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Back-scattering and nonlinear Radon transformation

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1 Introduction

We consider the Schrödinger operator $H_v = H_0 + v = -\Delta + v$ in $\mathbf{R}^n$ when $n \geq 3$ is odd. Although the arguments would work under less restrictive conditions on the potential v it will be assumed throughout that $v \in C_0^\infty(\mathbf{R}^n; \mathbf{R})$. Sometimes we will assume also that v is small in the sense that

$$\|v\| := \sum_{|\alpha|=n-2} \|v^{(\alpha)}\|_{L^1} \leq c_n, \tag{1}$$

where c_n denotes small constants which depend on the dimension only. This assumption will imply that H_v has a purely continuous spectrum.

2 Intertwining operators

Our presentation will rely heavily on the existence of intertwining operators between H_v and H_0. We need therefore to recall some facts about these operators and refer to [3], [4] and [5] for more references.

An operator $U : C_0^\infty(\mathbf{R}^n) \to \mathcal{D}'(\mathbf{R}^n)$ is called an intertwining operator (from H_v to H_0) if

$$H_v U = U H_0.$$

It is convenient to identify an operator with its distribution kernel. Then the intertwining equation takes the form of a partial differential equation:

$$(\Delta_x - \Delta_y) U(x, y) = v(x) U(x, y). \tag{2}$$

In addition to the equation above we are going to require boundedness of U in some weighted L^p-spaces. Let $\mathcal{D}'_S(\mathbf{R}^n \times \mathbf{R}^n)$ denote the Schur class. This is the set of all $U \in \mathcal{D}'(\mathbf{R}^n \times \mathbf{R}^n)$ such that U (viewed as an operator) and its transpose U^τ (with distribution kernel $U^\tau(x, y) = U(y, x)$) extend to continuous linear operators on L^p for every $p \in [1, \infty)$. Then U is defined also on $L^\infty(\mathbf{R}^n)$ if we view it as the adjoint of $U^\tau : L^1 \to L^1$. The elements of $\mathcal{D}'_S(\mathbf{R}^n \times \mathbf{R}^n)$ are measures, and this space is a Banach algebra under composition of operators.

We denote by $\mathcal{M}$ the set of U such that $(\partial_x + \partial_y)^\alpha U \in \mathcal{D}'_S(\mathbf{R}^n \times \mathbf{R}^n)$ for every α. This condition on U implies that any repeated commutator of U with constant coefficient vector fields is continuous on $L^p(\mathbf{R}^n)$ when $1 \leq p \leq \infty$. Hence U gives rise to operators on all the classical Sobolev spaces.

When v is small then there are invertible intertwining operators in $\mathcal{M}$. This result can not hold for arbitrary v since the existence of such operators imply that H_v has a purely continuous spectrum. We therefore have to consider other classes of intertwining operators as well. When $\theta \in S^{n-1}$ and $\lambda \geq 0$ we let $\mathcal{M}_{\theta,\lambda}$ be the set of all $U \in \mathcal{D}'(\mathbf{R}^n \times \mathbf{R}^n)$ such that $\langle y - x, \theta \rangle \geq 0$ in the suppport of U and $e^{-\lambda \langle y - x, \theta \rangle} U(x, y) \in \mathcal{M}$. These conditions imply that U is continuous in the weighted spaces

$$L^p_{\theta,\lambda}(\mathbf{R}^n) = \{f; e^{\lambda \langle x, \theta \rangle} f \in L^p(\mathbf{R}^n)\}.$$

We notice that $\mathcal{M}$ and $\mathcal{M}_{\theta,\lambda}$ are Fréchet algebras.

Theorem 1. *For every $\theta \in S^{n-1}$ there is a unique intertwining operator $U_\theta \in \cup_{\lambda \geq 0} \mathcal{M}_{\theta,\lambda}$ such that $U(x + t\theta, y + t\theta) \to 0$ as $t \to \infty$. This operator is invertible and $U^\tau_{-\theta}$ is its inverse. Moreover, if v is small then U_θ is an invertible element in $\mathcal{M}$.*

We remark also that $U_\theta(x, y) - \delta(x - y)$ is a locally integrable function. It will be seen later on (see formula (11)) that it is possible to express the back-scattering data in terms of the operators U_θ. When doing this we shall apply the following additional information about the intertwining operators.

Theorem 2. *Assume that v is small. Then*

$$\int |U_\theta(x, y) - \delta(x - y)|\, dy \to 0 \quad \text{as } |x| \to \infty,\ x/|x| \to \theta\ ,$$

and the same is true for the inverse of U_θ.

3 The back-scattering transform

We recall that the scattering operator is defined through

$$S = W_+^* W_-,$$

where

$$W_\pm = \lim_{t\to\pm\infty} e^{itH_v} e^{-itH_0}$$

are the wave operators. If $A = S - I$, then

$$A = W_+^*(W_- - W_+),$$

and it follows then readily from the definition of the wave operators that A is the strong limit as $\varepsilon \to +0$ of A_ε, where

$$A_\varepsilon = -i \int_{-\infty}^{\infty} e^{-\varepsilon|t|} e^{itH_0} W_+^* v e^{-itH_0}\, dt. \tag{3}$$

Let

$$\mathcal{F}u(\xi) = \hat{u}(\xi) = \int e^{-i\langle x,\xi\rangle} u(x)\, dx$$

be the Fourier transform and multiply (3) from the left by $\mathcal{F}$ and from the right by $\mathcal{F}^*$. Then

$$\hat{A}_\varepsilon(\xi,\eta) = -i \int_{-\infty}^{\infty} e^{-\varepsilon|t|} e^{it|\xi|^2 - it|\eta|^2} (\mathcal{F} W_+^* v \mathcal{F}^*)(\xi,-\eta)\, dt,$$

where $(\mathcal{F} W_+^* v \mathcal{F}^*)(\xi,-\eta)$ is the Fourier transform of

$$(W_+^* v)(x,y) = W_+^*(x,y) v(y).$$

When $\varepsilon \to 0$ one finds that

$$\hat{A}(\xi,\eta) = (-2i\pi)\delta(|\xi|^2 - |\eta|^2)(\mathcal{F} W_+^* v \mathcal{F}^*)(\xi,-\eta). \tag{4}$$

The formal computations presented here may be justified, and one can prove furthermore that

$$a(\xi,\eta) := (\mathcal{F} W_+^* v \mathcal{F}^*)(\xi,-\eta)$$

is continuous when $\xi + \eta \neq 0$, and it is rapidly decreasing when $|\xi+\eta| \to \infty$. Hence the right-hand side of (4) makes sense, and we see that the scattering operator determines the function $a(\xi,\eta)$ in the set where $|\xi| = |\eta|$. We notice also that

$$\overline{a(-\eta,-\xi)} = (\mathcal{F} v W_+ \mathcal{F}^*)(\xi,-\eta).$$

The back-scattering data are obtained if one restricts the function

$$(a(\xi,\eta) + \overline{a(-\eta,-\xi)})/2$$

to the diagonal in $\mathbf{R}^n \times \mathbf{R}^n$ and replaces ξ by $\xi/2$ afterwords:

Definition 3. The back-scattering data is the function

$$\tilde{v}(\xi) = (\mathcal{F}(vW_+ + W_+^* v)\mathcal{F}^*)(\xi/2, -\xi/2)/2.$$

Since the linearization of the operator $W_+ v$ at $v = 0$ is multiplication by v, it follows that the linerization of $v \mapsto \tilde{v}$ at $v = 0$ is the Fourier transformation. This motivates us to compose the mapping $v \mapsto \tilde{v}$ with the inverse Fourier transform:

Definition 4. The back-scattering transform Bv of v is the inverse Fourier transform of $\tilde{v}$.

Remark. Since $\tilde{v}$ is rapidly decreasing at infinity, in view of a previous remark, it follows that Bv is a smooth function. Moreover, since $\tilde{v}(-\xi)$ is the conjugate of $\tilde{v}(\xi)$ it follows that Bv is real.

The following result gives an expression for Bv in terms of v and W_+:

Theorem 5. *The back-scattering transform is given by*

$$(Bv)(x) = 2^n \operatorname{Re}\Big(\int v(x-y) W_+(x-y, x+y)\, dy \Big), \tag{5}$$

where the integral is interpreted in the distribution sense. (This makes sense since v is compactly supported.)

Proof. Let W_j be a sequence in $\mathcal{S}(\mathbf{R}^n \times \mathbf{R}^n)$ which converges to W_+ in $\mathcal{S}'(\mathbf{R}^n \times \mathbf{R}^n)$ as $j \to \infty$. Then the right-hand side of (5) is the limit in $\mathcal{S}'(\mathbf{R}^n \times \mathbf{R}^n)$ of $\operatorname{Re} P_j(x)$, where

$$P_j(x) = 2^n \int (vW_j)(x-y, x+y)\, dy.$$

Since

$$\hat{P}_j(\xi) = (\mathcal{F}(vW_j))(\xi/2, \xi/2) \to \overline{a(-\xi/2, -\xi/2)} \quad \text{as } j \to \infty$$

it follows that the Fourier transform of $\operatorname{Re} P_j$ tends to $\tilde{v}$ as $j \to \infty$. This proves (5). $\square$

4 The wave equation approach

It is often convenient to define the back-scattering data in terms of the wave group associated to the operator $\Box_v = \partial_t^2 - \Delta + v$ in $\mathbf{R}^{n+1}$. Let $f = K_v(t)u$, where $u \in C_0^\infty(\mathbf{R}^n)$, be the unique solution to

$$\Box_v f = 0, \quad \text{when } t > 0, \quad f = 0,\ f_t' = u \text{ when } t = 0.$$

Then $K_v(t)$ may be viewed as a smooth function of $t \geq 0$ with values in $\mathcal{D}'(\mathbf{R}^n \times \mathbf{R}^n)$, $K_v(0) = 0$ and $\dot{K}_v(0) = \delta(x-y)$, where $\dot{K}_v$ is the derivative with respect to t. We define $K_v(t) = 0$ when $t < 0$. We shall need two lemmas:

Lemma 6. *Let $K_v(t)$ be as above. Then*

(i) One has $K_v(t) = U_\theta \circ K_0(t) \circ U_\theta^{-1}$.

(ii) If a and b are real numbers with $a + b \neq 0$ and if $\mu \geq 0$ is an arbitrary integer, then one may write

$$K_v(x,y,t) = \sum_{|\alpha| \leq N} (b\partial_x - a\partial_y)^\alpha R_\alpha(x,y,t), \quad t \geq 0,$$

where the derivatives with respect to x and y of order $\leq \mu$ of $R_\alpha(x,y,t)$ are continuous functions of t with values in $L^\infty(\mathbf{R}^n \times \mathbf{R}^n)$.

(iii) If (1) holds then the derivatives in (ii) are exponentially decaying in t, uniformly with respect to $(x,y) \in \mathbf{R}^{2n}$ as long as (x,y) stays in a compact set ,

A combination of (i) of the preceding lemma with explicit expressions for $K_0(t)$ gives rise to the following result about the behaviour of $K_v(t)$ as $t \to \infty$.

Lemma 7. *One may write*

$$K_0(x,y,t) = \sum_{\nu=0}^{n-2} \partial_t^\nu \Gamma_\nu(x,y,t),$$

where

$$\|\Gamma_\nu(\cdot,\cdot,t)\|_{L^\infty} = O(t^{(1-n)/2}) \quad \textit{as } t \to \infty.$$

The following observation gives an important link between K_v and the intertwining operators.

Theorem 8. *One has*

$$\int_{\langle y,\theta\rangle=t} U_\theta(x,y)\,dy = \delta(\langle x,\theta\rangle - t) - \int K_v(x,y,t-\langle y,\theta\rangle)v(y)\,dy, \tag{6}$$

where the integrals are interpreted in the distribution sense.

Proof. We shall assume first that v is small. Let us define $L_\theta : C_0^\infty(\mathbf{R}) \to C^\infty(\mathbf{R}^n)$ by $(L_\theta g)(x) = g(\langle x,\theta\rangle)$. If we view L_θ as a distribution then L_θ is equal to the first term in the right-hand side of (6). Since $e^{\lambda\langle x,\theta\rangle}L_\theta g \in L^\infty$ for every λ it is true also that $U_\theta \circ L_\theta$ is defined, and its distribution kernel is the left-hand side of (6). In order to get a suitable representation of the last term in (6) we introduce the mapping $A_\theta : C_0^\infty(\mathbf{R}^n\times\mathbf{R}) \to C_0^\infty(\mathbf{R}^n\times\mathbf{R}^n\times\mathbf{R})$ by

$$(A_\theta\varphi)(x,y,t) = \varphi(x,t+\langle y,\theta\rangle)v(y).$$

We then define $T_\theta \in \mathcal{D}'(\mathbf{R}^n \times \mathbf{R})$ by

$$\langle \varphi, T_\theta\rangle = \langle A_\theta\varphi, K_v\rangle,$$

where K_v is viewed as an element in $\mathcal{D}'(\mathbf{R}^n \times \mathbf{R}^n \times \mathbf{R})$.

Set

$$S_\theta = (U_\theta - I)\circ L_\theta + T_\theta,$$

and view S_θ as an element in $\mathcal{D}'(\mathbf{R}^n \times \mathbf{R})$. Since

$$T_\theta(x,t) = \int K_v(x,y,t-\langle y,\theta\rangle)v(y)\,dy,$$

the lemma is equivalent to the assertion that $S_\theta = 0$. When proving that this holds we first notice that $\Box_v \circ A_\theta = A_\theta \circ \Box_v$. Hence

$$\langle \varphi, \Box_v T_\theta\rangle = \langle \Box_v A_\theta\varphi, K_v\rangle = \langle A_\theta\varphi, \Box_v K_v\rangle.$$

Since $\Box_v K_v(x,y,t) = \delta(x-y)\delta(t)$ we find that $\Box_v T_\theta = vL_\theta$, and the intertwining property of U_θ implies that

$$\Box_v(U_\theta \circ L_\theta) = U_\theta \circ (\Box_0 L_\theta) = 0.$$

Combining these observations we see that

$$\Box_v S_\theta = 0. \tag{7}$$

Since v is compactly supported, and since $|x-y| \le t$ in the support of $K_v(t)$, it follows that T_θ must be continuous from $C_0^\infty(\mathbf{R})$ to $\mathcal{E}'(\mathbf{R}^n)$ and

$\langle x, \theta\rangle \le t$ in the support of T_θ. An application of (ii) of Lemma 6 with $b = 0$, $a = 1$ implies then that T_θ is continuous from C_0^∞ to C_0^∞. Hence T_θ is continuous from $C_0^\infty(\mathbf{R})$ to $L^\infty(\mathbf{R}^n)$. Since $\langle x, \theta\rangle = t$ in the support of L_θ and $\langle y - x, \theta\rangle \ge 0$ in the support of U_θ we see that

$$\langle x, \theta\rangle \le t \quad \text{in } \operatorname{supp}(S_\theta) \tag{8}$$

Let us consider now $R_\theta = U_\theta^{-1} \circ S_\theta$. This is defined since S_θ must be continuous from $C_0^\infty(\mathbf{R})$ to $L^\infty(\mathbf{R}^n)$. The bounds given by Lemma 6 show that $R_\theta \in \mathcal{S}'(\mathbf{R}^n \times \mathbf{R})$. It follows from (7) and the intertwining property that $\Box_0 R_\theta = 0$. Since $R_\theta \in \mathcal{S}'$, and since $\langle x, \theta\rangle \le t$ in its support, one may conclude now that

$$R_\theta(x, t) = \sum_\alpha x^\alpha g_\alpha(t - \langle x, \theta\rangle),$$

where the sum is finite, $g_\alpha \in \mathcal{S}'(\mathbf{R})$ and $t \ge 0$ in the support of g_α. Since R_α is continuous from $C_0^\infty(\mathbf{R})$ to $L^\infty(\mathbf{R}^n)$ it follows that R_θ can only depend on $\langle x, \theta\rangle$ and t. Hence

$$R_\theta(x, t) = \sum_j \langle x, \theta\rangle^j g_j(t - \langle x, \theta\rangle),$$

where the g_j have the same properties as the g_α. The support conditions on the g_j together with the fact that $\Box_0 R_\theta = 0$ imply now that $g_j = 0$ when $j > 0$.

In order to prove that $R_\theta = 0$ we consider translations $\tau_y f(x) = f(x+y)$. Then

$$\tau_{\lambda\theta} \circ R_\theta \circ \tau_{-\lambda} = R_\theta.$$

By applying Theorem 2 it is easily shown that R_θ must be the limit in $\mathcal{D}'$ as $\lambda \to +\infty$ of $\tau_{\lambda\theta} \circ S_\theta \circ \tau_{-\lambda}$. Another application of Theorem 2 shows that this must have the same limit as $T_{\theta,\lambda} = \tau_{\lambda\theta} \circ T_\theta \circ \tau_{-\lambda}$. In order to prove that $S_\theta = 0$ it remains there only to prove that

$$T_{\theta,\lambda} \to 0 \quad \text{in } \mathcal{D}'(\mathbf{R}^n \times \mathbf{R}) \text{ as } \lambda \to \infty. \tag{9}$$

Let $\varphi \in C_0^\infty(\mathbf{R}^n \times \mathbf{R})$ and set

$$\varphi_\lambda(x, y, t) = \varphi(x - \lambda\theta, t - \lambda + \langle y, \theta\rangle)v(y).$$

Then

$$\langle \varphi, T_{\theta,\lambda}\rangle = \langle \varphi_\lambda, K_v\rangle.$$

An application of Lemma 7 shows that the right-hand side is $O(\lambda^{(1-n)/2})$ as $\lambda \to \infty$. This settles (9) and we have proved the theorem in the case when v is small. In the general case one replaces v by cv where $c \in \mathbf{R}$. Then it can be proved that both sides of (6) can be viewed as an entire analytic function of c with values in $\mathcal{D}'(\mathbf{R}^n \times \mathbf{R})$. Since this vanishes for small c we have proved the theorem in the general case. □

Now we have made all preparations necessary in order to express the back-scattering transform Bv in terms of K_v. In order to avoid some technical complications we limit ourselves to the case when v is small

Theorem 9. *Assume that* $v \in C_0^\infty(\mathbf{R}^n;\mathbf{R})$ *is small in the sense that* (1) *is fulfilled. Then*

$$(Bv)(x) = v(x) - \tfrac{1}{2}\Big(-\tfrac{\Delta_x}{4\pi^2}\Big)^{(n-1)/2} \iiint K_v(y,z,\langle 2x-y-z,\theta\rangle)v(y)v(z)\,dy\,dz\,d\theta,$$

where the integral has to be interpreted in the distribution sense (cf. Lemma 6).

Proof. Let us introduce the generalized wave functions

$$\varphi(x,\xi) = \int U_{-\xi}(x,y)e^{i\langle y,\xi\rangle}\,dy,$$

where we have extended U_θ to a homogeneos function of θ. Then one can show that

$$\varphi(x,\xi) = (W_+\mathcal{F}^*)(x,\xi). \tag{10}$$

Recalling Definition 3 we may now express the back-scattering in terms of the A_θ. We obtain

$$\tilde{v}(\lambda\theta) = \tfrac{1}{2}\iint v(y)(U_\theta(y,z) + U_{-\theta}(x,y))e^{-i\lambda\langle x+y,\theta\rangle/2}\,dy\,dz, \tag{11}$$

when $\theta \in S^{n-1}$ and $\lambda \in \mathbf{R}$. Notice that both sides are even in (λ,θ). We shall now carry out some formal computations which may easily be justified.

We first write

$$\begin{aligned}(Bv)(x) &= (2\pi)^{-n} \int e^{i\langle x,\xi\rangle} \tilde{v}(\xi)\, d\xi \\ &= (2\pi)^{-n} \iint\limits_{S^{n-1}\times \mathbf{R}_+} t^{n-1} e^{it\langle x,\theta\rangle} \tilde{v}(t\theta)\, dt\, d\theta \\ &= (2\pi)^{-n} 2^{-1} \iint\limits_{S^{n-1}\times \mathbf{R}} t^{n-1} e^{it\langle x,\theta\rangle} \tilde{v}(t\theta)\, dt\, d\theta \\ &= (2\pi)^{-n} 2^{-1} (-\Delta_x)^{(n-1)/2} \Big(\iint\limits_{S^{n-1}\times \mathbf{R}} e^{it\langle x,\theta\rangle} \tilde{v}(t\theta)\, dt\, d\theta \Big).\end{aligned}$$

We combine this expression for Bv with (11) and make use of the identity

$$\int_{\mathbf{R}} e^{it\langle x,\theta\rangle} e^{-it\langle y+z,\theta\rangle/2}\, dt = 2\pi\delta(\langle x-(y+z)/2, \theta\rangle).$$

This leads to

$$(Bv)(x) = \tfrac{1}{4}\Big(\tfrac{-\Delta_x}{4\pi^2}\Big)^{(n-1)/2} \iiint v(y) U_\theta(y,z) \delta(\langle x-(y+z)/2, \theta\rangle)\, dy\, dz\, d\theta.$$

The proof is then completed by (6) and the following computations

$$\begin{aligned}&\tfrac{1}{2} \int U_\theta(y,z)\delta(\langle(x-(y+z)/2, \theta\rangle)\, dz \\ &= \int U_\theta(y,z)\delta(\langle z,\theta\rangle - \langle 2x-y,\theta\rangle)\, dz \\ &= \delta(\langle y,\theta\rangle - \langle 2x-y,\theta\rangle) - \int K_v(y,z,\langle 2x-y-z,\theta\rangle) v(z)\, dz \\ &= \tfrac{1}{2}\delta(\langle x-y,\theta\rangle) - \int K_v(y,z,\langle 2x-y-z,\theta\rangle) v(z)\, dz\end{aligned}$$

□

We finally remark in this section that the back-scattering data have been expressed in three different ways:

(i) in terms of v and W_+ by (5);

(ii) in terms of v and the family U_θ by (11);

(iii) in terms of v and K_v as in Theorem 9.

5 The inversion problem

We shall now make some remarks on the inverse back-scattering problem. We refer to the papers [1], [2], and [6] and the references given therein for some recent discussions.

It will be assumed througout in this section that v is small. Since the intertwining operators may always be expressed in convergent power series in v it follows from (11) that Bv can be expanded in a power series in v. This can also be seen from Theorem 9, since iteration techniques in the construction of K_v give such an expansion. Letting $B_j v$ denote the part of Bv which is homogenoeus of degree j in v we have therefore an expansion

$$Bv = \sum_1^\infty B_j v, \tag{12}$$

with convergence in $\mathcal{D}'(\mathbf{R}^n)$ for small v. We notice that $B_1 v = v$.

It is natural to ask for a Banach space X continuously embedded in $\mathcal{D}'(\mathbf{R}^n)$ such that (12) is convergent in that space. This would make it possible to approach the inversion problem with methods from nonlinear functional analysis. The analysis carried out in the construction of intertwining operators indicates that one should try with

$$X = X_n = \{v \in L^1_{\text{loc}}(\mathbf{R}^n; \mathbf{R});\ v^{(\alpha)} \in L^1 \text{ when } |\alpha| = n-2\}$$

with the norm $\|v\|$ given by (1). In fact, this is the norm one has to control when one constructs the operators U_θ and proves that these are in the Schur class.

We can now formulate the following problem:

Problem A**:** Can one find a constant C_n, depending on n only, such that the inequalities

$$\|B_j v\| \leq C_n^j \|v\|^j \tag{13}$$

hold for every j when $v \in C_0^\infty(\mathbf{R}^n)$?

Recent investigations of R. Lagergren indicate that Problem A has an affirmative answer in the case when $n = 3$. Some details of the proof remain to be checked. The proof is based on Theorem 9. One expresses $K_{v,j}$, the part of K_v which is homogeneous of degree j in v, in terms of integral formulas involving K_0 and v, and then one estimates $\nabla B_j v$ by carrying out partial integrations in the formulas so that each factor v is differentiated exactly once . Although the idea is simple the method offers problems of combinatorial nature.

We shall finish our discussions by making some remarks on a somewhat simpler problem
Problem A_j: Can one find constants $C_{n,j}$ such that

$$\|B_j v\| \leq C_{n,j} \|v\|^j \tag{14}$$

when $v \in C_0^\infty(\mathbf{R}^n)$?

When $j = 1$ then $B_1 v = v$. Hence $C_{n,1} = 1$. We shall prove now that the constants $C_{n,2}$ also exist:

Theorem 10. *There are constants $C = C_{n,2}$, when $n \geq 3$ is odd, such that*

$$\|B_2 v\| \leq C \|v\|^2, \quad v \in C_0^\infty(\mathbf{R}^n; \mathbf{R}).$$

We need some preparations in order to prove the theorem. Since $n \geq 3$ is odd it is true that $\Delta_x - \Delta_y$ has a fundamental solution in the form

$$E(x,y) = E_n(x,y) = c_n \delta^{(n-2)}(x^2 - y^2).$$

Lemma 11. *Assume that $u \in \mathcal{S}'(\mathbf{R}^n \times \mathbf{R}^n)$ satisfies the following:*

(i) $(\Delta_x - \Delta_y)u = 0$*;*

(ii) u *is rotation symmetric in* x *and* y *separately;*

(iii) $|x| \leq |y|$ *in the support of* u*;*

(iv) u *is homogeneous of degree* $2 - 2n$*.*

Then $u = 0$*.*

Proof. An application of Asgeirsson's theorem shows that $|x| = |y|$ in the support of u. Since $(\xi^2 - \eta^2)\hat{u}(\xi, \eta) = 0$, the rotation invariance together with homogeneity implies that $\hat{u} = c\delta(\xi^2 - \eta^2)$ for some constant c. Computing derivatives in polar coordinates one finds that $(\Delta_\xi - \Delta_\eta)^{n-2}\hat{u}(\xi, \eta) = 0$. Hence $(x^2 - y^2)^{n-2}u(x,y) = 0$. Taking homogeneity into account one can prove that this implies that $u = c' E(x,y)$ for some c'. Since $(\Delta_x - \Delta_y)u = 0$ it follows therefore that $u = 0$. □

Let us denote by $k_0(x,t)$ the forward fundamental solution of $\Box_0$ so that $K_0(x,y,t) = k_0(x-y,t)$.

Lemma 12. *There is a constant $c_n' \neq 0$ such that*

$$\Delta_y^{(n-1)/2} \int_{S^{n-1}} k_0(x, \langle y, \theta \rangle)\, d\theta = c_n' E(x,y).$$

Proof. Set $w(x,y) = \Delta_y^{(n-1)/2} \int k_0(x, \langle y, \theta\rangle)\, d\theta$. Since

$$(\Delta_x - \Delta_y) \int k_0(x, \langle y, \theta\rangle)\, d\theta = -\delta(x) \int \delta(\langle y, \theta\rangle)\, d\theta,$$

and since

$$\Delta_y^{(n-1)/2} \int \delta(\langle y, \theta\rangle)\, d\theta = c_n'' \delta(y),$$

it follows that

$$(\Delta_x - \Delta_y) w(x,y) = c_n' \delta(x,y)$$

for some $c_n' \neq 0$. Since $u = w - c_n' E$ satisfies the hypotheses of the preceding lemma it follows that $w = c_n' E$. □

Proof of Theorem 10

It is an immediate consequence of Theorem 9 and the previous lemma that there is a constant c_n'' such that

$$(B_2 v)(x) = c_n'' \iint E(y-z, 2x-y-z) v(y) v(z)\, dy\, dz. \tag{15}$$

We write the right-hand side as

$$c_n'' \iint E(y-z, y+z) v(x+y) v(x+z)\, dy\, dz.$$

The explicit formula for $E = E_n$ above allows us to replace $B_2 v$ by $Q(v,v)$, where we define

$$(Q(v_1, v_2))(x) = \iint \delta^{(n-2)}(yz) v_1(x+y) v_2(x+z)\, dy\, dz. \tag{16}$$

Let us introduce

$$v^*(x) = \sum_{|\alpha|=n-2} |v^{(\alpha)}(x)|$$

and write

$$\partial_x^\alpha v_1(x+y) v_2(x+z) = (\partial_y + \partial_z)^\alpha v_1(x+y) v_2(x+z).$$

A combination of (16) and Lemma 13 below gives us then the inequality

$$\begin{aligned} &|(\partial_x + \partial_y)^\alpha (Q(v_1, v_2))(x)| \\ &\le C \iint v_1^*(x+y) v_2^*(x+z)(|y|^{2-n} + |z|^{2-n}) \delta(yz)\, dydz. \end{aligned}$$

Let us consider the contribution from $|y|^{2-n}$. In polar coordinates $y = r\omega$ we may write this as a constant times

$$\iiint v_1^*(x + r\omega)v_2^*(x + z)\delta(\omega z)\, dr\, d\omega\, dz.$$

We are going to integrate this expression with respect to x. If one take $x + z$ as a new variable of integration instead of x one gets the expression

$$\int \cdots \int v_1^*(x + r\omega - z)v_2^*(x)\delta(\omega z)\, dr\, d\omega\, dz\, dx.$$

By integrating first with respect to z and r one finds that the integral may be estimated from above by a constant times $\|v_1\|\|v_2\|$. Since the contribution from $|z|^{2-n}$ may be treated in the same way this finishes the proof.

It remains to prove the following lemma.

Lemma 13. *Assume that α is a multi-index of length $|\alpha| = n - 2$. Then*

$$(\partial_x + \partial_y)^\alpha \delta^{(n-2)}(xy) = \sum_{|\beta|=|\gamma|=n-2} \partial_x^\beta \partial_y^\gamma (u_{\alpha\beta\gamma}(x, y)\delta(xy)), \tag{17}$$

where the $u_{\alpha\beta\gamma}$ are Borel measurable functions satisfying the estimates

$$|u_{\alpha\beta\gamma}(x, y)| \le C(|x|^{2-n} + |y|^{2-n}). \tag{18}$$

Proof. Let us denote by $h_a(x)$ any function that is homogeneous of degree $-a$ and smooth away from the origin. Let b and c be nonnegative integers with $b + c = n - 2$. We claim that

$$\delta^{(n-2)}(xy) = \sum_{|\beta|=b,|\gamma|=c} \partial_x^\beta \partial_y^\gamma h_{n-2-b}(x)h_{n-2-c}(y)\delta(xy). \tag{19}$$

We prove this first in the set where $|x||y| \ne 0$. Consider the formula

$$\delta^{(n-2)}(xy) = \langle y/|y|^2, \partial_x\rangle^b \langle x/|x|^2, \partial_y\rangle^c \delta(x, y).$$

By commuting the factors involving ∂_y to the left of the coefficents involving $y/|y|^2$ one finds that

$$\delta^{(n-2)}(xy) = \sum_{|\beta|=b,|\gamma|\le c} \partial_x^\beta \partial_y^\gamma h_{n-2-|\gamma|}(y)h_c(x)\delta(xy).$$

Since $h_{n-2-|\gamma|}(y)\delta(xy)$ is homogeneous of negative degree in y we may write this as a linear combination of $\partial_y^\mu y^\mu h_{n-2-|\gamma|}(y)\delta(xy)$, with $|\mu| = c - |\gamma|$.

This leads to (19) in the set where $|x||y| \neq 0$. Since both sides are separately homogeneous of degree $1-n$ in x and y it follows that (19) holds everywhere.

Consider now $\partial_x^\mu \partial_y^\nu \delta^{(n-2)}(xy)$ when $|\mu| + |\nu| = n-2$. By choosing $b = n-2-|\mu|$ and $c = n-2-|\nu|$ we find that the conclusion of the lemma must be true for $\partial_x^\mu \partial_y^\nu \delta^{(n-2)}(xy)$, and the lemma follows then by expanding $(\partial_x + \partial_y)^\alpha \delta^{(n-2)}(xy)$ in such terms. □

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