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EIGENVALUE ASYMPTOTICS FOR NEUMANN LAPLACIAN IN DOMAINS WITH ULTRA-THIN CUSPS †

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ABSTRACT. Asymptotics with sharp remainder estimates are recovered for number $\mathbf{N}(\tau)$ of eigenvalues of the generalized Maxwell problem and for related Laplacians which are similar to Neumann Laplacian. We consider domains with ultra-thin cusps (with $\exp(-|x|^{m+1})$ width; $m>0$) and recover eigenvalue asymptotics with sharp remainder estimates.

1. Introduction. We are interested in eigenvalue asymptotics for Maxwell operator $\mathcal{A}$ in $X\subset\mathbb{R}^d$. Namely, let $\mathfrak{H}=L^2(X,\Lambda^k)\oplus L^2(X,\Lambda^{k+1})$, Λ^k is a space of exterior forms of a degree $k=0,\dots,d$[1)], and let for

$$\phi=\sum_I \phi_I dx_I,\quad dx_I=dx_{i_1}\wedge\dots dx_{i_k}:I=(i_1,\dots,i_k),i_1<\dots<i_k \tag{1}$$

we define $\|\phi\|^2=\sum_I\|\phi_I\|^2_{L^2(X)}$ and $\|\begin{pmatrix}\phi\\ \psi\end{pmatrix}\|^2=\|\phi\|^2+\|\psi\|^2$. Let us consider an operator

$$\mathcal{A}=\begin{pmatrix}0 & \beta^\dagger\mathcal{D}^*\alpha^\dagger\\ \alpha D\beta & 0\end{pmatrix} \tag{2}$$

with domain $\mathfrak{D}(\mathcal{A})=\{\begin{pmatrix}\phi\\ \psi\end{pmatrix}\in\mathfrak{H},\mathcal{A}u\in\mathcal{H},\iota_{\partial X}\phi=0\}$ where $\mathcal{D}=i\langle dx,D\rangle\wedge:$ $C^\infty(X,\Lambda^k)\to C^\infty(X,\Lambda^{k+1})$ is the operator of the exterior differentiation and $\mathcal{D}^*:C^\infty(X,\Lambda^{k+1})\to C^\infty(X,\Lambda^k)$ is the formally adjoint operator[2)]; for ϕ in form (1) we have $\mathcal{D}\phi=\sum_j(iD_j\phi_I)dx_j\wedge dx_I,$; $\mathcal{D}^*\phi=\sum_{1\le p\le k}(iD_{i_p}\phi_I)(-1)^p dx_{I\setminus i_p}$,

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1) At given point, $\dim\Lambda^k=\frac{d!}{k!(d-k)!}$ and we consider complex rather than real spaces

2) We can define action of $\mathcal{D}$, $\mathcal{D}^*$ and $\mathcal{A}$ at distributions as well

$\alpha(x) \in \mathcal{L}(\Lambda^{k+1}, \Lambda^{k+1})$; and $\beta(x) \in \mathcal{L}(\Lambda^k, \Lambda^k)$ are nondegenerate matrices smoothly depending on x and constant close to infinity (or quickly stabilizing to constant), $\iota_Y : C^\infty(X, \Lambda^k(X)) \to C^\infty(X, \Lambda^k(Y))$ is the restriction of the exterior form to submanifold Y[3)]

There is no problem with self-adjoint expansion of such an operator defined on functions with compact support first provided ∂X, α and β are smooth enough. Furthermore, we assume that α and β are Hermitian positive matrices (otherwise one can reach it by means of the unitary transformation $\begin{pmatrix} T_1 & 0 \\ 0 & T_2 \end{pmatrix}$ with unitary matrices $T_1(x)$ and $T_2(x)$.

Obviously for eigenfunctions $\begin{pmatrix} \phi \\ \psi \end{pmatrix}$ with non-zero eigenvalues automatically $\mathcal{D}\alpha^{-1}\psi = \mathcal{D}^*\beta^{-1}\phi = 0$, and further, $\iota_{\partial X}\alpha^{-1}\psi = 0$. Further, $\begin{pmatrix} \phi \\ -\psi \end{pmatrix}$ is an eigenfunction with an eigenvalue $-\tau$ and moreover, the number of eigenvalues of operator A belonging to $(0,\tau)$ is

$$\mathbf{N}_k(\tau) = \mathbf{N}^-_{L_{k,0}}(\tau^2) - \mathbf{N}^-_{L_{k,0}}(0+0) \tag{3}$$

where $\mathbf{N}^-_{L_{k,0}}(\lambda)$ is the number of the eigenvalues of operator $L_{k,0}$ generated by a quadratic form $Q(\phi) = \|\alpha D\beta\phi\|^2 + \|\alpha_1^{-1} D^*\beta^{-1}\phi\|^2$ on the space $\mathcal{H}_{k,0} = \{\phi \in L^2(X,\Lambda^k), \mathcal{D}^*\beta^{-1}\phi = 0\}$ with domain $\mathfrak{D}(Q) = \{\phi \in \mathcal{H}_{k,0}, Q(\phi) < \infty, \iota_{\partial X}\beta\phi = 0\}$, which are less than λ; later in our conditions it will be a finite number; $\alpha_1 \in \mathcal{L}(\Lambda^{k-1}, \Lambda^{k-1})$ is a nondegenerate matrix (usually constant close to infinity).

Furthermore, considering the same form $Q(u)$ on the space $\mathcal{H}_k = \{\phi \in L^2(X,\Lambda^k)\}$ with domain $\mathfrak{D}(Q) = \{\phi \in \mathcal{H}_k, Q(\phi) < \infty, \iota_{\partial X}\beta\phi = 0\}$, we get an operator L_k; one can check easily that for $\tau > 0$ eigenspaces $\mathcal{H}_k(\tau)$ and $\mathcal{H}_{k,0}(\tau)$ of $L_k, 0$ and L_k satisfy

$$\mathcal{H}_k(\tau) \subset \mathcal{H}_{k,0}(\tau),\ \mathcal{H}_k(\tau) \ominus \mathcal{H}_{k,0}(\tau) = \beta^{-1}\mathcal{D}\alpha_1^{-1}\big(\alpha_1^{-1}\mathcal{D}^*\beta^{-1}\mathcal{H}_k(\tau)\big) \tag{4}$$

and that $\alpha_1^{-1}\mathcal{D}^*\beta^{-1}\mathcal{H}_k(\tau)$ is an eigenspace of the operator $L_{k-1,0}$ generated by a form $\|\beta^{-1}\mathcal{D}\alpha_1^{-1}\phi'\|^2 + \dots$ on the space $\mathcal{H}_{k-1,0} = \{\phi' \in L^2(X,\Lambda^{k-1}), \mathcal{D}^*\alpha_1\phi = 0, \iota_{\partial X}\alpha_1^{-1}\phi = 0\}$. Therefore

$$\mathbf{N}(\tau) = \mathbf{N}^-_{L_k}(\tau^2) - \mathbf{N}^-_{L_k}(0+0) - \mathbf{N}^-_{L_{k-1,0}}(\tau^2) - \mathbf{N}^-_{L_{k-1,0}}(0+0) \tag{5}$$

This reduction is not necessarily correct for non-smooth problems unless we are able to prove that $\mathfrak{D}(L_k) \subset H^2_{\mathrm{loc}}(X)$ which is not true even for $\alpha = \beta = I$ and $X = W \oplus \mathbb{R}^{d-2}$ with a sector W with an angle between π and 2π.

[3)] One can see easily that if Y is smooth of codimension 1 and $\phi \in L^2(X,\Lambda^K)$, $\mathcal{D}\phi \in L^2$ then $\iota_Y\phi$ is well-defined; if $Y = \{x_1 = 0\}$ in appropriate coordinates and ϕ is of the form (1) then $\iota_Y\phi = \sum_{I\not\ni 1}\phi_I|_Y dx_I$.

Note that for $k = d-1$, $\alpha = I$, $\beta = I$ and $\psi = u dx_1 \wedge \cdots \wedge dx_d$ we obtain exactly eigenvalue problem for the Neumann Laplacian.

These formulae lead us to standard (known) asymptotics for compact domains with smooth boundaries:

$$\mathbf{N}(\tau^2) = c_0\tau^d + O(\tau^{d-1}) \tag{6}$$

(here and below we omit indices k and may be 0) and even

$$\mathbf{N}(\tau^2) = c_0\tau^d + c_1\tau^{d-1} + o(\tau^{d-1}) \tag{7}$$

provided

$$\det\Bigl(\tau^2 - \beta\mathcal{D}(\xi)^\dagger\alpha^2\mathcal{D}(\xi)\beta\Bigr) = \tau^r\left(\tau^2 - g(x,\xi)\right)^s \tag{8}$$

where $\mathcal{D}(\xi) = i\langle dx,\xi\rangle\wedge$ is a principal symbol of $\mathcal{D}$, g is a metrics on X and standard billiard condition holds.

The same results hold for other types of compact domains: the irregularity of the boundary should be of the type described in [Ivr1], sect. 10.2, inner cone condition should be fulfilled [IF], and (5) should hold.

2. Thin cusps. Heuristics. Let us consider domains with cusps; I consider one cusp for simplicity: we assume that unbounded part of X is $\{x : x'' \in f(x')\Omega$ where $x = (x';x'')$, $x' \in X' = \mathbb{R}^{d'}$ which is the *base* of the cusp (or $X' = \mathbb{R}^+$ for $d' = 1$), Ω (which is *cross-section*) is a bounded domain in $\mathbb{R}^{d''}$ with smooth boundary, $d = d' + d''$, $1 \le d' \le d-1$, $f(x') > 0$ and decays as $|x'| \to \infty$. Look first for operators L_k. As we know, boundary conditions are very important for the Laplace operator: if we have Dirichlet boundary condition then the spectrum of operator is discrete (provided cusp shrinks at infinity).

On the other hand for operator with a Neumann boundary condition spectrum is discrete only if cusp is very thin ($\log f \asymp |x'|^{1+m}$ with $m > 0$) and for such operators asymptotics with sharp remainder estimate are derived in [Ivr2].

So basically we should determine first if for operator L_k condition is "Dirichlet"-like or "Neumann"-like at infinity; then for "Neumann"-like cusp assume that it is ultra-thin and write the cusp contribution; for "Dirichlet"-like cusp we need to describe non-Weyl contribution.

Let us consider the space $\Phi_k = \{\phi \in C^\infty(\Omega,\Lambda^k)\mathcal{D}''\beta\phi = \mathcal{D}''^*\beta^{-1}\phi = 0, \iota_{X''}\beta\phi = 0\}$ (so we consider full forms with coefficients depending on x'' only); one can see easily that $\dim\Phi_k$ does not depend on the choice of β and

$$\dim\Phi_k = \sum_j \frac{d'!}{j!(d'-j)!}\dim\Phi''_{k-j} \tag{9}$$

where $\Phi''_j = \{\phi \in C^\infty(\Omega, \Lambda^j_\Omega), \mathcal{D}''\phi = \mathcal{D}''^*\phi = 0, \iota_\Omega = 0\}$. From the point of view of operator L_k the cusp is "Dirichlet"-like iff $\dim\Phi_k = 0$; we will discuss this case later and assume that $\dim\Phi_k \geq 1$. Let us consider operator

$$\ell_k(x', D') = \mathbf{P}_k \beta M_k(x', D')^* \alpha^2 M_k(x', D')\beta + \beta^{-1} M_{k-1}(x', D')\alpha_1^{-2} M_{k-1}(x', D')^* \beta^{-1}\phi \tag{10}$$

acting from $C^\infty(X', \Phi_k)$ to $C^\infty(X', \Phi_k)$ with

$$M(x', D') = \mathcal{D}' + \frac{d''}{2}(D' \log f)\wedge) \tag{11}$$

acting from : $C^\infty(X', \Phi_k)$ to $C^\infty(X', \Lambda^{k+1}_\Omega)$, $M(x', D')^*$ a formally adjoint operator and P_k orthogonal projector from $L^2(\Omega, \Lambda^k)$ to Φ_k. Then the cusp term for operator L_k will be

$$N_{k,c}(\tau) = (2\pi)^{-d'} \iint \mathbf{n}_k(x', \xi', \tau^2)\,dx'd\xi' \tag{12}$$

where $\mathbf{n}_k(x', \xi', \tau)$ is the eigenvalue counting function for finite-dimensional[4)] symbol $\ell_k(x', \xi')$ of an operator $\ell_k(x', D')$.

Applying then (5) we get cusp contribution for operator $L_{k,0}$ given by the formula (10) with $\mathbf{n}_k(x', \xi', \tau^2)$ replaced by $\mathbf{n}_{k,0}(x', \xi', \tau^2)$ which is the eigenvalue counting function for $\mathcal{L}(\Phi_{k,0}, \Phi_{k,0})$-valued symbol $\ell_{k,0}(x', \xi') = \mathbf{P}_{k,0}(x', \xi')\ell_{k,0}(x', \xi')$; here $\Phi_{k,0}(x', \xi') = \{\phi \in \Phi_k, M_{k-1}(x', \xi')^*\beta^{-1}\phi = 0$ and $\mathbf{P}_{k,0}(x', \xi')$ is the orthogonal projection on $\Phi_{k,0}(x', \xi')$.

One can see easily that

$$\dim\Phi_{k,0} = \dim\Phi_k - \dim\Phi_{k-1,0}, \ \dim\Phi_{0,0} = 0 \tag{13}$$

and (9),(13) yield that

$$\dim\Phi_{k,0} = \sum_j \frac{(d'-1)!}{j!(d'-1-j)!}\dim\Phi''_{k-j}. \tag*{(9)$'$}$$

Furthermore, for $\alpha = I$, $\beta = I$ near infinity

$$N_{k,c}(\tau^2) = \dim\Phi_k N_c(\tau^2), \ N_{k,0,c}(\tau^2) = \dim\Phi_{k,0} N_c(\tau^2) \tag{14}$$

where $N_c(\tau^2)$ is a cusp contribution to the asymptotics for Neumann Laplacian.

One can expect that L_k or $L_{k,0}$ are "Neumann-like" iff $\dim\Phi_k = 0$ or $\dim\Phi_{k,0} = 0$ respectively.

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4) $\mathcal{L}(\Phi_k, \Phi_k)$-valued

3. Thin cusps. Results.

THEOREM 1. *Let either $L = L_k$, $\dim \Phi_k \ge 1$ or $L = L_{k,0}$, $\dim \Phi_{k,0} \ge 1$. Let*

$$\alpha = I,\ \beta = I \qquad \text{as } |x| \ge C \tag{15}$$

and let

$$|\nabla^\alpha \log f| \le C|x'|^{m+1-|\alpha|} \tag{16}$$

$$-\log f \asymp |x'|^{1+m}, \quad |\nabla^\alpha \log f| \asymp |x|^m \tag{17}$$

with $m > 0$. Then

(i) For $d'' \ge 2$ the following asymptotics holds

$$\mathbf{N}(\tau^2) = c_0\tau^d + N_c(\tau^2) + O\bigl(\tau^{d-1}\bigr) + O\bigl(\tau^q\bigr) \tag{18}$$

where c_0 and c_1 (see below) are standard Weyl coefficients, $N_c(\tau) \asymp \tau^q$ is defined by (12), $p = \frac{d'(m+1)}{m}$, $q = \frac{(d'-1)(m+1)}{m}$ and we omit operator-related indices k and may be 0;

(ii) For $d'' = 1$ the following estimate holds

$$\mathbf{N}(\tau^2) = c_0\tau^d + N_c(\tau^2) + O\bigl(\tau^{d-1}(\log\tau)^{\frac{d-1}{m+1}}\bigr) + O(\tau^q). \tag{19}$$

(iii) Moreover, if $d'' \ge 2$, (8) holds $L = L_{k,0}$ (and similar condition for $L = L_k$) and standard Hamiltonian condition is fulfilled, then one can replace $O(\tau^{d-1})$ in asymptotics (18) by $(c_1 + o(1))\tau^{d-1})$. One can find in theorem 0.1 [Ivr2] *how to improve $O(\tau^q)$.*

THEOREM 2. *Let $d'' = 1$, $m \ge d-2$ and conditions (15) – (17) be fulfilled and let $\log f$ be positively homogeneous of degree $m+1$. Then*

(i) Asymptotics

$$\mathbf{N}(\tau^2) = c_0\tau^d + \nu(\tau^2) + N'_c(\tau^2) + O\bigl(\lambda^{\frac{d-1}{2}}\bigr) \tag{20}$$

holds with $\nu(\tau^2) = c_3\tau^{d-1}(\log\tau)^{\frac{d-1}{m+1}} + O(\tau^{d-1})$ the Weyl expression for second order term in domain $\{f(x')\tau \le 1\}$.

(ii) Moreover, if $m > d-2$, (8) holds $L = L_{k,0}$ (and similar condition for $L = L_k$) and standard Hamiltonian condition is fulfilled, then one can replace $O(\tau^{d-1})$ in Asymptotics (20) by $(c_1 + o(1))\tau^{d-1})$. One can find in theorem 0.3 [Ivr2] *how to improve $O(\tau^q)$ for $m = d-2$.*

REMARK 3. One can weaken condition (15). Moreover, same asymptotics hold for manifolds and for manifolds with compact boundary one can skip condition (15).

4. Thick cusps. Sketch. In the case of 'Dirichlet'-like' cusp condition of being ultra-thin is no longer necessary and in this case one can derive asymptotics with sharp remainder estimates exactly in the same type as in [Ivr1], section 12.1. The only one difficulty is in the case of operator $L_{k,0}$, $\dim \Phi_{k,0} = 0$ and $\dim \Phi_k \ge 1$ but one can overcome them taking α_1 fast growing rather than constant at infinity.

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