $\varepsilon \in]0, \varepsilon_0]$, $h \in [\varepsilon/\gamma_0, h_0]$, one has*

$$(2.33)\quad \|u^\varepsilon\|^2 \le C\left[\|Q\Gamma_0(u^\varepsilon)\|^2_{L^2(X_{T_0})} + \|u^\varepsilon\|^2_{L^2((0,T_0)\times V)}\right] \ \forall\, u^\varepsilon \in I_h^\varepsilon$$

3. By Proposition 2.1, we shall obtain the inequality (1.10), if we are able to estimate the L^2 norm of the first component $\Gamma_0(u^\varepsilon)$ of the Bloch-wave near the set K.

The formula (2.29) shows that $\Gamma(u^\varepsilon)$ satisfies the equation

$$(\varepsilon\partial_{x_d} + L)\Gamma(u^\varepsilon) \in O(\varepsilon^\infty L^2) \text{ (microlocally in } W_1). \tag{2.34}$$

By (2.27) this equation is very closed to the homogeneized equation $(\underline{\rho}(x)\partial_t^2 - \Delta_g)[\Gamma_0(u^\varepsilon)] = 0$.

As one can see, all the difficulty in our problem is thus to obtain an estimate on the first Dirichlet data of $\Gamma(u^\varepsilon)$ on the boundary $x_d = 0$, in order to apply propagation arguments to the equation (2.34).

Proposition 2.2 *If $\gamma_0, \varepsilon_0, h_0$ are small enough, there exists a constant C such that for any $\varepsilon \in]0, \varepsilon_0[$, $h \in [\varepsilon/\gamma_0, h_0]$ the following estimate holds true*

$$\|\Gamma_0(u^\varepsilon)_{|x_d=0}\|_{L^2(X_{T_0}\cap x_d=0)} \leq C\,\varepsilon/h\|u^\varepsilon\| \quad \forall\, u^\varepsilon \in I_h^\varepsilon\,. \tag{2.35}$$

■

The above estimate is obtain as a consequence of a uniform Lopatinski estimate on $w^\varepsilon = \mathcal{T}(u^\varepsilon) = \begin{bmatrix} w_0^\varepsilon \\ w_1^\varepsilon \end{bmatrix}$. More precisely, we have

Theorem 2.1 *Let Q be a scalar tangential o.p.d. with essential support in W_0 ; if $W_0, \gamma_0, \varepsilon_0, h_0$ are small enough, there exist $s_1 < 0$ and a constant C such that for any $u^\varepsilon \in I_h^\varepsilon$ the following estimate holds true*

$$\|Q(t, x, \varepsilon\partial_{x'}, \varepsilon\partial_t)(w_1^\varepsilon)_{|x_d=0}\|_{L^2(X_{T_0}\cap x_d=0, H^{s_1}(\mathbb{T}^d))} \leq C\|u^\varepsilon\| \tag{2.36}$$

■

Notice that w^ε satisfies the equation (2.15), with Dirichlet data $w^\varepsilon_{0|x_d=0} = 0$ on the boundary.

The weaker estimate

$$\|Q(w_1^\varepsilon)_{|x_d=0}\| \leq C\,\varepsilon^{-1/2}\|u^\varepsilon\| \tag{2.37}$$

is easy to obtain (it is sufficient to commute the equation (1.4) with the normal vector field $\frac{\partial}{\partial n}$).

The proof of (2.36) is the most technical part of our work. It involves a detail study of how the spectral theory of $\mathbb{M}^0(x, \xi', \tau; y, \partial y)$ (see (2.15)) depends on the parameter (x, ξ', τ). This involves both the real and the complex part of the Bloch variety.

4. The last part is devoted to the proof of the following proposition.

Proposition 2.3 *Let $Q(\varepsilon, t, x, \varepsilon\partial_{x'}, \varepsilon\partial_t)$ be a zero order opd equal to Id near K, with essential support in W_1. There exist $\gamma_0, \varepsilon_0, h_0$, and a constant C_0 such that, for any $\varepsilon \in]0, \varepsilon_0]$, $h \in [\varepsilon/\gamma_0, h_0]$ and $u^\varepsilon \in I_h^\varepsilon$, the following estimate holds true*

$$\|Q\Gamma_0(u^\varepsilon)\|^2_{L^2(X_{T_0})} \leq C_0 \left[\|\Gamma_0(u^\varepsilon)_{|x_d=0}\|^2_{L^2(X_{T_0}\cap x_d=0)} + \|u^\varepsilon\|^2_{L^2(0,T_0)\times V} \right] \tag{2.38}$$

■

This estimate is obtain by rather classical arguments in the theory of control of linear waves, for the rescale equation

$$\begin{cases} \left(h\frac{\partial}{\partial x_d} + \mathcal{L}\right) \begin{bmatrix} g_0 \\ g_1 \end{bmatrix} \sim 0 & \qquad \begin{bmatrix} g_0 \\ g_1 \end{bmatrix} = \begin{bmatrix} \Gamma_0(u^\varepsilon) \\ \frac{h}{\varepsilon}\Gamma_1(u^\varepsilon) \end{bmatrix} \\ \mathcal{L} = \frac{h}{\varepsilon} \begin{pmatrix} 1 & 0 \\ 0 & h/\varepsilon \end{pmatrix} \circ L \circ \begin{pmatrix} 1 & 0 \\ 0 & \varepsilon/h \end{pmatrix} & \end{cases} \tag{2.39}$$

We verify that $\mathcal{L}$ is still a h-pseudo differential operator, with ε/h as parameter. (We use this rescaling in order to be able to use propagation arguments in the range $\varepsilon \ll h$).

We finally remark that the validity of (2.1), hence the proof of Theorem 0.1, is a direct consequence of the Proposition 2.1, 2.2 and 2.3.

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